Industry Dynamics and Search in the Labor Market

Pietro Garibaldi
University of Torino and Collegio Carlo Alberto

Espen R. Moen
Norwegian School of Management

January 8, 2009

Abstract

The paper proposes a model of on- and off-the-job search that combines convex hiring costs and directed search. Firms permanently differ in productivity levels, their production function features constant or decreasing returns to scale, and search costs are convex in search intensity. Wages are determined in a competitive manner, as firms advertise wage contracts (expected discounted incomes) so as to balance wage costs and search costs (queue length). An important assumption is that a firm is able to sort out its coordination problems with their employees in such a way that the on-the-job search behavior of workers maximizes the match surplus. Our model has several interesting features. First, it is close in spirit to the competitive model, with a tractable and unique equilibrium, and is therefore useful for empirical testing. Second, the resulting equilibrium gives rise to an efficient allocation of resources. Third, the equilibrium is characterized by a job ladder: unemployed workers search for low-productivity, low-wage firms. Workers in low-wage firms search for firms slightly higher on the productivity/ ladder, and so forth up to the workers in the second most productive firms who only apply to the most productive firms. Finally, the model rationalizes empirical regularities of on-the-job search and labor turnover. First, job-to job mobility falls with average firm tenure and firm size. Second, wages increase with firm size, and wage growth is larger in fast-growing firms.

1 Introduction

In the real economy, firm-and industry dynamics play an important role. Firms are born, expand and contract. Resources are allocated from less productive to more productive firms, and thereby improve the allocation of resources. There is substantial evidence that reallocation of resources on firms is important for economic growth, and Baily, Hulten and Campbell (1992) argues the about half of overall productivity growth in the U.S. manufacturing in the 80ies can be attributed to this. Existing empirical evidence also shows that industry dynamics is associated with large worker flows, not only in and out of unemployment, but even more importantly as direct job to job movements (Haltiwanger, 1999; Foster et al., 2007; Bartelsman et al; 2005). Lentz and Mortensen (2005, 2006) decompose the effect of firm selection on the growth rate, and then estimate that it accounts for 58 percent of the growth rate

Several recent papers analyze models of industry dynamics (Hopenayn, 1992; Hopenayn and Rogerson, 1993, Melitz 2003, Klette and Kortum 2004). However, these papers typically do not take into account that the factor markets, and in particular the labor market, may contain frictions. (An exception here is Lents and Mortensen (2007), who do include a frictional labour market in a Klette-Kortum model of innovation-driven industry dynamics).

[In this paper we model on the job search as an optimal response to search frictions and differences in firm productivity. Our goal is thus to provide a simple competitively flavour model consistent with a variety of facts described in longitudinal data sets on firms and workers flows]

This paper studies the joint determination of worker flows and firm dynamics with on the job search. The model contains three key elements. First, it applies the *competitive search equilibrium* concept, initially proposed by Moen (1997). Thus, firms post wages and post a number of vacancies so as to minimize search -and waiting costs. Furthermore, the labor market is endogenously separated into submarkets so that in each submarket, all agents at the same side of the market are identical.

Second, we assume that firms have access to a search technology with *convex hiring costs* (Bertola and Cabalero, 1992; Bertola and Garibaldi, 2001). In the traditional search model (Mortensen and Pissarides, 1994) adjustment costs are linear. Together with constant returns to scale in production, this implies that the size of the firms typically is undefined. Our assumption of convex hiring costs allows firms with different productivity and with constant-returns to scale technology to coexist in the market.

Third, we follow Moen and Rosen (2004) and allow for *efficient contracting.* The contracts are thus designed so as to resolve any agency problems between employers and employees so that their joint income is maximized. In particular, this implies that the workers' on-the-job search behavior maximizes the joint surplus of the worker and the firm. This assumption simplifies the model enormously. Without this assumption, a worker's current wage will influence his search behavior. As shown by Shimer (2006), this opens up for multiple equilibria and generally makes on-the-job search models intractable.

Our analysis thus delivers a tractable model of on-the-job search, closely related to the competitive model, in which on-the-job search and wage differentials for identical workers is an optimal response to search frictions and hetroegenous firms. As a tool for empirical analysisour model is interesting because it includes, in a simple way, the effects of search frictions for industry dynamics. Finally, as our model gives rise to a (constrained) efficient allocation of resources, and hence is well suited as a benchmark for welfare analysis.

The equilibrium of the model is characterized by a sluggish employment growth toward a steady-state employment level. Low productivity firms pay low wages, face high turnover rates, and grow slowly towards a steady state with low employment. More efficient firms pay higher wages, post more vacancies, and grow more quickly to a steady state with a higher employment level. The equilibrium features a job ladder: unemployed workers disproportionately search for firms with the lowest productivity. Workers employed in these firms, in turn, search only for firms with higher productivity. Hence our model easily explain a set of stylized facts about industry dynamics and worker flows: 1) productivity differences between firms are large and persistent, 2) workers move from low-wage to high-wage occupations, 3) more productive firms are larger and pay higher wages than less productive firms, 4) job-to-job mobility falls with average firm size and worker tenure, 5) wages increase with firm size, and 6) wages are higher in fast-growing firms .

Pissarides (1994) was the first paper that studied on-the-job search in a Diamond-Mortensen-Pissarides type of matching model. As show by Shimer (2006), a problem with this model is that the bargaining is not convex, and this may give rise to a continuum of wages. Current papers by Bagger and Lentz, Lise et. al. (2008) still uses this model, and get around the problem of a non-convex bargaining set by introducing competitive bidding for the worker after successful on-the-job search.

Maybe the most used model of on-the-job search in empirical research is Burdett and Mortensen (1998) with its many follow-ups, for instance Postel-Vinay and Robin (2002). Moen and Rosen (2004) are the first to analyse competitive on-the-job search and the first to assume efficient on-the-job search. Shi (2008) studies competitive on-the-job search in a model with wage tenure contracts. Menzio and Shi (2008), in a paper written simulataneously and independently of the current paper, study the effects of business cycle fluctuations in a model with competitive on-the-job search.

Kiyotaki and Lagos (2006) study optimal assignment of workers to jobs in a model where matches differ in quality, but without entry of firms. Delacroix and Shi (2006) analyzes on-the-job search in an urn-ball type of model of the labor market, and also obtain a job ladder in a similar way as we do. However, in their model all agents on both side of the market are homogenous, and firms at most hire one worker, hence their model is ill suited to analyze industry dynamics.

The paper proceeds as flows. Section 2 briefly describes the empirical regularities we are interested and reviews the relevant literature. Section 3 introduces the structure of the model while sections 4 and 5 derive the main formulation of the model for different type of firms. Section 6 introduces the general equilibrium and spells out some key results. Section 7 presents the baseline simulation.

2 A Brief Look at some empirical regularities

We briefly review some of the key empirical regularities linked to industry dynamics and worker flows. We are certainly not meant to be exhaustive in this review, and the selection of facts outlined in closely associated to the theoretical approach we will propose in the rest of the paper.

1. *In any industry there is a large scale reallocation of input and output across producers.* The work of Davis and Haltiwanger (1999) summarize much of the literature on gross job flows that was carried during the last decade; they note that in the United States more than 1 in 10 jobs is created in a given year and more than 1 in 10 jobs is destryoed. Much of this reallocation reflects reallocation within narrowly defined sectors. Note that not only labor is being reallocated but also capital and output (Haltiwanger, 2000).A large fraction of the input and output gross creation is associated with entry of firms and a large fraction output and input gross destruction is associated to firm.

2. *Productivity differences between firms are large and persistent.* Bartelsman and Dome (2000) summarize most of the evidence based on longitudinal micro data on firm level productivity diffferential. They clearly argue that the most significant finding of this vast literature is the heterogeneity across establishments and firms in productivity in nearly all industries examined. Bernard et al. (2003) report the distribution across plants of value added per worker relative to the overall mean, and show that a substantial number of plants have productivity either less than a fourth or more than four times the average. These differences are also very persistent over time. Danish data analysed by Mortensen (2007) provide similar patterns.

3. *More productive firms are larger and pay higher wages.* Wage differentials across observationally equivalent workers are both sizable and persistent . The employer size-wage effect is perhaps the strongest such stylized fact: larger firms or plants pay higher wages. The literature includes especially influential work by Krueger and Summers (1988) and Brown and Medoff (1989), and has been surveyed by Oi and Idson (1999).

4. *Job to job mobility falls with worker tenure.* There is a well established relationship between job duration and tenure. Farber (1999) carefully reviews this literature and reports that a monotonically declining survival rates is one of the mot robust stylized facts in the labour market. Such monotonicity holds regardless of the reasons beyond job termination and is particularly significant for voluntary job to job movements.

5. *Job to job movements are associated to wage gains* . Bartel and Borjas (1978) early work showed that young men who quit experience significant wage gains compared both to to job stayers and to their own wage growth prior to the job change. More recently Light (2005) summarize empirical evidence based on the National Longitudinal Survey of Youth. He shows that the the typical worker holds about five jobs in the first 8 years of the career, but that workers vary considerably in their mobility rates. He also reports evidence that workers who change jobs voluntarily thorugh a job to

job transition receive significant contemporaneous wage boosts that, on average, are at least as large as the wage gains received by job stayers.

6. *and wages are higher in fast growing firms.* Belzil (2000) uses Danish data and shows that after controlling for individual and business cycle effects, job creation at the firm level is found to increase male wages.

7. *Wage differentials are associated to productivity differentials .* The evidence on this link is more scarce, since dataset able to observe output and wages for a variety of workers are not readily available. Iranzo et al. (2007) using Italian data show that the level of labour productivity is clearly associated to larger wages for both production and non production workers.

Remark 1 *As a benchmark structure one can set up and solve the model without market frictions but with adjustment cost, in the spirit of the competitive market with adjustment cost of Sargent (1987). Such model, fully discussed in section X, has the following propoerties: first, the resulting competitive equilibrium does not feature any on-the-job search. Second. it does not yield any link between firm and wage dynamics. Thus, in order to obtain a competitive model with on-the-job search, amd where wage differentials reflects firm producitivity differentials, adjustment costs of the Sargent (1987) model is not sufficient, one also has to include search frictions.*

Remark 2 *The structure we propose allow firms to speed up hiring through two channels: by increasing search effort (i.e. posting more vacancies) or by offering a higher wage. At the microeconomic level, a firm would choose a combination of the two instruments so as to minimize total costs. In the discussion section we show that with linear adjustment costs the resulting wage would be independent of the hiring rate, the productivity of the firm, and its long run size. The firm would ajust its hiring rate only though the number of vacancies posted. This suggests that in order to obtain a positive productivity-wage effect, the search cost of firms has to be convex. Furthermore, in general equilibrium with linear adjustment costs, only the most productive firms post vacancies. Again, convex search costs seem to be neccesary for firms of different (marginal) productivity to operate simultaneously in the labor market by posting vacancies.*

3 Model and equilibrium

The structure of our model is as follows

- Labor is the only factor of production. The labor market is populated by a measure 1 of identical workers. Individuals are neutral, infinitely lived, and discount the future at rate r.

