BOUNDING CERTAINTY EQUIVALENT FACTORS AND RISK ADJUSTED DISCOUNT RATES

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The Problem

The certainty equivalent (CE) and risk adjusted discount rate (RADR) models have long been used for evaluating stochastic, multiperiod cashflows. If $\tilde{X} = \{\tilde{X}_t\}$ is an uncertain cashflow with expectation $\bar{X} = \{\bar{X}_t\}$ and certainty equivalent $\hat{X} = \{\hat{X}_t\}$, cashflow values are respectively computed as

$$\text{VCE} = \sum_{t=0}^{T} \text{VCE}_t = \sum_t \hat{X}_t (1+i_t)^{-t} = \sum_t a_t \bar{X}_t (1+i_t)^{-t} \quad \ldots\ldots\ldots (1)$$

and

$$\text{VRADR} = \sum_{t=0}^{T} \text{VRADR}_t = \sum_t \bar{X}_t (1+r_t)^{-t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The set $\underline{i} = \{i_t\}$ in (1) contains riskfree rates, whereas $\underline{a} = \{a_t\}$ is the set of CE-factors; a_t adjusting time t expected cashflow into the time t CE. In (2), $\underline{r} = \{r_t\}$ is the set of RADRs, r_t transforming $\bar{X}_t$ directly into a present value. Thus $(1+r_t)^{-t}$ in RADR corresponds to $a_t\ (1+i_t)^{-t}$ in CE, both converting the expectation of $\tilde{X}_t$ into a time zero value.

Since the estimation of $\underline{a}$ and $\underline{r}$ may be difficult in practice, a simplifying device is to put à priori bounds on their values. Before estimation starts, the decision maker may for instance exclude $a_t \geq 1$ or $r_t \leq 0$. Moreover, he may choose only to consider certain time profiles of $\underline{a}$ or $\underline{r}$, like just monotone or constant ones. In both cases, however, particular assumptions are implicitly made about the properties of $\tilde{X}$ and the way it is valued. This paper explores the exact nature of such assumptions.

Risk adjustment factors in CE and RADR models are exogeneous. That is, neither one tells how to measure and price risk through $\underline{a}$ and $\underline{r}$, which are simply undefined inputs in (1) and (2). Consequently, the term valuation model is a misnomer, as some external valuation principle is required before VCE and VRADR can be computed.

Several authors have suggested proper risk adjustments at a fixed time t. However, different external models are used which all are rather ad-hoc in a market context of multiperiod uncertainty. Moreover, only some of the relationships between a_t and r_t versus $\tilde{X}_t$ and valuation are discussed. Specifically,

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Everett and Schwab (1979) do not specify any external model except in very general terms (p.62). Bar-Yosef and Mesznik (1977) use individual utility functions to show that no bounds can be put on a_t. Schwab (1978) derives some properties of a_t and r_t in a total risk context, whereas Ma (1980) shows that with negative beta in a single-period CAPM, a_1 may exceed unity. A similar approach is also used by Miles and Choi (1979), Kudla (1980), Berry and Dyson (1980) and others referenced therein, discussing CE and RADR models when $\tilde{X}_t$ is an outflow.

The next section of this paper generalizes and extends these results, using time-state-preference (TSP) as external valuation principle. Besides being theoretically superior and generally accepted for market contexts of multiperiod uncertainty, its careful description of uncertainty shows the economic contents of the bounds quite well.

As to the choice of time profiles for $\underline{a}$ and $\underline{r}$, the only paper this author knows is Robichek and Myers' (1966) on the implications of $r_t = \bar{r}$, still referenced in most finance textbooks. The third section discusses flat profiles for $\underline{r}$ as well as $\underline{a}$, subsequently showing the counterintuitive properties of $\underline{a}$ and $\underline{r}$ when X_t is an outflow. The final section summarizes the paper in detail.

The Range of a_t and r_t

In the TSP model, the value V of cashflow $\tilde{X} = \{\tilde{X}_t\}$ is

$$V = \sum_{t=0}^{T} V_t = \sum_{t=0}^{T} \sum_{s=1}^{S(t)} X_{st}\,\phi_{st} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

V_t is the TSP value of $\tilde{X}_t$, t and S(t) are respectively the number of periods and time t states, and ϕ_{st} is the present price of a prospective \$1 in state s at t. Finally, $\tilde{X}_t = \{X_{st}\}$ is the S(t) state-contingent payoffs of $\tilde{X}$ at t, paying X_{st} with certainty if and only if s occurs.

