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CONSISTENT RANKINGS BASED ON TOTAL AND DIFFERENTIAL AMOUNTS UNDER UNCERTAINTY*

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ABSTRACT

Rankings of decision alternatives based on total and on differential monetary amounts may in general be inconsistent under uncertainty. In the following cases, either approach is valid and hence yields consistent rankings: (i) with expected values, provided that the differential amounts have been coherently determined and are constant within states (but may differ across states); (ii) with exponential utility functions if the shared and differential amounts are statistically independent; or (iii) in a market valuation context, assuming diversification and implicit separate market values for differential and shared amounts.

INTRODUCTION

Formal recognition of uncertainty may challenge generally accepted tenets in managerial economics [2]. Recently, Vedder [12] examined the validity of focusing on differential costs and benefits between pairs of decision alternatives. The present paper argues that consistent rankings based on coherently determined differential amounts are not as restricted under uncertainty as an unreflected reading of Vedder's analysis may indicate. The first section demonstrates how apparent trouble with differential amounts and expected monetary values vanishes by recasting the decision problem to fit the framework of statistical decision theory. The second section shows how differential amounts are appropriate for exponential utility functions if differential and shared amounts are statistically independent. The third section points out that shared amounts are irrelevant in a market valuation context. A short summary, followed by a technical appendix, concludes the paper.

More specifically, let the subscript i indicate a relevant decision alternative, and let x_i, y_i, and z be random variables expressed in monetary units. Suppose the relationship

$$x_i = y_i + z \qquad (1)$$

holds for all decision alternatives. Then x_i is interpreted as a total amount, y_i as a differential amount, and z as a shared amount. Our problem is then to identify

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the cases in which a ranking of alternatives based on distributions of total amounts x_i is consistent with one using marginal distributions of differential amounts y_i.

DIFFERENTIAL AMOUNTS AND EXPECTED VALUES

In statistical decision theory [11] as well as in managerial information analysis [1], the concept of state denotes a particular, distinct configuration of the world, ideally leaving no relevant aspect of the decision maker's choice environment undescribed. The states are the primitive entities to which probabilities are assigned. The choice of any decision alternative leaves the probability distribution over states unchanged. Let j index a state, and let $P(\cdot)$ be a probability measure. In the state formulation, equation (1) may be written

$$x_{ij} = y_{ij} + z_j \tag{2}$$

such that the shared amount z is a constant between alternatives within a state, but may vary across states. By construction, a state model involves $P(x_{ij}) = P(x_{hj})$ for all decision alternatives i and h and all states j.

In contrast, the conditional decision-theoretic formulation [3, ch. 3] [5, ch. 8] [6] associates probabilities with events rather than states. More important, these probabilities may be affected by the selected decision alternative. Hence, a salient feature of the conditional formulation is that $P(x_{ik}) = P(x_{hk})$ does not necessarily hold where the subscript k denotes an event. In this model, equation (1) turns into

$$x_{ik} = y_{ik} + z_k. \tag{3}$$

The statistical and the conditional formulations are compatible in the sense that conversion from one formulation to the other one is, in principle, always possible in the finite case. The two approaches have different properties, however, when it comes to consistent rankings based on differential amounts. In the statistical formulation, either total or differential amounts apply to the risk-neutral case. In the conditional framework, however, only total amounts would be appropriate unless the shared amount z_k in (3) is constant across events. Therefore, when the shared amount is not a constant, only the statistical formulation is feasible if differential amounts are to be used for ranking. This point is illustrated below.

Consider Vedder's example [12] with costs of alternative advertising campaigns depending on demand level, where the demand probabilities are influenced by the decisions. With two events, L (low demand) and H (high demand), relative to two decision alternatives, A_1 and A_2, there are four conditional events, L_1, L_2, H_1, and H_2, each having different probabilities. For example, L_1 has the interpretation "if decision A_1 is taken, then demand will be low."

The data for the conditional events model are given in Table 1. Based on expected total costs, decision A_1 is the preferred alternative (\$64,000 < \$74,000), whereas A_2 is preferred (\$51,000 > \$40,000) if expected differential costs are used.

TABLE 1
Variable Costs (in thousand dollars) for Alternative Advertising Plans in Vedder's [12] Example Using Conditional Events Model

$$x_{ik} = y_{ik} + z_k$$

Advertising Plan	Events: Low Demand (L)	Events: High Demand (H)	Expected Value $E(x_i) = E(y_i) + E(z)$
A_1	60 = 50 + 10 (.9)	100 = 60 + 40 (.1)	64 = 51 + 13
A_2	50 = 40 + 10 (.2)	80 = 40 + 40 (.8)	74 = 40 + 34

Now the example is reformulated to fit into the state framework. Recalling that state probabilities are invariant with respect to decision alternatives, four distinct states, L_1L_2, L_1H_2, H_1L_2, and H_1H_2, are required. As an example, the state L_1H_2 has the interpretation "if decision A_1 is taken, then demand will be low; if decision A_2 is taken, then demand will be high." Also, let p be the probability of state L_1L_2, where the probabilities of all states are to be consistent with the event probabilities of Table 1. Thus, for example, the probability of the conditional event L_1 must equal the sum of the state probabilities for L_1L_2 and L_1H_2.