- The technology requires an entry cost equal to K. Conditional upon entry, the firm learns its productivity, which can take any value between y_1, and y_n with y_2, $y_1 < y_2 <$

... $< y_n$. The probability that each productivity is selected is α_i with $\sum_i \alpha_i = 1$. The productivity of a firm is a fixed throughout its life. Unemployed workers have access to an income flow y_0, which may denote unemployment benefits, the value of leisure, or the income when self-employed. We assume that $y_0 < y_1$.

- Firms post vacancies and wages to maximize expected profits. Vacancy costs $c(v)$ are convex in the number of vacancies posted. In the numerical examples we assume that $c(v) = \frac{v^2}{2c}$, where c is a constant. We further discuss this assumption at the end of this section.
- Firms die at rate δ and workers exogenously leave the firm at rate s.
- Search is directed. Firms post vacancies and wages to maximize expected profits. Firms face a relationship between the wage they set and the arrival rate of workers, which is derived from the indifference constraint of workers. Firms set wages so as to maximize profits given this relationship.
- Wage contracts are complete, and resolve any agency problems between employers and employees. In particular, the wage contract ensures efficient on-the-job search.

A submarket is characterized by an aggregate matching functions, brining together the searching workers and the vacant firms in that submarket. As will be clear below, the labor market endogenously separates into submarkets with identical agents on each side of the market. Suppose N workers search for V vacancies. We assume a Cobb-Douglas matching function, so that the number of matches is given by $x(N, V) = AN^{\beta}V^{1-\beta}$. The transition rates for workers and for firms are

$$\begin{aligned} p &= A\theta^{1-\beta} \\ q &= A\theta^{-\beta} \end{aligned}$$

where $\theta = V/N$ is the labor market tightness in the market. Inverting the first of the previous condition one gets that $\theta = A^{-\frac{1}{1-\beta}} p^{\frac{1}{1-\beta}}$ so that the transition rate for vacancies can be expressed as

$$q = A^{\frac{\beta}{1-\beta}} p^{-\frac{\beta}{1-\beta}} \tag{1}$$

3.1 Worker search

Let M_0 be the general present discounted value of an unemployed worker searching in a generic submarket that offers an NPV wate of W and an arrival rate of offers p. Then

$$rM_0 = y_0 + p[W - M_0] \tag{2}$$

Clearly, the workers will only search in those submarkets where the gain from search, $p[W - M_0]$, is maximized.

In what follows we rely on the joint income M_i generated by a worker and a firm of type i. Specifically, the joint income depends on the productivity of the firm and on the on-the-job search of the the worker during the employment spell. The corresponding asset equation reads

$$rM_i = y_i + (s + \delta)(M_0 - M_i) + p[W - M_j] \tag{3}$$

where again p and W are generic values of the arrival rate of jobs and the wages in the submarket in question. The expression deserves a few comments. The first term is the flow production value created on the job. Since the wage paid by firm is a pure transfer to the worker, it does not appear in the previous expression. In addition, the current job can be destroyed for exogenous reasons at rate $s + \delta$. In that case the worker turns into unemployment and receives M_0 while the firm gets zero, as reported in the equation. Finally, the worker will with probability rate p find a new job. In this event, the joint income is lost and the worker earn the present discounted value.

A core element in competitive search equilibrium is that workers choose search optimally, so that only firms that offer a combination of p and W that maximizes the searching workers income obtains vacancies. Suppose unemployed workers receive a (maximum) NPV income M_0 when searching. The asset value equation (2) then defines a relationship $p_0^w(W, M_0)$, defined by

$$p_0^w(W; M_0) = \frac{rM_0 - y_0}{W - M_0}$$

The function $p_0^w(W; M_0)$ defines combinations of NPV wages W and arrival rates p that give unemployed workers an expected income M_0.

As anticipated, we assume thoughout that the firm worker pair contract efficiently. In other words, the wage contract maximizes the joint income M_i. This simple assumption implies that the worker on the job search internalizes fully the loss of value incured by the firm when she finds a new job. The are various wage contracts that implement this behavior. For example, the worker pays the firm its entire pdv value up front and then gets a wage equal to y_1. In other words, the worker buys the job from the firm and acts thereafter a residual claimant. As an alternative contract, the worker gets a constant wage and pays a quit fee equal to the continuation value of the firm if a new job is accepted (see Moen and Rosen (2004) for more examples). In any event, the wage paid to the worker in the current job do not influence her on-the-job search behavior.[1]

Assume that M_i is the maximum joint income availiable in the market for a type i firm and a worker in that firm. As for the unemployed workers we can define a function $p_i^w(W; M_i)$, given by (from 3)

$$p_i^w(W; M_i) = \frac{(r + s + \delta)M_i - y_i - (s + \delta)M_0}{W - M_i} \tag{4}$$

[1]It follows from this that a worker in a low-type firm will never search for a job in another low-type firm, as these cannot offer a wage that exceeds the productivity in the current firm.

(which also holds for p_0^w). Finally, define

$$f(W; M_0, M_0, ..., M_n) = \min_i p_i^w(W; M_i) \tag{5}$$

Lemma 1 *The following is true:*

a) Single crossing: Two indifference curves $p_i^w(W; M_i)$ and $p_j^w(W; M_i)$, $i \neq j$ intersect once. At the intersection, p_i^w is steeper than $p_j^w(W; M_i)$ iff $i < j$.

b) Suppose $W > rM_0$ is offered in a submarket. Then the arrival rate of job offers in that submarket is $p^w = f(W; M_0, M_0, ..., M_n)$.

Proof: a) To be done (but trivial), of b). Suppose the lowest value of p_j^w is obtained for $j = k$.Suppose the arrival rate in the submarket is greater than p_k. Then the workers at level k can obtain a higher expected income than their M_k by applying in this submarket, which is inconsistent with efficient search.

3.2 Firms

The profit of a firm of type j, Π_j, can be written as

$$\Pi_j(t) = \int_t^\infty [(y_j - w_j(t))N_j - c(v)]e^{-(r+\delta)}t$$

At any point in time, the firm decides on the number of vacancies to be posted and the wages attached to them. This only influences profits through future hirings, and is independent of the stock of existing workers. It follows that at any point in time, the firm maximizes the value of search given by

$$\pi_j = -c(v) + v_j q[M_j - W_j]. \tag{6}$$

The first part is the flow cost of posting vacancies while the second part is the gain from search. The NPV to the firm of finding a worker is the difference between the joint income and the value of the job to the worker, taking into account the probability q that each vacancy v_i is matched. When setting wages, the firm takes into account the relationship between the NPV wage W the firm sets and the arrival rate of workers to the vacancy. In competitive search equilibrium, the preceived relationship between the posted wage W and the arrival rate of workers can be written as $q = q(p)$ and $p = f(W)$. To avoid uninteresting details we assume that the firm attatch the same wage to each of its vacancies. (Note though that we allow identical firms to set different wages.) The firm's maximization problem can thus be defined as follows:

Definition 1 *The firms maximiziation problem of firm j reads*

$$\max_{W.v} -c(v) + vq[M_j - W].$$
$$\textit{subject to}$$
$$\begin{aligned} q &= q(p) \\ p &= f(W) \end{aligned}$$

Let $i^W = i(W)$ denote the set of current employment levels of workers searching for firms that offer wages W.

Lemma 2 *For any given vector $M_0, ..., M_n$, $M_i < M_j$ if $i < j$, the following holds*

a) The st of solutions to firm type j's maximization problem is a vector $(W_j^0, W_j^1, ..., W_j^l)$ (ordered as an increasing sequence for convenience), where $l \leq j-1$
b) $i^W(W_j^l)$ is a singleton for all j, l
c) Denote $i^W(W_j^0)$ by i'. Then $i^W(W_j^1) = i'+1$, $i^W(W_j^2) = i'+2$,....,$i^W(W_j^1) = i'+k < j$.

The lemma implies that for any given vector $M_0, ..., M_n$ increasing in the index, the firms' maximisation problem may have several solution. However, if the maximization problem has more than one solution, then each solution implies that the firm attracts different worker types. Thus, all firms of a given type j that attracts workers hired in firms of a given type i offer the same wage W_{ij}.

It thus follows that the optimal wage advertizement of firms endogenously separate the market into submarkets with identical agents on both side of the market. Hence we can define a submarket $_{ij}$ as workers of type i searching for firms of type j. If there is no activity in a submarket ij, we say that the market is closed, otherwise it is open. The lemma inmplies that in each open submarket $_{ij}$, there exists a unique wage W_{ij}, a unique θ_{ij}, and thus also unique transition rates q_{ij} and p_{ij}. This helps dramatically when characterizing equilibrium

Finally, the expected profit of a firm of type j entering the market can be written as

$$\Pi_j = \frac{\pi_j}{r+\delta} \tag{7}$$

3.3 Flow equations

Let N_i denote the measure of workers in type i firms, τ_{ij} the fraction of type j firms searching for type i workers (in submarket $_{ij}$, and κ_{ij} the fraction of "type" i workers searching for type j firms (in empty submarkets both τ and κ are zero). Clearly

$$\sum_{i=0}^{n} N_i = 1 \tag{8}$$

$$\sum_{i=0}^{j-1} \tau_{ij} = 1 \text{ for all } j \tag{9}$$

$$\sum_{j=i+1}^{n} \kappa_{ij} = 1 \text{ for all } i \tag{10}$$

where k is the measure of firms in the economy. The flow equations read

$$\sum_{i=0}^{j-1} N_i p_{ij} \kappa_{ij} = [s + \delta + \sum_{k=j+1} p_{jk} \kappa_{jk}] N_j \tag{11}$$

for all j. The labor market tightness in each submarket is denoted by θ_{ij}. It follows that

$$\theta_{ij} = k \frac{\alpha_i \tau_{ij} v_{ij}}{\kappa_{ij} N_i}$$

3.4 General equilibrium

We are now in a position to define the general equilibrium.