As a riskless position results from buying claims to equal amounts in every state, the present price of a deterministic dollar at t is $\phi_t = \sum_s \phi_{st}$, the riskless discount rate i_t being $i_t = \phi_t^{-1/t} - 1$.

Having specified the external valuation model, TSP consistent risk adjustment factors a_t and r_t may be defined by requiring $VCE_t = VRADR_t = V_t$. Thus,

$$a_t = V_t/\bar{X}_t\phi_t = \sum_s X_{st}\phi_{st}/(\phi_t \sum_s X_{st} P_{st}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$r_t = (a_t\phi_t)^{-1/t} - 1 = (\bar{X}_t/V_t)^{1/t} - 1 = (\sum_s X_{st} P_{st}/\sum_s X_{st}\phi_{st})^{1/t} - 1 \quad \ldots (5)$$

Two points should be noticed. First, since P_{st} is the subjective probability of state s occurring at t, neither a_t nor r_t is a market parameter. Thus, unlike with the CAPM as external valuation principle, TSP consistent risk adjustments may vary across market participants.

Second, because the TSP model is value additive over time, both α_t, r_t and any bound depend on time t data alone. This happens because complete markets allow any cashflow $\tilde{Z}$ to be traded for any equally-valued pattern $\tilde{Y}$. Without this property, however, the value of $\tilde{X}_t$ would depend on the remaining $\tilde{X}_k$ $\in\tilde{X}$, $k\neq t$, and $\tilde{X}_t$ could not be separated out for independent valuation. Thus, although all the referenced papers determine α_t and r_t from time t data alone, that implicitly assumes time additivity in the external valuation model. Unlike the external models used by these authors, the economic environment behind (3) includes multiperiod completeness.

Turning next to the various restrictions on α_t and r_t implied by (3), the CE model permits uncertainty in time zero cashflow. Because $\text{VRADR}_0 = \bar{X}_0$ for any r_0, however, full certainty is implicitly assumed. The following presupposes $t>0$.

TABLE 1
BOUNDS ON RISK ADJUSTMENT PARAMETERS IMPLIED BY TSP VALUATION

Risk adjustment parameter	(a) $\bar{X}_t \to 0$	(b) $\bar{X}_t = 0$	(c) $(V_t\bar{X}_t)<0$	(d) $V_t \to 0$	(e) $V_t = 0$	(f) $V_t=\bar{X}_t\,\phi_t$	(g) $(V_t/\bar{X}_t\phi_t)>1$	(h) $(V_t/\bar{X}_t)>1$
α_t	$\to\infty$	undefined	<0	$\to 0$	0	1	>1	$>(1+i_t)^t$
r_t	$\to -1$	-1	imaginary at even dates, real otherwise	$\to\infty$	undefined	i_t	$<i_t$	<0

As shown by columns (a) and (b) of Table 1, α_t approaches infinity as $\bar{X}_t$ goes to zero, being undefined if $\bar{X}_t = 0$. Hence, it erroneously suggests that $\tilde{X}_t$ is extremely valuable per unit of expectation. The same lack of economic logic occurs in (5), where r_t approaches -100% as $\bar{X}_t$ goes to zero. For either model to be useful, therefore, $\bar{X}_t \neq 0$ must be assumed.[1]

The above implicitly assumes that V_t and $\bar{X}_t$ have equal signs, i.e., $V_t\bar{X}_t>0$. If not, α_t would approach $-\infty$ and not ∞ as $\bar{X}_t$ goes to zero. The RADR would still approach -1 at odd dates, but would turn imaginary when t is even. More generally, as suggested by (4) and column (c), $\alpha_t \lesssim 0$ requires V_t and $\bar{X}_t$ to have opposite signs. Clearly, this cannot be unless $\tilde{X}_t$ is a mixture of contingent inflows and outflows, paying a positive amount in at least one state and a negative amount in one or more other states.[2] With $\alpha_t < 0$ (corresponding to $V_t\bar{X}_t<0$) then r_t is imaginary for even t so r_t is economically meaningless. Thus, the RADR model is not viable, but the CE model is.

