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PROBABILITIES PROPORTIONAL TO TIME-STATE PRICES

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If probabilities and implicit prices of elementary state-contingent claims are proportional across states, any random income stream component is valued by discounting its expectation at the riskless rate. This lacking risk adjustment is explained by degeneracy in either market risk aversion or in randomness of aggregate consumption.

1. Introduction

As noted by Drèze (1970, p. 149), the normalized implicit prices in the time-state preference (TSP) general equilibrium model initiated by Arrow (1964) and Debreu (1959, ch. 7) have all the formal properties of a probability measure. However, Baron's (1979, p. 210) suggestion of interpreting these prices as an individual's state probabilities may be rather misleading, as it would be warranted only if implicit prices and probabilities are proportional (henceforth: PPP) across states. This note focuses exclusively on the case where such proportionality obtains, presenting the conditions and implications of PPP.

The next section outlines a TSP model and the PPP property. The following section shows that at any date for which PPP holds, the value of any risky income stream component is computed simply by discounting its expectation at the riskless rate. This lack of risk adjustment is subsequently explained by the fact that with state-independent preferences, PPP holds if and only if either there are some risk neutral individuals in the market or aggregate consumption is constant across states. Whereas different aspects of the results can be found scattered

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around in the literature, the final section notes that the unifying PPP approach may ease the understanding of previous contributions.

2. Model

Consider the TSP model, where the subscripts t and s denote, respectively, a date and a state at that date. Let $\tilde{X}=\{\tilde{X}_t\}$ be a stochastic income stream, and the component $\tilde{X}_t = X_{ts}$ with certainty if s occurs at t. Moreover, $\Pi_t^i = \{\Pi_{ts}^i\}$ is individual i's subjective probability distribution over time t states, as assessed at time zero. Assume there exists an equilibrium in the securities market, and let ϕ_{ts}^i be individual i's present implicit price of an elementary security paying \$1 at date t if and only if state s obtains. Then, the time zero market value of the income stream component $\tilde{X}_t$ is

$$V_t = \sum_s \phi_{ts}^i X_{ts} \tag{1}$$

for everybody, and the entire stream $\tilde{X}$ is valued by $V = \Sigma_t V_t$. Moreover, assuming the existence of a riskless asset, the riskless discount factor $\phi_t = \Sigma_s \phi_{ts}^i$ is also identical across investors.

Under PPP the ratio of implicit prices to probabilities equals the riskless discount factor. That is,

$$\phi_{ts}^i / \Pi_{ts}^i = \phi_t \tag{2}$$

as $\phi_{ts}^i / \Pi_{ts}^i = \Sigma_s \phi_{ts}^i / \Sigma_s \Pi_{ts}^i$ by corresponding addition.

3. Implications

Proposition 1. If implicit prices are proportional to their respective state probabilities, then an income stream component is valued by discounting its expectation at the riskless rate.

The result follows directly from substituting (2) into (1), yielding $V_t = \phi_t \Sigma_s \Pi_{ts}^i X_{ts}$, or, using the expectation operator E,

$$V_t = \phi_t E^i(\tilde{X}_t). \tag{3}$$

An indirect demonstration builds on Theorem 1 of Rubinstein (1976, p. 409), who showed the existence of some positive random variable $\tilde{Y}=\{\tilde{Y}_t\}$, such that

$$V_t=\phi_t\left[E^i(\tilde{X}_t)+\text{cov}^i(\tilde{X}_t,\tilde{Y}_t)/E^i(\tilde{Y}_t)\right], \tag{4}$$

where cov is the covariance operator. As

$$Y_{ts}\equiv\phi_t^{-1}E^i(\tilde{Y}_t)\phi^i_{ts}/\Pi^i_{ts} \tag{5}$$

substituting (2) into (5) yields $Y_{ts}=E^i(\tilde{Y}_t)$ for all s and i. Hence, the covariance term in (4) is zero under PPP, and (4) simplifies into (3).

Drèze (1970, p. 160) first observed that riskless discounting follows from PPP (actually, he discussed equality rather than proportionality between state prices and probabilities). Hirshleifer (1970, p. 234) found that PPP implies that a riskless position is not only feasible but also optimal. Both using a 'large market' argument based on replicating individuals within a single-period economy, but without invoking PPP, Malinvaud (1972, sect. 5) argued for maximization of expected profit, and Caspi (1974, sect. V) showed how a competitive equilibrium would yield each trader his expected endowment. More recently, Breeden and Litzenberger (1978, sect. V) noted that valuation requires only the expected payoffs of a security, provided aggregate consumption is known and that beliefs are homogeneous.

4. A closer look

Proposition 2. With state-independent preferences and implicit prices proportional to their respective state probabilities, individuals are either risk neutral or do not face any aggregate risk at all at their optimal positions in the market.