Definition 2 *General equilibrium is defined as a vector of asset values $M_0, M_1, ..., M_n$, a matrix of wages $W_{ij},$, a matrix of arrival rates $p_{ij}, i, j \leq n$ such that*

1. *Firms maximize profits*

2. *The expected profit of entering the market is equal to the entry cost K, i.e.,*

$$E\Pi_j = K$$

3. *The flow equations (8), (9), (10) and (11) are all satisfied.*

4 Characterizing equilibrium and equilibrium properties

In this section we characterize equilibrium. We first look at the case with two firms, and then at the general case with n firms.

Let market ij denote the market for workers currently employed in firms of type i searching for jobs in firms of type j. We thus require that $j > i$. Equation (3) now reads

$$(r + s + \delta)M_i = y_i + \max_{j>i} p_j(W_{ij} - M_j) + (s + \delta)M_0 \tag{12}$$

Consider a firm of type j that search for workers of type i (later we will decide whether this is optimal). Inserting for p_i^w and $p(q)$ from (1) into the expressio n for the flow profit π_j in (6) gives

$$\pi_j = -c(v) + v_{ij} A^{\frac{\beta}{1-\beta}} (p_i^w(W_{ij})^{\frac{\beta}{1-\beta}} [M_j - W_{ij}].$$

The first order condition for the wage is obtained by setting the elasticity of the last term equal to zero. Using (4) it follows that

$$\frac{\beta}{1-\beta} = \frac{W_{ij} - M_i}{M_j - W_{ij}}$$

The first order condition for v_{ij} follows directly. After rearranging the first order conditions for maximum can thus be written as

.

$$\begin{aligned} W_{ij} &= M_i + \beta(M_j - M_i) && (13) \\ c'(v) &= (1 - \beta)(M_j - M_i)q && (14) \\ q &= q(p(W_j; W_i) && (15) \end{aligned}$$

The first conditions is the traditional efficient rent sharing in competitive search equilibrium. In the matching literature, it also refers to the Hosios condition, since the share β is the elasticity of the matching function. The second condition equates the marginal cost of vacancy posting to its expected benefit. The third equation defines the arrival rate of job offers in this submarket. The fourth equation repeats the joint income.

To summarize, firms in this submarket post a measure of vacancies v_1 independently of their employment status. This feature follows directly from the constant returns to scale assumptions.Note that also in steady state the firm is characterized by continuous job turnover, even though employment does not grow. The wage contract posted by the firm is also constant throughout the life of the firm and does not feature any transitional dynamics.

Remark 3 *Note that as long as $y_i > y_0$, firms of type i are active in equilibrium. Since workers search equally well on and off jobs, the joint income of a worker and a firm of type i is then strictly greater than M_0. The firm will thus offer a wage $W = M_0 + (1 - \beta)(M - M_0)$ and attract some workers.*

Denote the resulting optimal profit flow by π_{ij}, and define $\pi_j = \max_j \pi_{ij}$. It follows that a submarket ij only is active if $\pi_{ij} = \pi_j$. The net present value of profits Π_j is then given by (7). With quadratic costs, the NPV profit reads

$$\Pi_j = \frac{[(1-\beta)(M_j - M_i)q(Rij)]^2 c}{r+s+\delta}$$

Finally, free entry of firms implies that

$$E^j \Pi_j = K$$

Proposition 1 *The equilibrium exists*

We are not able to show that the equilibrium is unique. Since the matching function is Cobb-Douglas, one would expect the equilibrium to be unique. However, there are complementarities between the layers, an increase in the number of firms searching for a given firm type increases the joint income of that firm type.

If there are more than one equilibrium, we will assume that the market picks the equilibrium which give rise to the highest aggregate income, defined as

$$W = \int_0^\infty [\sum_{j=0}^n N_j y_j - \sum_{i=0}^n \sum_{j=i+1}^n \alpha_j k c(v_{ij}) - xK] e^{-rt} dt$$

where x is the inflow of firms. Note that if we introduce a market maker as in Moen (1997), the market maker would pick the most efficient of the equilibrium candidates given the law of motions for employment and firm growth.

With this purification of the equilibrium concept we are able to show that the equilibrium is efficient:

Proposition 2 *The equilibrium is efficient*

Our next lemma characterizes wage distributions and search behaviour of workers and firms

Lemma 3 *In any n-firm equilibrium the following holds*

Proposition 3 *a) Workers in a firm of type j always search for jobs with strictly higher wages than workers employed in firms of type $l < j$.*

b) Firms of type j always offer a strictly higher wage than firms of type l if $j > l$.

Almost trivially, we can show the following result, which we refer to as the maximum separation result

Lemma 4 *Let I_k denote the set of worker types searching for firms of type k. Consider I_k and I_l, $k > l$. Then*

Proposition 4 *- All elements in I_k are greater than or equal to all elements in I_l.*

- I_k and I_l have at most one common element.

It follows that the market, to the largest extent possible, separates workers and firms so that the low-type workers search for the low-type firms. Note the similarity with the non-assortative matching results in the search litterature (Shimer and Smith (2001), Eeckout and Kirkcher (2008). If the production technology is linear in the productivities of the worker and the firm, it is optimal that the high-type firms match with the low-type workers and vice versa. Similarly, in our model it is optimal that the workers in a firm with a high current productivity search for vacancies with high productivity, and vice versa.

From an efficiency point of view, the result can be understood as follows:, recall that if vacancies are filled quickly that requires long worker queues, and the flip-side of the coin is that workers find jobs slowly. It is therefore optimal that the most "patient" workers, i.e., the workers with the highest current wage, search for the most "impatient" firms, the firms with the highest productivity. It is also trivial to exstend the efficiency result above to the n-firm case.

Note that the growth rate of a firm of a given type depends on the wage that it offers. Thus, firms of different productivities may offer different wages and attract workers at different speeds, as an efficient response to search frictions. Furthermore, the size of a firm in a given market converges to a steady state level. Thus, firms do not grow indefinitely. In order to obtain indefinite growth, one may alter the cost function of posting wages. This will be done in a later session.

4.1 Equilibrium with two types of firms

Consider the special case with two types of firms. In this case we can get some more structure and therefore also some more results regarding the opening and closing of submarkets.

Our first observation is that the $_{12}$ market is always open. Suppose not. Then a high-type firm that opens vacancies with a wage slightly above y_1 would attract applications for all workers employed in type 1 firms. The firm would thus obtain an infinitely high arrival rate q of job offers, and would make infinitely high profit. A deviation from equilibrium would thus surely be profitable.

The next question is whether the $_{02}$ market will open up (stairways to heaven). If not, we say that we have a pure job ladder. Whether or not we have a pure job ladder depends on parameter values. However, with very mild restrictions on $c(v)$ (that $\lim_{v\to\infty} c'(v)/v = \infty$) we can show the following proposition:

Proposition 5 *a) Suppose K is high, so that few firms enter the market. Then high-type firms search both for unemployed and employed workers.*

b) Suppose there exists a pure job ladder for some values of K. Provided that the number of vacancies is not too flexible (c'' is sufficiently large around the equilibrium point), then there exists a K^* such that there is a pure job ladder for $K < K^*$ while both the $_{12}$ and the $_{02}$ market open up if $K < K^*$.

We need the qualifier in order to ensure that the measure of vacancies in the economy goes to zero when the measure of firms goes to zero. The proposition thus states that the pure jobladder equilibrium only prevails if the frictions in the market are sufficiently low. DO WE REALLY NEED THE QUALIFIER?

Thus, contrary to what one may expect, a pure job ladder (if it emerges at all) emerges when there are many jobs relative to workers and hence the unemployent rate is low.

Our second question regards the relationship between the share of high-type firms in the equilibrium. Let a balanced increase in α_2 denote an increase in α_2 where other variables (for instance the entry cost K) is adjusted so that the number of firms k is kept constant. We are able to show the following result:

Proposition 6 *a) For high values of α_2, both the $_{02}$ and the $_{12}$ submarkets are active. For low values of α_2, only the $_{12}$ market is active.*

b) Consider balanced changes in α_2. Suppose $c''(e)$ is large. Then there exists a unique $\alpha = \alpha^*$ such that the $_{02}$ market is open if and only if $\alpha_2 > \alpha^*$.

Two remarks regarding b) is warranted. The first regards the fact that we are only considering balanced changes. The reason is the following. Suppose that $\alpha_2 = \alpha^*$, so that there is a pure job ladder and the 2-firms are just indifferent by entering the $_{12}$ and the $_{02}$ market. Consider an increase in α_2. This has two effects on equilibrium. First it becomes more crowded in the $_{12}$ market, and this favours the $_{02}$ market. However, if we let k vary, it follows that k will increase, and as seen in proposition (5) this favours the pure job ladder equilibrium. In general we are not able to show which force is the stronger, and hence cannot garantee that there is a unique switching point. However, with balanced changes we can.

The second comment regards the the requirements on $c''(e)$. When α_2 increases, the direct effect is that p_{01} decreases, as there are fewer low-type firms. Our proof of uniquness of the switching point depends on p_{01} being decreasing in α_2. However, low-type firms may post more vacancies, and in principle this may imply that p_{01} increases in α_2. As we have not been able to rule this out, we instead put restrictions on $c''(e)$ so that the flexibility of e is not too big.

5 Infinitely many firm types

In this section we let the number of worker types go to infinity. The simplest way of doing this may be to assume that the types are symetrically distributed on the interval $[y_{\min}, y_{\max}]$, and then go to the limit as the number of firm types goes to infinity. Alternatively, we may chose a more general approach as in Peters (2008). a general "splitting of types.

We have not yet done the existence proof for the limit. However, our conjecture is that the limit exists and that the maximum separation result also holds in equilibrium. Clearly, the unemployed workers cannot appy only to the lowest type of firms, since the number of jobs in this firm has mass zero.

From a revealed preference argument, it follows that firms with different productivities advertize different wages. It also follows that the sets I_y and $I_{y'}$ have at most one element in common. Furthermore, if chosing from a continuous and monotone set of $(p, W(p))$ combinations, a high-type worker is always more willing to trade off p for W than is a lower-type worker. This leads us to the following conjecture:

Conjecture 7 *In the limit, there exists a mapping from current productivity y_i to future productivity $f(y_i)$. Furthermore f is continuous and strictly increasing in y_j, and*

$$\begin{aligned} \lim_{y_j \to y_{\max}^-} f(y) &= y_{\max} \\ \lim_{y_j \to y_{\min}^+} f(y) &= y_z \end{aligned}$$

Claim 8 *The following is true in any equilibrium: Unemployed workers search for jobs with productivity on an interval $[z, y^z]$. Employed workers search for jobs with productivity above y^z, where $y^z \leq y^{\max}$. The assignement mapping is a single-valued function $y_2 = F(y)$ which is continuous and strictly increasing in y_1.*

We want to characterize the equilibrium when there is a given number of firms , f, in the economy. One way to do this may be to start with the outflow function $p(y)$. Alternatively we may start with the inflow function $q(y)$. Note that, since the number of vacancies per firm is constant, we there is a one-to one mapping from f to p.