Unlike for $\bar{X}_t$, a V_t which is close or equal to zero is unproblematic in the CE model (columns (d) and (e)). However, r_t approaches infinity as V_t goes to zero, suggesting that risk increases beyond all bounds, which is nonsense.[3] Furthermore, as r_t is undefined for $V_t = 0$, bounds on a real-valued RADR are only meaningful if the CE factor is strictly positive (i.e., $V_t\bar{X}_t>0$).

Having discussed the cases where an à priori bound on at least one of a_t and r_t is problematic, consider next the uncomplicated column (f). If $X_{st} = \bar{X}_t$ in every state, then $\tilde{X}_t$ is deterministic, and $V_t = \bar{X}_t\phi_t$ from (3). Moreover, a stochastic $\tilde{Y}_t$ with $E(\tilde{Y}_t) = \bar{X}_t$ may also have that property, like when probabilities are proportional to prices (Bøhren and Ekern, 1981). In both cases, $a_t = 1$ and $r_t = i_t$ from, respectively, (4) and (5), such that no risk adjustment is called for in either model. If the single-period CAPM were used as an external valuation model, the analogy would be that the value of a deterministic date 1 cashflow equals that of a stochastic one with the same expectation if the latter's beta is zero.

In column (g), $V_t/\bar{X}_t\phi_t > 1$. As the denominator $\bar{X}_t\phi_t$ is the present value of a time t claim to $\bar{X}_t$ in every state, $V_t/\bar{X}_t\phi_t > 1$ implies that the stochastic $\tilde{X}_t$ with expectation $\bar{X}_t$ is more valuable than its deterministic counterpart. That may occur if $\tilde{X}_t$ pays off relatively much in highly valued states. Loosely speaking, ϕ_{st} is high if the total market payoff is low in that state, if investors in aggregate put a high probability on state occurrence, or if they regard a marginal dollar as particularly desirable if that state occurs. In the CAPM, the explanation would be that the beta of $\tilde{X}_1$ is negative.

Because $\bar{X}_t$ and V_t have equal signs and differ from zero when $V_t/\bar{X}_t\phi_t > 1$, a real r_t exists. Moreover, as $a_t > 1$, then by (6) the riskfree rate exceeds the risk adjusted one.

The final column of Table 1 assumes $V_t/\bar{X}_t > 1$, causing a_t to exceed unity by $(1+i_t)^t-1$ and r_t to be negative. Compared to column (g), an even stricter bound is now put on X_t, as its present value must exceed time t expected cashflow (i.e., a present deterministic dollar is worth less than a time t lottery having the same expectation). Clearly, this cannot be if valuation includes risk aversion and positive time preference. Despite some suggestions that a negative, real RADR may be of interest (Joy and Grube, 1981, p.155), it therefore seems fair to conclude that the case has insignificant practical importance. Correspondingly, $(1+i_t)^t$ may safely be considered an upper bound for a_t.

As to the general relationship between the two risk adjustment parameters and the valuation of $\tilde{X}_t$, it follows from (4) and (5) that

$$\partial a_t/\partial r_t = (\partial r_t/\partial a_t)^{-1} = a_t(-t/(1+r_t)) = -t\phi_t{}^{-1}(V_t/\bar{X}_t)^{1+1/t} \quad \ldots\ldots(6)$$

Assuming that V_t and $\bar{X}_t$ have equal signs and differ from zero, (6) is negative. Hence, TSP consistent (a_t, r_t) pairs always move in opposite directions if underlying time t valuation parameters change.

Time Profiles of $\underline{a}$ and $\underline{r}$

The previous section refers to à priori bounds at a fixed point in time, offering no insight into the different problem of excluding particular time patterns before estimation starts. The first part of this section discusses the common practice of setting $a_t = a$ or $r_t = r$ for every t, stressing the relationship between the two. Finally, the nonintuitive properties of $\underline{a}$ and $\underline{r}$ under cash outflows is explored.

The constant RADR of $\underline{r} = \{ r \}$ implies from (5) that

$$\bar{X}_t/V_t = (1+r)^t, \ t = 1,2, \dots, T. \qquad (7)$$

Accordingly, expected cashflow to present value must be a geometric series with parameter (1+r). As a meaningful RADR requires $\bar{X}_t V_t > 0$ at every date (see previous section), a positive r presupposes that $\bar{X}_t/V_t$ increases from one day to the next, i.e., the relative $\bar{X}_t$ change must exceed that of V_t.[4]

Substituting (7) into (4), a constant RADR corresponds to

$$a_t = (1+i_t)^t/(1+r)^t \qquad (8)$$

With $i_t = i$, this is the well known result of Robichek and Myers (1966): if $r_t = r$, then $VRADR_t = VCE_t$ only holds if $\underline{a}$ is a geometric series with the parameter (1+i) / (1+r).