TABLE 2
Variable Costs (in thousand dollars) for Alternative Advertising Plans in Vedder's [12] Example Using States Model

$$x_{ij} = y_{ij} + z_{ij}$$

Advertising Plan	States: L_1L_2 (p)	States: L_1H_2 (0.9 − p)	States: H_1L_2 (0.2 − p)	States: H_1H_2 (p − 0.1)	Expected Value $E(x_i) = E(y_i) + E(z_i)$
A_1	60 = 50 + 10	60 = 50 + 10	100 = 60 + 40	100 = 60 + 40	64 = 51 + 13
A_2	50 = 40 + 10	80 = 40 + 40	50 = 40 + 10	80 = 40 + 40	74 = 40 + 34

Table 2 displays the appropriate data and computations for the state model. In states L_1L_2 and H_1H_2, the shared costs z_j are, respectively, \$10,000 and \$40,000. Carefully observe, however, that in states L_1H_2 and H_1L_2 the alleged "shared" costs z_j vary within states. Equation (2) is replaced by

$$x_{ij} = y_{ij} + z_{ij} \tag{2'}$$

causing $E(z_1) \neq E(z_2)$. Hence, the correctly determined expected "differential" cost for the state model is not $E(y_i)$ but $E(x_i)$ itself. Thus the consistency between total and differential amounts is trivial. When (2) holds, however, the applicability of differential amounts for risk-neutral decision makers is obvious from the fact that the expectation of a sum always equals the sum of the expectations.

DIFFERENTIAL AMOUNTS AND RISK AVERSION

Suppose the decision maker has a nonlinear utility function u(x) for money. If he is risk averse, then u(x) is a concave function, which is monotonically decreasing if x is expressed as costs and monotonically increasing if x denotes benefits. Vedder [12] correctly asserts that use of differential amounts can lead to erroneous choices in such a case. He fails to provide information about the class of utility functions for which the use of total or differential amounts does not matter, given that differential and shared amounts are independent.

Let $C(\bullet)$ be the certainty equivalent operator defined for the marginal distribution of the operand. Consistent ranking of a pair of decision alternatives indexed by i and h requires

$$C(x_i) > C(x_h) <=> C(y_i) > C(y_h)$$

or, using (1),

$$C(y_i + z) > C(y_h + z) <=> C(y_i) > C(y_h).$$

For this to hold for any random variable z, the certainty equivalent must be separable, i.e.,

$$C(y_i + z) = C(y_i) + C(z). \tag{4}$$

It is well known [4, p. 167] [7] [8] that a necessary but not sufficient condition for (4) to hold for any random variable z is that the utility function u(x) is either linear $u(x) = x$ or exponential

$$u(x) = -e^{\gamma x}. \tag{5}$$

Assuming risk aversion, γ is a positive constant if x denotes costs and a negative constant if x denotes benefits. Of course, a positive linear transformation of (5) works just as well.

Combining (1) and (5), and ignoring subscripts,

$$u(x) = -e^{\gamma x}$$
$$= -e^{\gamma(y+z)}$$
$$= (e^{\gamma z})(-e^{\gamma y}).$$

If the amounts y_i and z are statistically independent, then by taking expectations,

$$E(u(x)) = E(e^{\gamma z})E(u(y)). \qquad (6)$$

As $E(e^{\gamma z})$ is a positive constant, rankings based on expected utility of total amounts $E(u(x_i))$ or on expected utility of differential amounts $E(u(y_i))$ will be consistent when y_i and z are statistically independent. Note in particular that when z is a constant across states, then independence between y_i and z is automatically satisfied. In that case, total and differential amounts will always yield consistent rankings for an exponential utility function.