For a given outflow function from unenmployment we can calculate y^z mechanically. Furthermore, we can calculate the distribution $N(y)$ of workers on firms as well as the unemployment rate. This follows from the fact that the number of vacancies in each firm is constant. We can also calculate $y_2 = F(y)$. The exact details on how to do this is not yet specificed.

From the envelope theorem it follows that $M'(y) = \frac{1}{r+s+p+\delta}$. Furthermore, $M(y) = y^{\max}/(r+s+\delta)$. It follows that we can calculate $M_{p()}(y)$. From the first order condition we know that

$$W(F(y)) - M_{p()}(y) = \beta(M_{p()}(F(y_1) - M_{p()}(y))$$

It follows that

$$(r+s+\delta)M_{p()}(y) = y_1 + sM_0 + p(y)\beta[M_{p()}(F(y)) - M_{p()}(y)]$$

Define Γp as follows

$$\Gamma p = \frac{(r+s+\delta)M_{p()}(y_1) - y_1 - sU}{\beta[M_{p()}(F(y)) - M_{p()}(y)]}$$

The equilibrium is given by $\Gamma p = p$.The next issue is then how to characterize $f(y)$.

6 Search costs decreasing in firm size

The model presented above implies a constant gross hiring rate of workers. As the firms grow, so does the number of separations, and as a result there exists a steady state level of employment. Thus, the growth rate of any given firm is decreasing with firm age and size, and is hence violating Gibraths law.

The dynamic properties of the firms depend crucially on the functional form of the cost of hiring vacancies. To see this, uppose the cost of posting vacancies can be written as

$$c = Nc(\frac{v}{N})$$

The cost function can be rationalized as follows: Suppose $c(v_i)$ is the individual effort cost of assisting in the hiring process by excerting v_i units of effort. Since $c()$ is convex, it is optimal to distribute effort equally over the work force so that each worker contrinutes $v_i = v/N$ units, where v is the total effort level of the firm, i.e., the number of vacancies. Total effort cost is then $Nc(\frac{v}{N})$. In what follows we assume that the effort cost is quadratic, hence we can write

$$c(v, N) = \frac{v^2}{2cN}$$

In all other aspects the model proceeds exactly as before. For a given NPV income the workers' on-the-job search behaviour can be described in exactly the same way as above. In particular, there exists a function $f(W; \widetilde{M}_0, ..., \widetilde{M}_n)$ giving the arrival rate of the worker as a function of the npv wage offered and the asset value of the searching workers. Furthermore, by applying the same arguments as above it follows that the labor market endogenously separates into submarkets as described above.

Let $\widetilde{v} = v/N$ denote the vacancy rate as a fraction of employment. However, the joint expected income of a match is different.

Lemma 5 *Consider a firm of type j that searches for workers hired in type i, searching as if her NPV income is $\widetilde{M}_i$. Let an asterix denote optimal values. Then the following holds:*

Proposition 9 *a) The relevant joint income for the worker and the firm writes*

$$\widetilde{M}_{ij} = \frac{y_j + \max_k p_{jk}(W_{jk} - \widetilde{M}_{ij}) + (s+\delta)\widetilde{M}_0 + [\widetilde{v}^*_{ij} q^*_{ij}(1-\beta)(\widetilde{M}_{ij} - \widetilde{M}_i) - \widetilde{v}^{*2}_{ij}/2c]}{r + s + \delta} \tag{16}$$

b) The optimal wage reads

$$W^*_{ij} = \beta(\widetilde{M}_{ij} - \widetilde{M}_i) \tag{17}$$

and $q^* = q(p(W^*_{ij}, \overline{M}_i)$.

c)The optimal fraction of vacancies to workers, $\widetilde{v}$, reads

$$\widetilde{v}^*_{ij} = (1-\beta) q^*_{ij}(\widetilde{M}_{ij} - \widetilde{M}_i)c \tag{18}$$

Note that the only difference between M_{ij} defined by (12) and $\widetilde{M}_{ij}$ defined by (16) is the last term in the nominator. This term reflects what we refer to asl *breeding.* More workers imply that hiring will be more easy in the future, and this increases the value of a match. For the problem to be well defined, the effect of breeding must not be too large, since the value of a job then will be infinite.Note also that in the firmt order conditions (17) and (18) $\widetilde{M}ij$ is taken as given. We are now able to show the following result

Proposition 10 *Suppose $rM_i > y_i - s(M_i - M_0)$ (some on-the-job search takes place). Then the firms' maximization problem is well defined provided that c is sufficiently small (the search cost is sufficiently high). Furthermore, given that the maximization problem is well defined it has a unique solution $\widetilde{v}_{ij}^*$,W_{ij}^*.*

Thus, it follows that the fraction of vacancies to employees is constant. The growth rate of the firm (conditioned on survival) is then $v_{ij}^* q_{ij}^* - (s + p_{jk})$, independently on firm size. Note that the growth rate may be positive or negative. Note also that the wage rate is independent of firm size, while profits from future hirings is proportional to firm size

Define $\Pi_j = \max_{i<j} \Pi_{ij}$. In equilibrium submarket ij is open if and only if

$$\Pi_{ij} = \Pi_j$$

Finally, free entry of firms implies that

$$E^j \Pi_j = K$$

It can be shown that the equilibrium exists and is efficient. (TO BE DONE)

7 Endogenizing productivity differences - Out?

Suppose the firms can choose between (y_1, K_1) and (y_2, K_2), $y_1 < y_2$, $K_1 < K_2$. Furthermore, suppose that with no on-the-job search, the parameters are such that all firms choose the lowest investments.

Conjecture 11 *Suppose we allow for on-the-job search. Then the resulting equilibrium is a pure job-ladder equilibrium, determined by the zero profit conditions*

$$\begin{aligned} \Pi_1 &= K_1 \\ \Pi_{12} &= K_2 \end{aligned}$$

Scetch of proof: First note that the 2-market cannot be empty. Suppose it is. Then a firm that opens up can fill its vacancies infinitely quickly, and thus as long as $y_2 > y_1$ can obtain unbound profit. Furthermore, by assumption it will not be profitable for firm 2 to search for unemployed workers, since in this market they will be dominated by type-1 firms.

Suppose the firms could choose investment-output combinations from a menu defined by the function $y = F(K)$, where $F(K)$ is increasing and concave and bounded above by $\overline{y}$. Then we conjecture that the equilibrium is a pure jobladder equilibrium with infinitely many steps.

8 Basic Calibration and Comparative Static

Table 1 and 2 report the basic parameter values for our calibration. The calibration is based on quarterly statistics and the pure interest rate is 1 percent. The productivity level in low type firm is set to a baseline reference value of $y_1 = 1$, while the premium for the high type is 10percent. The flow value of unemployment z is 0.6, a value far the replacement rate observed in real life labour markets. The matching function is Cobb Douglas with an elasticity β equal to 0.5. The parameter of the search cost is 0.15, while the entry cost k is 5, a value roughly equal to five times times the output produced by a low productivity job.The sum of the separation s and the firm death rate is 0.06. The proportion of low productivity firms is rather high at 0.93.The rest of the parameters are reported in 1

The baseline equilibrium features an unemployment rate equals to 5.8 percent and a job finding probability equal to 1, in line with the basic quarterly statistics in the United States labour market. Unemployment flows are 5.7 percent, consistent with quarterly job creation rate in the US manufacturing sector compiled by Davis and Haltiwanger. Job to job mobility is slightly below 5 percent. In Table 1 most of the unemployed workers search for low productivity firms, as indicated by $k_{01} = 0.96$. Similarly, high productivity firms search mainly among the employed sector, as indicated by the fraction of firms hiring from the employment pool ($\tau = 0.97$) The equilibrium allocation is described in the central part of Table 1. The job finding rate for unemployed workers p_{01} is the largest among the various job finding rates, but the bulk of workers in the labor market is employed in type 2 firms. Indeed, type 2 firms absorb 75 percent of the total workforce. As a result, the submarket 02, albeit significant, represents a fringe of the entire economy.

As we mentioned above, the labor market features unemployment flows and job to job flows that are comparable in absolute magnitude, and the *job ladder* mechanism is clearly present in the simulated economy. Workers start out in low productivity firms and eventually graduate to high type jobs through on the job search. Eventually, firm and match specific shocks at rate δ and s induce another round of job ladder. The bottom part of the Table 1 features also an important relationship between firm size and firm wages, where the latter are measured in terms of PDV wages. Clearly, high type firms are larger in size and pay higher wage.

The idea of the baseline simulation from Table 1 to Table i2 is to show that an increase in the share of low productivity firms α lead the economy to move toward a pure job ladder equilibrium. Indeed, the only parameter that changes between Tables and 1 and 2 is α. Recall that in the baseline specification of Table 1 the equilibirum value of τ is very close to one and as a result the submarket 02 is very small. A small increase in α, similarly to that experienced from Table 1 to Table i2 leads to an equilibrium value of $\tau > 1$, a value that is not consistent with all three submarkets being operative. In other words, as α is increased with respect to the value assigned in Table 1, the economy moves to a pure job ladder equilibrium. In moving from Table 1 to Table 2 α increases from 0.93 to 0.94, suggesting that α^* in our numerical example is inside this small interval. The economy described in Table 2 does look very similar to that described in Table 1, even though two only submarkets are operative. Note also that the equilibrium value of unemployment M_0^* does slightly fall as α increases. This is not surprising since in a pure job ladder equilibrium the number of low productivity firms is higher.