Considering first a securities market that either is complete or could freely be made so, $\phi^i_{ts}=\phi_{ts}$ for all individuals. Moreover, the first-order conditions for interior solutions of the individuals' portfolio optimization problems are that [e.g., see Drèze (1970, p. 151)]

$$\phi_{ts}=\Pi^i_{ts}\lambda^i_{ts}/\lambda^i_0. \tag{6}$$

Here, λ^i_{ts} is marginal utility in state s at time t, λ^i_0 is expected marginal utility of current wealth, and the superscript on ϕ_{ts} has been dropped because of the completeness assumption. Thus, the implicit prices are products of a probability and a ratio of marginal utilities.

From (2), PPP is then only consistent with the homogeneous probability beliefs of $\Pi^i_{ts} = \Pi_{ts}$. Moreover, using (2) and (6), $\lambda^i_{ts}/\lambda^i_0 = \phi_t$ for all i and s under PPP.

Individuals' preferences for consumption C^i_{ts} are assumed to be represented by time additive and state-independent monotonically increasing but non-strictly concave utility functions $U^i_t(C^i_{ts})$. As more precisely $\lambda^i_{ts} = \mathrm{d}U^i_t(C^i_{ts})\mathrm{d}C^i_{ts}$, this derivative can only be a constant across states if the utility function is either linear or being evaluated at a fixed value of its argument for all states at the date t. These two cases inducing PPP are discussed separately below.

If all individuals are risk neutral, then PPP is satisfied. Risk neutral valuation may also be imposed on a risk averter by one or more risk neutral traders who are not being constrained by short-selling restrictions or other boundary conditions. As such individuals will pick up all risk [Mossin (1973, p. 107)], remaining risk averters will in fact face no risk in the market. Then, with fixed probabilities, prices will adjust such that in equilibrium, PPP prevails for everybody.

PPP can also be caused by a deterministic aggregate consumption, as it will then be optimal for each individual to have state-independent consumption. The advantage of pooling states with the same aggregate social wealth was first demonstrated by Borch (1960, p. 170), and has later been shown by Mossin (1973, p. 108), Caspi (1974, Theorem 1), Hakansson (1978, Theorem 2) and Breeden and Litzenberger (1978, Theorem 1). The latter authors (1978, Lemma 2) also note that PPP holds for the subset of states having the same aggregate consumption. Johnsen (1979, Proposition 2) comments that constant social endowments are equal to a pure martingale property, which is another way of characterizing PPP.

Thus, degeneracy in either randomness of aggregate consumption or in at least one individual's risk aversion implies PPP.

Switching to a single-period CAPM framework,

$$V = \phi\left[E(\tilde{X}) - \theta \operatorname{cov}(\tilde{X}, \tilde{R})\right], \tag{7}$$

where time subscripts are omitted, θ is the market price of risk, and $\tilde{R}$ is the cash flow of the market portfolio. By (7), the market price of an elementary state s claim is $\phi_s = \phi\{\Sigma_k \Pi_k X_k - \theta \Sigma_k \Pi_k [X_k - E(\tilde{X})][R_k - E(\tilde{R})]\}$ which simplifies into

$$\phi_s / \Pi_s = \phi\{1 - \theta[R_s - E(\tilde{R})]\}. \tag{8}$$

Hence, consistently with Proposition 2, PPP holds if and only if either the market pays nothing for risk-bearing ($\theta = 0$) or, by portfolio separation, no individual faces any risk whatsoever because the market return is deterministic ($R_s = E(\tilde{R})$).

Relaxing the completeness assumption, the first-order optimality condition is

$$V^j = \sum_t \sum_s \phi^i_{ts} X^j_{ts} \tag{9}$$

for all securities j. Formally, Proposition 1 goes through, using possibly heterogeneous probability beliefs. However, to ensure the existence of an equilibrium, additional restrictions beyond those inducing PPP are required, and most of the apparent increased generality disappears. In case of some risk neutral individuals in the economy, everybody has to agree on the expected cashflow $E^i(X^j_t) = E(X^j_t)$. If there are no risk neutral individuals at an interior solution, then there can be neither any randomness in aggregate consumption nor heterogeneous, probability beliefs for PPP to hold.

5. Concluding remarks

The PPP propositions may be combined and rephrased without explicit reference to the PPP property: expected profit is a sufficient statistic if and only if either the market price of risk is zero or if the social risk is zero. As such, various versions of the results or parts thereof have been known for some time. Beyond presenting a synthesis, our main purpose has been bringing out how the PPP property provides an illustrative link between the two parts of the equivalence relation above. The note also applied PPP to Rubinstein's theorem and to a CAPM economy. By using PPP as the starting point, underlying restrictions on probability beliefs, on risk aversion and on social endowments followed from the analysis rather than being á priori assumptions as in previous approaches.

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