Comparing (7) and (8), a particular $\bar{X}_t V_t$ series causes a constant RADR, but the corresponding shape of $\underline{a}$ depends on the riskfree rate as well. Thus, a_t exceeds unity and increases if $\tilde{X}_t$ is more valuable than a riskfree claim to $\bar{X}_t$, being less than 1 and decreasing otherwise.

Consider next a time-independent CE-factor of $\underline{a} = \{ a \}$. As

$$V_t/\bar{X}_t \, \phi_t = a, \qquad (9)$$

this implicitly assumes that the ratio between the present value of $\tilde{X}_t$ and that of a deterministic $\bar{X}_t$ always equals a. Moreover, $\underline{a} = \{ a_t \}$ means

$$r_t = (1+i_t)\, a^{-1/t} - 1 \qquad (10)$$

Thus, if $i_t = i$, the $\underline{r}$ series decreases when $a < 1$, staying constant at $r_t = i$ for $a = 1$.

Reviewing, the pair $\{ \underline{a} = \{a\} , \underline{r} = \{ r \} \}$ is only consistent in the trivial case of $V_t = \phi_t \bar{X}_t$ and $\phi_t = (1+i)^{-t}$ for all t, yielding $a = 1$ and $r = i$. This may indicate that a_t and r_t reflect fundamentally different valuation properties. Still, Robichek and Myers (1966, p.728) consider $\tilde{X}_t$ and $\tilde{X}_{t+\Delta}$ equally risky if $a_t = a_{t+\Delta}$ $(\Delta > 0)$, suggesting that the CE-factor may serve as a proper indicator of risk adjustment. However, if instead the RADR were to be used as risk proxy, $\tilde{X}_{t+\Delta}$ would be classified as less risky than $\tilde{X}_t$, as $r_{t+\Delta} < r_t$ by (10).

Clearly, in order for either $\underline{a}$ or $\underline{r}$ to be flat over time, very restrictive assumptions with a weak intuitive backing are required. As shown elsewhere (Bøhren

1981), a CE model with $\underline{a} = \{a\}$ has the same weakness as the simulation procedure of Hertz (1964) and the related analytical model of Hillier (1963). On the other hand, $\underline{r} = \{r\}$ in the RADR is shown to share the problems of the $\bar{X}$-based internal rate of return model. Thus, although $\underline{a} = \{a\}$ and $\underline{r} = \{r\}$ rely on different assumptions about the periodic payoffs and market values of $\tilde{X}$, it is not obvious which of the two is more restrictive. However, whereas the problems of a constant RADR are very frequently stressed in the literature, those of a constant CE-factor are not. In fact, the model comparison is usually made by contrasting $r = \{r\}$ to an unbounded $a = \{a_t\}$. From our analysis, that seems as unfair as criticizing $\underline{a} = \{a\}$ relative to an unbounded $\underline{r} = \{r_t\}$.

Finally, if $\tilde{X}_t$ is an outflow with negative $\bar{X}_t$ and V_t, the properties of risk adjustment parameters change radically. Differentiating (4) and (5) with respect to V_t;

$$\partial a_t/\partial V_t = 1/\bar{X}_t \phi_t < 0 \quad (11)$$

$$\partial r_t/\partial V_t = (-\bar{X}_t/tV_t^2)(\bar{X}_t/V_t)^{(1/t)-1} > 0 \quad (12)$$

Thus, a high TSP value per unit of $\bar{X}_t$ is reflected by a large RADR and a low CE-factor, such that if two outflows have the same $\bar{X}_t$, the most valuable one (i.e., the one causing the least cash drain in highly valued states) gets the strongest risk adjustment. Clearly, this counterintuitive relationship is simply due to the way the two models are constructed. As $VRADR_t = \bar{X}_t/(1+r_t)^t$, this ratio increases towards zero from below as r_t rises. Correspondingly, value is less negative the smaller is a_t in $VCE_t = a_t\bar{X}_t/(1+i_t)^t$. Such conceptual problems occur because, unlike TSP, the two models are not stated in explicit price and quantity format.