Table 1: Baseline Calibration with three submarkets

Parameter	Notation	Value
Pure Discount Rate	r	0.010
Separation Rate	s	0.040
Firm Bankruptcy Rate	δ	0.020
Bargaining Share	β	0.500
entry cost	k	5.000
low type proportion	α	0.9300
high type productivity	y_1	1.000
low type productivity	y_2	1.100
unemployed income	z	0.550
search cost parameter	c	0.150
matching function parameter	A	7.000
matching function elasticity	β	0.500
Equilibrium Values		
Joint Income 1	M_1	100.8178
Joint Income 2	M_2	101.3549
unemployment flow value	rU	0.9991
unempl. job finding rate in low type	p_{01}	0.9939
on the job finding rate	p_{12}	0.2324
unempl. job finding rate directly to high type	p_{02}	0.6234
Equilibrium Quantities		
Unemployment	n_0	0.0585
Employment in Low productivity type	n_1	0.1915
Employment in High productivity type	n_2	0.7500
Proportion of unemployed in submkt 01	k_{01}	0.9633
Number of Firms	f	0.0628
Proportion of high type firms in submarket 12	τ	0.9714
Worker Flows		
Unemployment Flows	$n_0 * (p_{01} + p_{02})$	0.0573
Job to Job Flows	$n_1 * p_{12}$	0.0445
Firm Size, PDV Wages and Profits		
Profits in submarket 01	Π_{01}	3.6174
Profits in submarket 02	Π_{02}	23.3687
Profits in submarket 12	Π_{12}	23.3687
Firm Size in submarket 01	N_{01}	0.1031
Firm Size in submarket 02	N_{02}	2.0363
Firm Size in submarket 12	N_{12}	5.4628
Wages in submarket 01	W_{01}	100.3659
Wages in submarket 02	W_{02}	100.7196
Wages in submarket 12	W_{12}	102.0753

Source: Authors' calculation

Table 2: Baseline Calibration with two submarkets

Parameter	Notation	Value
Pure Discount Rate	r	0.010
Separation Rate	s	0.040
Firm Bankruptcy Rate	δ	0.020
Bargaining Share	β	0.500
entry cost	k	5.000
low type proportion	α	0.9400
high type productivity	y_1	1.000
low type productivity	y_2	1.100
unemployed income	z	0.550
search cost parameter	c	0.150
matching function parameter	A	7.000
matching function elasticity	β	0.500
Equilibrium Values		
Joint Income 1	M_1	100.7446
Joint Income 2	M_2	101.2824
unemployment flow value	rU	0.9983
unempl. job finding rate in low type	p_{01}	0.9797
on the job finding rate	p_{12}	0.2319
Equilibrium Quantities		
Unemployment	n_0	0.0577
Employment in Low productivity type	n_1	0.1937
Employment in High productivity type	n_2	0.7486
Number of Firms	f	0.0713
Worker Flows		
Unemployment Flows	$u*(p_{01}+p_{02})$	0.0565
Job to Job Flows	$n_1 * p_{12}$	0.0449
Firm Size, PDV Wages and Profits		
Profits in submarket 01	Π_{01}	3.8171
Profits in submarket 12	Π_{12}	23.5327
Firm Size in submarket 01	N_{01}	0.1222
Firm Size in submarket 12	N_{12}	6.2383
Wages in submarket 01	W_{01}	100.2870
Wages in submarket 12	W_{12}	100.6362
Source: Authors' calculation		

8.1 Comparative Static

Figures 1 and 2 describe the equilibrium of the model following an increase in the proportion of low productivity firms, α. The parameters used in the simulations are identical to those of Table 1, with the only exception of α that ranges from close to zero to close to 1. In each panel in Figure 1 and 2, the horizontal axis ranges from 0 to 1. The first panel shows that as α reaches α^*, or a share of low productivity firms sufficiently higherm the submarket 02 shut down and the economy moves to a pure job ladder equilibrium where firms search only for employed workers. Such features is accounted for by a value of $\tau = 1$. In the second panel of 1 we report the share k_{01}, or the share of unemployed workers searching for low productivity firms. As α reaches α^* such proportion becomes very close to 1.

The fourth panel of 1 reports the comparative static with respect to the value of unemployment rM_0 as the economy increases the share α. The value of unemployment clearly falls monotonically. This is probably the most important and clear result following the increase in α. An economy with a larger proportion of low productivity firm is an economy that brings lower utility to non employed workers. This simple result is true regardless of the type of equilibrium in which the economy settles, as demonstrated by the monotonic fall in rM_0 across all ranges of α. Similar results hold for the joint income and the wage obtained by workers, which decline monotonically as the low productivity share α increases. The last two panels on Figure 1 shows the effects effects on profits following an increase in the proportion of low productivity firms. Profits in both type 1 and type 2 firm increase. Consider first low productivity firms. The increase in α, by reducing the the share of high productivity firms in the economy, increases the employment and profit opportunity in low productivity firms. It is a simple competition effect due to the fact that there are fewer firms with superior technology. The effect on high productivity firms is similar, since existing firms face lower competition from firms of similar technology. The latter effect is milder in the pure job ladder equilibrium.

The effects of α on the number of firms is rather non linear. As long as α is lower than α^*, an increase in the proportion of low type firms reduce the number of entrants. As the economy switches to the pure job ladder equilibrium, the number of firms increase dramatically.

Figure 2 focuses on the aggregate labour market. To understand the overall effect it is important to first look at n_2, employment in high productivity firms, displayed in panel 5 of Figure 2 . Following an increase in α, there are fewer high productivity firms, and thus aggregate employment in these of firms fall. This monotonic falls is the counterpart of the fall in the value functions displayed in Figure 1. The market composition in terms of employment changes in favor of low productivity firms, and as a result employment n_1 increases substantially. If one looks at employment n_1 as a sort of first step toward employment in good firms, the increase in this employment is akin to an increase in "bad employment". The fall in overall unemployment should also not be surprising, especially if we look at the sum of n_0 and n_1 as the pool of workers that are waiting to move to high employment n_2.

Turning back to job finding rate, the first three panels of Table 2 show that all job finding rates fall as the proportion of low productivity firms falls.

The increase in the share of low productivity firms induce an increase of both vacant firms $V_1(0)$ and searching workers into the submarket 01, the latter being obtained by the product $k_{01}n_0$. The simulation shows that the effects from searching worker effects dominates despite the fall in in n_0 and thus the job finding rate p_{01} falls. The effect on p_{02} is similar. As α increases, both high productivity firms and searching workers falls, but the effects obtained by the reduction in searching firm is stronger. The reduction in p_{12} is simpler, since in such submarket there is an increase of searching workers and a reduction in vacant firms $V_2(0)$. Finally, there is an increase in job to job flows, as the increase in α implies that the only channel to reach employment in high productivity firms is through a passage through employment n_1

Figures 3 and 4 focus on the comparative statics following an increase in the productivity of high type firms y_2. The lowest value of y_2 in the simulation is 1.08, a value lower than that displayed in Table 2. The simulation clearly shows that as long as y_2 is less than 1.09, the economy settles in a pure job ladder equilibrium and the market 02 shuts down. The first two panels in Figure 3 show also that as the high productivity premium increases, the economy moves out of a pure job ladder equilibrium toward an equilibrium in which the market 02 is operative. Not surprisibgly, the equilibrium value of unemployment increases monotonically, as well as the joint income and the PDV wages in both low and high productivity firms. The increase in the value of low productivity jobs is linked to the expectation of a capital gain associated to a future move toward a high productivity job that, as a result of the larger y_2 has higher value. The effects of y_2 on profits depends entirely on the type of equilibrium in which the economy settles. For values of $y > 1.09$, all three submarkets are operative and the larger productivity y_2 increase profits in high type firms and lower profits in low productivity firms. The opposite happens when the pure job ladder equilibrium prevails. In a pure job ladder equilibrium the increase in y_2 increases the demand for employed workers n_1 with obvious positive impact on firm 1 profits. With convex hiring costs it is possible that such increase in demand leads to an overall increase in costs, depsite the larger produtctivity y_2.

Figure 4 focuses on aggregate quantities. The clear and simpler effect is the increase in aggregate employment in high productivity firms. This result is the counterpart of the increase in the value functions described in the panel of 3.Employment in low productivity firms falls, since more and more workers move to better jobs. The increase in the job finding rate p_{12} is consistent with such change. Things are more complicated when we look at the unemployment level. To understand this effect one has first to realize that the fall in n_1 employment is quantitatively very sizeable, as displayed in the fifth panel in Figure 4. In the pure job ladder equilibrium low productivity firms are now smaller and reduce the demand for unemployed workers. As a result the job finding rate p_{01} falls and and n_0 increases. As the economy moves to the equilibrium in which all submarket opens up, high productivity firms hire directly from the unemployed and induce a reduction in the supply of searching workers for low productivity jobs. In other words there is a jump in p_{02} and p_{01} increases, since unemployed are a more scarce resource in the 01 submarket. As the high productivity premium increases further, the overall effect on unemployment is ambiguous and non monotonic.

Figures 5 6 report the comparative static following an increase in the entry costs. Results are straightforward. The value of unemployment falls monotonically, as do all PDV wages. An economy with larger costs is an economy with more frictions and barriers to entry and does yield lower utility. The number of firms fall while profits for incumbent firms increase. The latter ìresult is not surprising, since free entry implies that expected profits match the entry costs. All job finding rates fall as does fall employment in high productivity firms. There is as a consequence an increase in employment and unemployment at lower productivity level.