To clarify the importance of the independence restriction, note that if the random variable x (ignoring subscripts) is normally distributed with mean μ_x and variance σ_x^2, then its corresponding certainty equivalent is

$$C(x) = \mu_x + .5\gamma\sigma_x^2 \qquad (7)$$

for the exponential utility function (5) [4, pp. 201-202]. Assume y and z have a bivariate normal distribution with means μ_y and μ_z, variances σ_y^2 and σ_z^2, and correlation coefficient ϱ. Hence, if $x = y + z$ (ignoring subscripts), $\mu_x = \mu_y + \mu_z$ and $\sigma_x^2 = \sigma_y^2 + \sigma_z^2 + 2\varrho\sigma_y\sigma_z$. Substituting these values into (7), it is seen that

$$C(x) = (\mu_y + \mu_z) + .5\gamma(\sigma_y^2 + \sigma_z^2 + 2\varrho\sigma_y\sigma_z)$$
$$= (\mu_y + .5\gamma\sigma_y^2) + (\mu_z + .5\gamma\sigma_z^2) + \gamma\varrho\sigma_y\sigma_z$$
$$= C(y) + C(z) + \gamma\varrho\sigma_y\sigma_z.$$

Thus, (4) is not satisfied unless $\varrho = 0$. A ranking based on exponential utility function and differential amounts would therefore be erroneous when differential and shared amounts are correlated.

Let w be the decision maker's total wealth exclusive of the project being considered. If the project's random monetary consequences x_i are to be evaluated independently of w, then reasoning analogous to the one leading to (4) shows that the utility function must be either linear or exponential. Hence, if a proper ranking does not require the use of expected utility of total asset position $E(u(w + x_i))$, then the project's differential amounts y_i and total amounts x_i are equally applicable for project selection, under the stated independence assumption.

Consider Vedder's [12] second example, regarding daily costs of two alternative machines (projects) under two different breakdown incidences (states), as given by Table 3. Using the utility values of the second column of Table 4, Vedder computed the certainty equivalent total costs of $200 for machine A and $175 for machine B, whereas the corresponding certainty equivalent differential costs were $0 and $25, respectively. Hence, total costs indicated a preference for machine B and differential costs a preference for machine A.

TABLE 3
Daily Costs (in dollars) of Alternative Machines in Vedder's [12] Example Using Model
$x_{ij} = y_{ij} + z_j$

Decision Alternative	Incidence of Breakdown: Low (.5)	Incidence of Breakdown: High (.5)
Machine A	100 = -50 + 150	300 = 150 + 150
Machine B	175 = 25 + 150	175 = 25 + 150

TABLE 4
Utility Scale Values for Selected Daily Costs

Costs in Dollars	Utility Values Used by Vedder [12, Table 4]	Exponential Utility Function Computed by (10) in the Appendix
300	0.00	0.0000
216.937	NA	.35565
200	.44	.4165
175	.50	.5000
150	.85	.5766
100	.88	.7113
66.937	NA	.7883
25	.90	.8742
0	.925	.9198
-50	1.00	1.0000

Next, use an exponential utility function, with values presented in the rightmost column of Table 4 (details of the computations are given in the Appendix). The expected utility of total costs for machine A is computed as (.5)(.7113 + 0.0000) = .35565, with a corresponding certainty equivalent of \$216.937. Based on differential costs for machine A, its expected utility is given by (.5)(1.0000 + .5766) = .7883, with a corresponding certainty equivalent of \$66.937. The certain total and differential costs of machine B are \$175 and \$25, respectively. Hence, both the total and the differential computations indicate a superiority of machine B of \$216.937 − \$175 = \$66.937 − \$25 = \$41.937.

DIFFERENTIAL AMOUNTS AND MARKET VALUATION

So far, the problem of differential amounts has been examined independently of any market structure considerations. Obviously, the total amount x_i might be interpreted as the random cashflow from a two-component project package whose first subproject, y_i, differs between the composite projects and whose second component, z, is common to all relevant project packages. Now, suppose the distributions of y_i and of z can be duplicated in the market by constructing appropriate portfolios of marketed assets. Under such spanning conditions, then the existence of arbitrage opportunities will insure additive valuation [9], i.e.,

$$V(x_i) = V(y_i) + V(z) \tag{8}$$

where $V(\bullet)$ is a market valuation operator. As a specialized example, the market values may be determined according to the Capital Asset Pricing Model [10].

Thus, where implicit values for the random variables x_i, y_i, and z exist separately in the market, then rankings based on market values of total amounts x_i and of differential amounts y_i will be identical. This result holds in the market context, regardless of risk aversion, provided that the decision maker takes adequate diversification actions to obtain a desirable risk profile.

CONCLUSIONS

Rankings of decision alternatives based on total and on differential monetary amounts may in general be inconsistent under uncertainty. In the following cases, either approach is valid and hence yields consistent rankings: (1) with expected values, provided that the differential amounts have been coherently determined and are constant within states (but may differ across states); (ii) with exponential utility functions if the shared and differential amounts are statistically independent; or (iii) in a market valuation context, assuming diversification and implicit separate market values for differential and shared amounts.