In most capital budgeting projects, cash outflows in the first years are followed by inflows later on. Consequently, a_t is first a decreasing and subsequently an increasing function of cashflow value (r_t increases, then decreases). For financing projects, where inflows precede outflows, the opposite is true. In either case, the problem of à priori disregarding some numerical values for a_t and r_t or certain time profiles for $\underline{a}$ and $\underline{r}$ gets still another dimension of complexity.

Summary

One way of reducing estimation complexity is by bounding the estimators à priori. This paper explores the validity of such an approach for the parameter sets $\underline{a} = \{a_t\}$ and $\underline{r} = \{r_t\}$ of respectively the certainty equivalent (CE) and risk-adjusted discount rate (RADR) models.

As neither model defines its inherent risk adjustments, $\underline{a}$ and $\underline{r}$ were specified from time-state-preference (TSP). Unlike the valuation principles previously used, the time additive property of the TSP justifies a definition of a_t and r_t in terms of just time t parameters. Moreover, its careful description of multiperiod uncertainty brings out the economic content of the bounds quite clearly.

If the analyst restricts his attention to certain general classes of $\underline{a}$ and $\underline{r}$ profiles, particular assumptions are implicitly made about the cashflow $\tilde{X} = \{\tilde{X}_t\}$, its expectation $\overline{X} = \{\overline{X}_t\}$, and market values $V = \sum_t V_t(\tilde{X}_t)$. Additionally, $\underline{a}$ also depends on the pattern of riskfree rates $\underline{i} = \{i_t\}$. In general, it is difficult to specify the type of profile before estimation starts, and intuition may not be a too helpful guide. Although a const: nt RADR is clearly less general than an unbounded $\underline{a}$, a more fair standard of comparison is the constant $\underline{a} = \{a\}$. The latter involves different assumptions about the cashflow being valued, but they may be as restrictive as those implied by $\underline{r} = \{r\}$.

Besides focusing on certain general time profiles of $\underline{a}$ and $\underline{r}$, the numerical values of the time t parameters a_t and r_t may also be restricted. However, several cases exist where à priori bounds are either ineffective or intuitively troublesome. First, bounding r_0 is useless, as the value of $\tilde{X}_0$ in RADR is always $\overline{X}_0$. Second, if $\tilde{X}_t$ is a mixture of contingent inflows and outflows, r_t may be imaginary when t is even, a_t being negative whatever the value of t. Third, a_t is undefined and $r_t = -100\%$ when $\overline{X}_t = 0$, a_t approaching plus or minus infinity as $\overline{X}_t$ goes to zero. Clearly, that involves a meaningless economic interpretation of a_t and r_t.

In sum, these problems suggest that no à priori bounds are helpful when $\overline{X}_t$ is close to zero, and that restrictions in the RADR model may only reduce estimation complexity when $\overline{X}_t$ and V_t differ from zero and have equal signs.

If $\tilde{X}_t$ has the value of a riskfree claim, then $a_t = 1$ and $r_t = i_t$. When $\tilde{X}_t$ pays off relatively well in highly valued states, a_t may exceed unity, causing r_t to be less than i_t. Finally, $a_t > (1+i_t)^t$ and r_t is negative when a present, deterministic dollar is less valuable than a claim to a time t lottery having the same expectation. The relevance of this extreme case seems insignificant, being infeasible if market valuation is characterized by risk aversion and positive time preference.

Capital budgeting projects typically have cash outflows in early years and inflows later on. Counterintuitively, a_t is first a decreasing function of cashflow values, subsequently increasing when the cashflow changes sign (r_t first increases, then decreases). In such cases, the problem of à priori disregarding some numerical values for a_t and r_t or certain shapes for $\underline{a}$ and $\underline{r}$ gets still another dimension of complexity.

NOTES

1 It is not true in general that when $\overline{X}_t$ approaches zero, V_t also does. For instance, going from investor a to b, the former's probability beliefs may cause his subjective $\overline{X}_t^a$ to be closer to zero than b's $\overline{X}_t^b$ (the state-contingent payoffs X_{st} of $\tilde{X}_t$ are investor independent; only probability beliefs in $\tilde{X}_t$ may vary). The market value V_t, however, does not change from one investor to the next.

2 If $\overline{X}_t > 0$, contingent outflows dominate inflows in value terms, yielding $V_t < 0$. For $\overline{X}_t < 0$, the converse is true.