Discussion

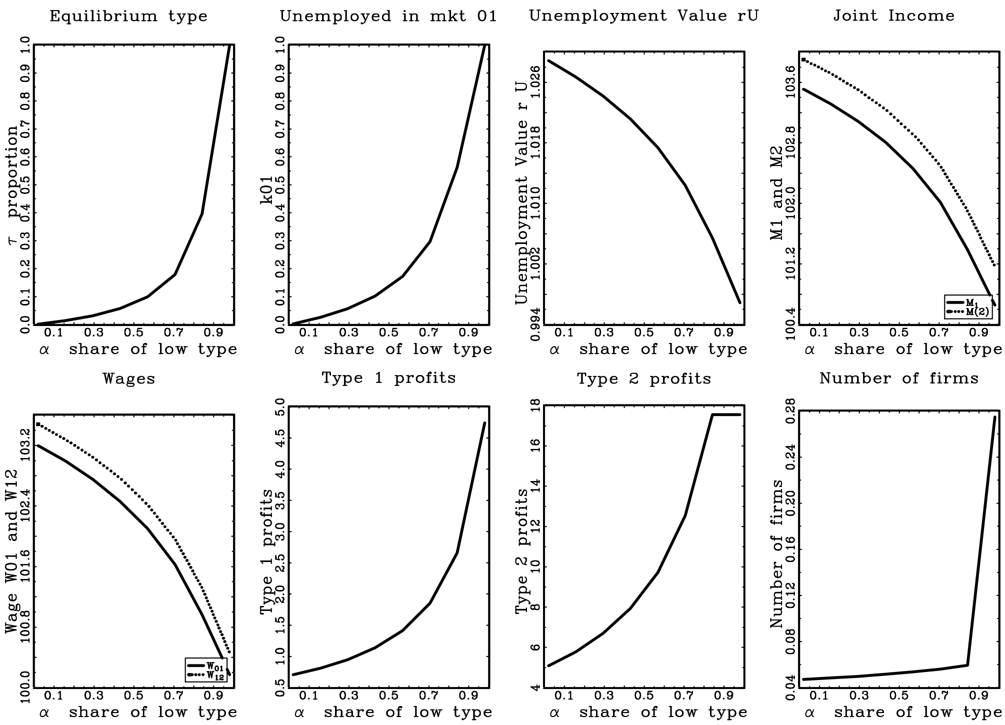

Figure 1: Increase in α, the proportion of low productivity firms. Value Functions

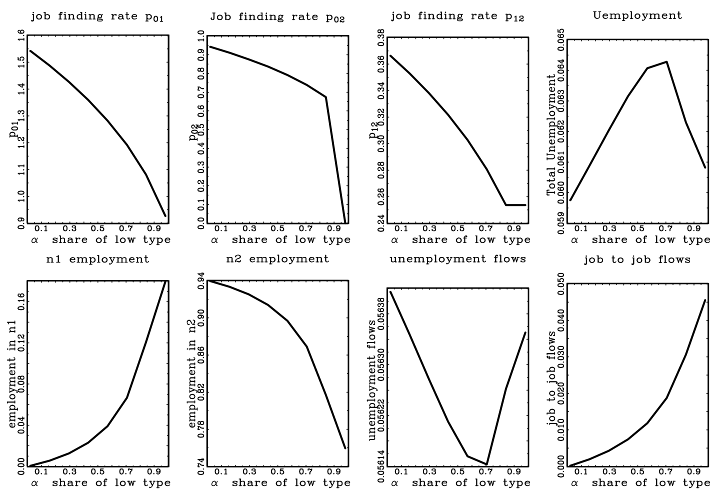

Figure 2: Increase in α, the proportion of low productivity firms. Aggregate Quantities

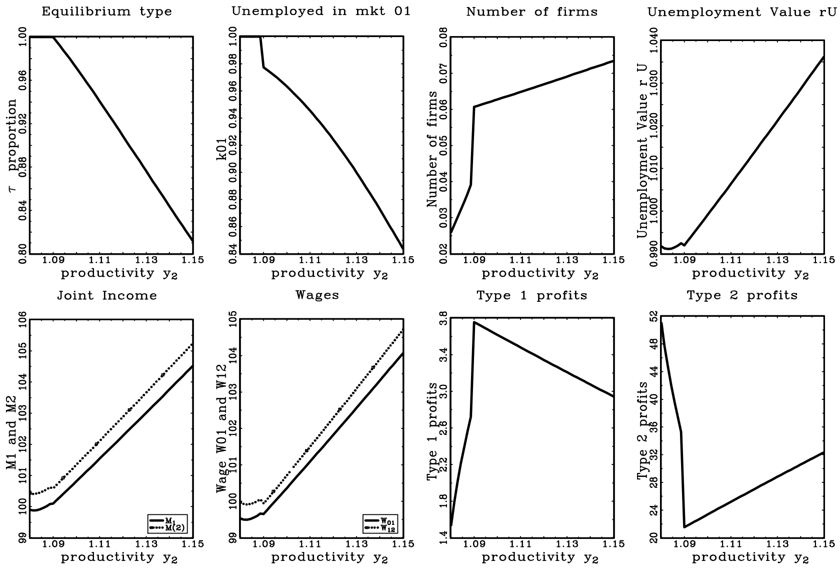

Figure 3: Increase in productivity y_2 of high type firms. Value Functions

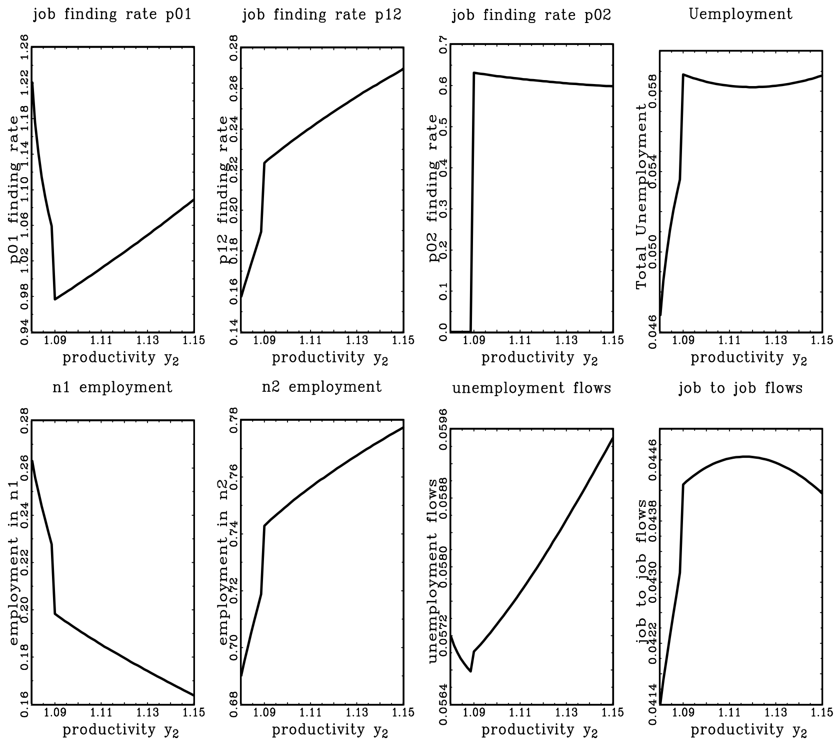

Figure 4: Increase in productivity y_2 of high type firms. Aggregate Quantities

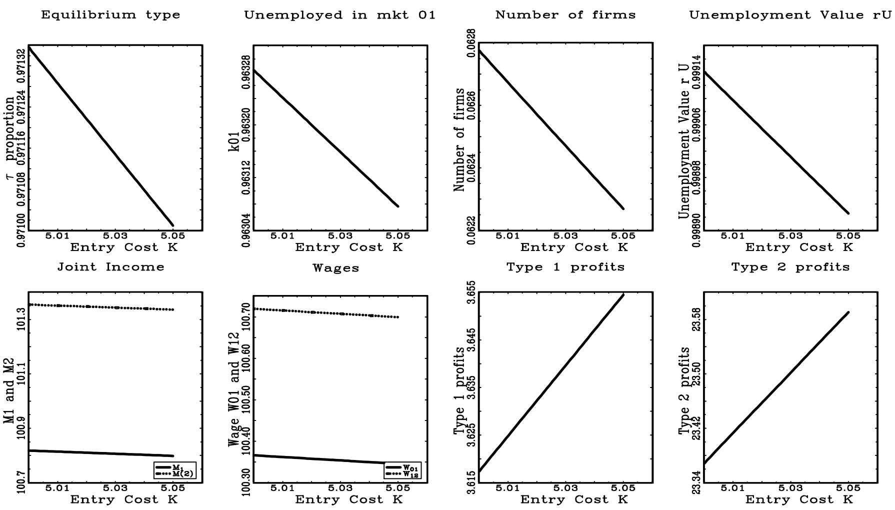

Figure 5: Increase in entry cost k. Value Functions

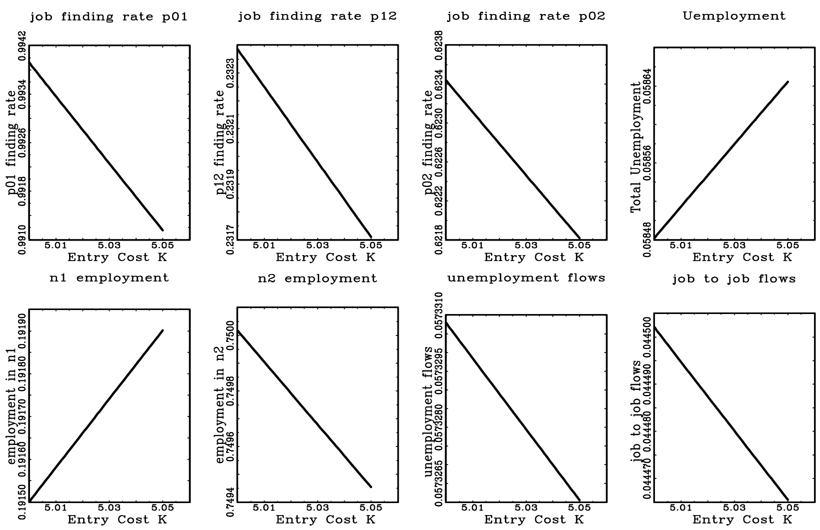

Figure 6: Increase in entry cost k. Aggregate quantities

8.2 Competitive equilibrium wage dynamics

We assume first that the labor market is perfect and in equilibrium there is a single wage paid to the entire workforce. There is full employment but to obtain the equilibrium wage we need to derive labour demand. Type i firm's instantaneous profit is $(y_i - w)N_i - cv_i^\alpha()$. Dynamics reads $\dot{N} = v() - sN$. If $w \geq y_i$ the firm leaves the market and obtains zero profit. The firm takes as given the wage and chooses vacancies to maximize profits. The Hamiltonian writes

$$H = (y_i - w)N_i - cv_i^\alpha() + \lambda_i(v - sN)$$

First order conditions reads

$$\begin{aligned} v_i() &= (\frac{\lambda_i}{\alpha c})^{\frac{1}{\alpha-1}} \\ \lambda_i &= \frac{y_i - w}{r+s+\delta} \end{aligned}$$

The profit of a type i firm entering the market is

$$\Pi_i = \frac{v_i \lambda_i}{r+\delta} = \frac{(\alpha c)^{\frac{1}{1-\alpha}} \lambda^{\frac{1+\alpha}{r+\delta}}}{r+\delta}$$

The free entry condition uniquely pins down w

$$E\Pi_i = K$$

We can now show the following result:

Proposition 12 *a) Suppose $\alpha \to 1^+$. Then in the limit only the most productive firms are active, and they pay a wage*

$$w = y_{\max} - c(r+s+\delta)$$

b) Suppose $\alpha > 1$. Then firm i is active provided that y_i is sufficiently close to $y\max$.

In the competitive setting with adjustment costs different firms can coexist in the market as long as the productivity differential is not too high. Since the wage is unique there is no links between firms dynamics and wage differentials. There is also no on the job search. This suggests that most of the empirical regularities discussed above can not be rationalized in the competitive setting. In the rest of the paper we show that the combination between convex adjustment costs at the firm level and labour market imperfections do deliver most of such implications.

9 The vacancy-wage trade-off

Before turning to the general equilibrium with imperfect labour market, we analyze the microeconomics of a firm that has some ability to fix wages. This section shows that a positive wage size effect requires adjustment costs to be convex. We suppose that hiring can be obtained through two means, v and w. In other words the firm has the ability to attract a given amount of workers in two ways. Either by advertising effort v, where v is a measure of efficiency unit of search. Alternatively the firm can attract workers with higher wage w. These feature are common to the competitive search equilibrium that we will be using.

If we assume that the firm needs to hire an amount of labour h (h is a proxy of firm size in this section), the relationship between h, v and w is given by the following function

$$h = q(w)v$$

where v and w are defined as above. The function $q(w)$ denote the arrival rate of workers per efficiency unit of search, increasing and concave and v, the number of efficiency unit. For a given level of hiring h the previous condition determines a technological trade off between vacancies and wages. Let us assume that the firm needs to hire an amount h and needs to minimize total costs. Labor costs are naturally given by hw and suppose the cost of efficiency units is given by $c(v) = cv^\gamma$ where γ is a positive constnat. The formal problem of the firm of obtaining a hiring flow of h is then given by

$$\min hw + cv^\gamma \qquad \text{s.t. } q(w)v = h$$

The associated Lagrangian is

$$L = hW + cv^\gamma - \lambda[q(w)v - h]$$

with first order conditions

$$\begin{aligned} h &= \lambda q'(w)v \\ c\alpha v^{\gamma-1} &= \lambda q(w) \end{aligned}$$

or

$$c\alpha v^\gamma = h\frac{q(w)}{q'(w)}$$

Substituting in $v = h/q$ gives

$$\begin{aligned} c\alpha(h/q)^\gamma &= h\frac{q(W)}{q'(W)} \\ \frac{q'}{q^{(1+\gamma)}} &= \frac{h^{1-\gamma}}{c\gamma} \end{aligned}$$

which uniquely determines w as a function of γ. Assuming $q = w^\beta$, the left hand right hand side reads $\frac{q'}{q^{(1+\gamma)}} = \beta w^{-(1+\gamma\beta)}$ so that the wage paid by the firm will be or

$$w = h^{-\frac{1-\gamma}{1+\gamma\beta}}k$$

where $k = (c\alpha)^{\frac{1}{1+\gamma\beta}}$.

We establish an important result

Remark 4 *A positive link between wage and firm size requires γ to be less than one.*

The model relies on a convex hiring cost to prevent firms from posting an infinite number of vacancies upon entry in the labor market. This assumption can be justified along several dimension.

The measure v is closer to search effort from the firm standpoint rather than to a measure vacancies. When a firm double its search intensity it does not typically double the number of applicants, in a way similar to diminishing returns. The counterpart of this simple reasoning is a convex hiring cost.

Another way to justify convexity relies on labour market frictions linked to firm's optimal scale. Changing firm scale requires a costly look for talent that is not easily available in local markets. Our modeling strategy can be thus seen as a reduced form of this extremely costly search for talent.

At a more technical level, convex hiring costs can be seen as a generalization of Burdet Mortensen (1999) model with on the job search. In their model, the arrival of workers to firms is exogenously set. Our specification allows for some flexibility, and we let firms to increase this arrival rate through search effort. Note also that the structural estimates provided by Yashiv (2000a,b) are fully consistent with a marginal cost increasing in the stock of vacancies.

Finally, most models of endogenous search effort focus on the worker side. In such models (Pissarides, 2000) workers' cost of effort is typically model as convex function with respect to individual effort. Our function is the analogous approach on the firm side.

10 APPENDIX:

10.1 Proof of existence

The strategy for the proof is to construct a mapping for which the equilibrium of the model is a fixed point, and then apply Brouwer's fixed point theorem.

To this end, let $\overline{\kappa}$ denote a matrix describing submarket choices of workers κ_{ij}, $\kappa_{ij} = 0$ if $i \geq j$, and $\sum_{j=i+1}^{n} \kappa_{ij} = 1$ for all i. Similarly, let $\overline{\theta}$ denote a matrix of labor market tightnesses θ_{ij}, $\theta_{ij} = 0$ if $i \geq j$. We require that $0 \leq \theta_{ij} \leq \theta^{\max}$ for all $i < j$, where $\theta^{\max}$ will be defined below. Finally, let the real number k denote the measure of firms in the economy. We require that $k \leq k^{\max}$. It follows that the set $D^n \in R^{2(n+1)^2+1}$ of allowed vectors $(\overline{\kappa}, \overline{\theta}, k)$ is closed and convex.

We want to construct a continous mapping $\Gamma : D^n \to D^n$, and proceed as follows: Let $\overline{p}$ denote the matrix of transition probabilities $p_{ij} = A\theta_{ij}^{1-\beta}$. Analogous with (12), define

$$(r + s + \delta)M_{ij} = y_i + (s + \delta)M_0 + p_{ij}\beta(M_j - M_{ij}) \tag{19}$$

(where M_0 is replaced with M_0). Let $\overline{M}$ denote the matrix of values M_{ij}. Given the matrix $\overline{p}(\overline{\theta})$, the matrix $\overline{M}$ is uniquely defined as a continous function of $\overline{\theta}$, $\overline{M}(\overline{\theta})$. To see this, first note that M_n is independent of $\overline{\theta}$. Suppose M_{ij} and $M_i = \max_j M_{ij}$ are uniquely defined as continous functions of $\overline{\theta}$ for all $i, j > i$, for all $i > k$. It then follows from equation (19) and the definition of M_i that this also holds for M_{kj}, $j > k$, and M_k. Thus it holds for all i, j such that $j > i$.

The gross income flow of a firm of type j of posting a vacancy in submarket i is given by $\rho_{ij} = q(\theta_{ij})(1-\beta)(M_j - M_{ij})$. Define $\rho_j = \max_i \rho_{ij}$. Now define θ^a_{ij} implicitely by the function

$$q(\theta^a_{ij})(1-\beta)(M_j - M_{ij}) = \rho_j$$

The equation thus shows the values of θ_{ij} such that the firm of type j is indifferent between searching in submarket i and in the best submarket given $\overline{\theta}$. Finally, let v^a_j be defined by the equation $c'(v^a_j) = \rho_j$ (the optimal number of vacancies given ρ_j). It follows that both $\overline{\theta}^a$ and v^a_i are continous functions of $\overline{\theta}$.

Given the initial vector $\overline{\theta}$, equation (11) uniquely defines $N_0, N_1, ...N_n$ as continuous functions of $\overline{\theta}$. In each submarket, aggregate consistency requires that

$$N_i \kappa_{ij} \theta_{ij} = k\alpha_i \tau_{ij} v_{ij} \tag{20}$$

Sum over i. This gives

$$\sum_{i<j} N_i \kappa_{ij} \theta_{ij} = \sum_{i<j} k\alpha_i \tau_{ij} v_{ij}$$

We now insert $v_{ij} = v^a_j$ into this equation. Define the constant ζ_j by the expression

$$\sum_{i<j} N_i \kappa_{ij} \theta_{ij} = \zeta_j k \alpha_i v^a_j \tag{21}$$

Finall, define

$$\widehat{\theta}_{ij} = \zeta_j(\widehat{\theta})\theta^a_{ij}(\widehat{\theta})$$

This is our updating rule for rule for θ unless the upper bound $\theta^{\max}$ binds, in which case $\widehat{\theta}_{ij} = \theta^{\max}$.

Consider the searching workers. Suppose M_i is obtained for $j \in J_i$. For all $j \notin J_i$, define

$$\widehat{\kappa}_{ij} = \frac{M_{ij}}{M_i} \kappa_{ij}$$

Note that $\widehat{\kappa}_{ij}$ is continuous in $\overline{\theta}$ and κ_{ij}. Define the constant η_i by the experssion

$$\eta_i \sum_{j \in J_i} \kappa_{ij} + \sum_{j \notin J_i} \widehat{\kappa}_{ij} = 1$$

For all $j \in J_i$, the updating rule reads

$$\widehat{\kappa}_{ij} = \eta_i \kappa_{ij}$$

Finally, the expected profit of a firm of type i entering the market and searching for a firm $j \in J_i$ reads

$$\Pi_i = \frac{1}{r+\delta}\left\{v_i^a q(\theta_{ij})(M_{ij} - M_i)(1-\beta) - c(v_i^a)\right\}$$

The expected profit of entering, given the initial parameter values, is

$$E\Pi = \sum_i \alpha_i \Pi_i$$

The updating rule for k reads

$$\widehat{k} = k\frac{E\Pi}{K}$$

unless the upper bound $k^{\max}$ binds, in which case $\widehat{k} = k^{\max}$.

We have thus constructed a mapping $\Gamma : D^n \to D^n$, which by construction is continous. It follows from Brouwers fixed point theorem that the mapping has a fixed point.

Our next step is to show that a fixed point of Γ is an equilibrium of our model. Denote the fixed point by D^*. First, given the asset value matrix M_{ij}, the firm sets the optimal sharing rule by construction. Furthermore, by the very definition of $\widehat{\theta}$ it follows that the all firm are indifferent by entering any submarket $_{ij}$. Thus, the firms' search behaviour is optimal.

Second, from the updating rule for κ it follows that if $\kappa_{ij}^* > 0$, then it is optimal workers in firm j to search for a position in firm j.

Third, we have to show that the model is internally consisten, and satisfies (20). At the fixed point, $\zeta_j = 1$ for all j. Hence (21) is satisfied. However, this means that the weights τ_{ij} give us enough degrees of freedom to satisfy (20).

By construction, the labor market tightness θ_{ij}^* is defined even in sumbarkets where $\kappa_{ij} = 0$, i.e., even in empty submarkets. We have thus ruled out the situations where no agents enter a submarket which potentially may be active because noone else enter the market.

Finally, we characterize the bounds. Consider the equilibrium with $\alpha_n = 1$ (only firms of the highest productivity). It is trivial to show that this equilibrium exists. Define $k^{\max}$ and $\theta^{\max}$ as the equilibrium values of k and θ in this equilibrium, respectively. By construction, $\theta_{ij}^* < \theta^{\max}$ for all i, j, and that $k^* < k^{\max}$.

10.2 Proof of efficiency

The welfare function reads

$$W = \int_0^\infty [\sum_{j=0}^n N_j y_j - \sum_{i=0}^n \sum_{j=i+1}^n \alpha_j kc(v_{ij}) - aK]e^{-rt}dt$$

The law of motions are defined as

$$\begin{aligned}
\dot{N}_j &= \sum_{i=0}^{j-1} x(\kappa_{ij}N_i, \alpha_j k\tau_{ij}v_{ij}) - \sum_{i=j+1}^{n} x(\kappa_{ji}N_j, \alpha_i k\tau_{ji}v_{ji}) - (s+\delta)N_j \\
\dot{k} &= a - \delta k
\end{aligned}$$

The initial conditions take care of the requirement that $\sum_i N_i = 1$. The controls are a, κ_{ij}, τ_{ij} and v_{ij}. All κ_{ij}, τ_{ij} have to be between zero and 1, this will be discussed later. The current-value Hamiltonian reads

$$\begin{aligned}
H &= \sum_{j=0}^n N_j y_j - \sum_{i=0}^n \sum_{j=i+1}^n \alpha_j kc(v_{ij}) - aK \\
&\quad + \sum_{j=0}^n \lambda_j [\sum_{i=0}^{j-1} x(\kappa_{ij}N_i, \alpha_j k\tau_{ij}v_{ij}) - \sum_{i=j+1}^{n} x(\kappa_{ji}N_j, \alpha_i k\tau_{ji}v_{ji}) - (s+\delta)N_j] \\
&\quad + A[a - \delta f]
\end{aligned}$$

The controls are chosen so as to maximize H. Note that $x(u,v) = Au^\beta v^{(1-\beta)}$ it follows that $x_v = (1-\beta)Au^\beta v^{-\beta} = (1-\beta)q(\theta)$. We thus get the following first order conditions for vacancy creation:

$$c'(v_{ij}) = (1-\beta)q(\theta_{ij})[\lambda_j - \lambda_i] \tag{22}$$

The first order conditions for the other controls read

$$\begin{aligned}
A &= K && (23) \\
p_{ij}(\lambda_j - \lambda_i) &= \max_k p_{ik}(\lambda_i - \lambda_k) \text{ if } \kappa_{ij} > 0 && (24) \\
q_{ij}(\lambda_j - \lambda_i) &= \max_k q_{kj}(\lambda_j - \lambda_k) \text{ if } \tau_{ij} > 0. && (25)
\end{aligned}$$

Equation (25) and equation (22) implies that $v_{ij} = v_{kj} = v_j$ in all submarkets where firm j is active. Finally, the value functions for the adjungated variables are given by (in steady state)

$$\begin{aligned}
(r+s+\delta)\lambda_j &= y_j + \beta \max_{k>j} p_{jk}(\lambda_k - \lambda_j) + (s+\delta)\lambda_u && (26) \\
(r+\delta)A &= \sum_{j=1}^n \alpha_j[(1-\beta)v_j \max_k q_{jk}(\lambda_k - \lambda_j) - \frac{v_1^2}{2c}]
\end{aligned}$$

It follows that the first order conditions of the planner is exactly equal to the market solution. More than that, the maximization problem for the controls is exactly equal to the maximization problem of the firm. Thus, the planner's solution and the decentralized solution is the same.

10.3 Proof of proposition 5

Consider a situation where K is arbitrarily high. For firms to be willing to enter, it follows that Π_2 must be arbitrarily high. However, this can only be the case if the arrival rate of jobs in the market the firm searches for is arbitrarily high, i.e., k must be arbitrarily small.

It follows that p_{01} is arbitrarily low, and thus also that N_1 is arbitrarily low. Suppose all high-type firms were searching for employed 26workers. Since the ratio of high-to low type firms is bounded, the labor market tightness in this market would be bounded. Hence the firms could not obtain an arbitrarily high profit, and we cannot be in equilibrium. In the unemployed submarkets, the labor market tightness by contrast is arbitrarily small and the firms get an arbitrarily high profit.

Suppose then that for some K', there exists a pure job ladder. We want to show that then there is a pure job ladder for all $K < K'$. First we keep the number of vacancies per firm constant. By a revealed preference type of argument one cans show that k is decreasing in K.

We first want to show that $el_k p_{01} > el_k p_{02}$. First note that

$$\begin{aligned} el_k p_{01} &= el_k(\frac{\alpha_1 k}{u})^{1-\beta} = (1-\beta)(1-el_k u) \\ el_k p_{12} &= el_k(\frac{\alpha_2 k}{N_1})^{1-\beta} = (1-\beta)(1-el_k N_1) \end{aligned}$$

Now

$$N_1 = \frac{p_{01} u}{s+\delta+p_{12}} \tag{27}$$

$$u = \frac{s+\delta}{s+\delta+p_{01}} \tag{28}$$

For given stocks u and N_1, $el_k p_{01} = el_k p_{12}$. From (27) and (28) it then follows that $el_k N_1 > el_k u$ (since $p_{01} u = (s+\delta)(1-u)$ is increasing in k). It thus follows that $el_k p_{01} > el_k p_{12}$.

Note that $el_k(\lambda_2 - \lambda_0) < el_k(\lambda_1 - \lambda_0)$ (since λ_1 is increasing in k). From (24) it follows that $el_k p_{02} > el_k p01$. From (26) it follows that we can write

$$\begin{aligned} \lambda_2 - \lambda_0 &= \frac{y_2 - y_0}{r+s+(1-\beta)p_{02}} \\ \lambda_2 - \lambda_1 &= \frac{y_2 - y_1}{r+s+(1-\beta)p_{12}} \end{aligned}$$

Taking elasticities give

$$\begin{aligned} el_k(\lambda_2 - \lambda_0) &= el_p \frac{y_2 - y_0}{r+s+(1-\beta)p_{02}} el_k p_{02} \\ el_k(\lambda_2 - \lambda_1) &= el_p \frac{y_2 - y_1}{r+s+(1-\beta)p_{12}} el_k p_{12} \end{aligned}$$

Now $el_x \frac{1}{a+x} = -\frac{x}{a+x}$ (for any constant a) which is decreasing in k. Furthermore, we have seen that $el_k p_{02} > el_k p_{12}$. It thus follows that

$$el_k(\lambda_2 - \lambda_1) > el_k(\lambda_2 - \lambda_0)$$

Finally, from (25) it follows that the result holds if

$$el_k q_{02} + el_k(\lambda_2 - \lambda_0) < el_k q_{12} + el_k(\lambda_1 - \lambda_0)$$

Since $el_k p_{02} > el_k p_{12}$ it follows that $el_k p_{02} < el_k p_{12}$, and we have just shown that $el_k(\lambda_2 - \lambda_1) > el_k(\lambda_2 - \lambda_0)$. The result thus follows.

10.4 Proof of claim related to α^*

a) Sketch of proof: For any number $\varepsilon > 0$. Consider a high-type firm that sets $w = y_1 + \varepsilon$. As $\alpha_0 \to 1$, the arrival rate of workers to this firms goes to infinity, independently of which wage $w \in (y_1, y_2)$ the other high-type firms choose. Thus profits go to infinity. If a high-type firm searches for unemployed workers, the arrival rate of workers to the firm will be bounded, and hence also profit. The claim thus follows. By a similar argument, it also follows that at least some high-type firms searches for employed workers as long as $\alpha > 0$.

b) We want to show the following claim: For a given number of firms k, there exists a unique α^* with the following property: If $\alpha > \alpha^*$ there exists a pure job ladder. If $\alpha < \alpha^*$ some high-type firms search for unemployed workers. We start by assuming that the number of vacancies per firm is constant.

Consider first the case where $\alpha \to 1$. Note that λ_{12} is limited above. We want to show that $\lim_{\alpha \to 1} q_{12} = \infty$. Suppose not, and suppose instead that q is bounded by $\overline{q}$. Since λ_{12} is limited above by $\overline{\lambda} = y_2/(r+s+\delta)$ it follows that v is limited above by $\frac{\overline{\lambda}\overline{q}}{c}$.

Let $\overline{N_1}$ denote the value of N_1 in the limit as $\alpha \to 1$. Clearly $\overline{N_1} > 0$ and $rM_0 > z$. It follows that

$$\begin{aligned} \lim_{\alpha \to 1} q_{12} &= \lim_{\alpha \to 1} A[\frac{(1-\alpha)kv}{N_1}]^{-\beta} \\ &\leq \lim_{\alpha \to 1} A[\frac{(1-\alpha)k\overline{v}}{N_1}]^{-\beta} \\ &= \infty \end{aligned}$$

Hence q cannot be limited above. But then it follows that the profit of searching for employed workers goes to infinity as α goes to zero.

Consider then the profitability of a high-type firm searching for unemployed workers. Since λ_{02} is bounded above by $\overline{\lambda} = y_2/(r+s+\delta)$, the profit can only goes to infinity if q_{12} does. Suppose it does. Then workers applying to this job has a job finding rate of $p = 0$ and thus receives $rM_0 = z$. However, the workers would then prefer to search for the low-type firm and we cannot be in equlibrium. It follows that it is more profitable to search for employed than for unemployed workers if α is sufficiently close to 1.

Suppose then $a \to 0$. It follows that $N_1 \to 0$. We want to show that the proportion of high-type firms searching for employed workers goes to 0. Suppose not, and suppose the share is bounded below by $\tau^{\min} > 0$. Suppose that in the limit, $v_{12} > 0$. It follows that

$$\begin{aligned}
\lim_{\alpha \to 0} q_{12} &= \lim_{\alpha \to 1} A[\frac{(1-\alpha)k\tau v_{12}}{N_1}]^{-\beta} \\
&\leq \lim_{\alpha \to 1} A[\frac{(1-\alpha)k\tau^{\min} v_{12}}{N_1}]^{-\beta} = 0
\end{aligned}$$

Note also that $v_{12} = 0$ if and only if $q_{12}\lambda_{12} = 0$. Thus both if $v_{12} = 0$ in the limit and when it is not the assumption that $\tau^{\min} > 0$ is inconsistent with (25).

Finally we want to show that for any $\alpha > 0$, $\tau > 0$. Suppose not. Then there exists an $\alpha > 0$ such that $\tau = 0$. If (25) is satisfied we must have that $v_{12} < \infty$. But then it follows that $q_{12} = \infty$, hence (25) cannot be satisfied. Again we have derived a contradiction.

Finally we want to show that there exists a unique α^* as described above. That there exists a α^* such that (25) is satisfied with equality for $\tau = 1$ follows from continuity and the results just laid out. What is left is to show that this α^* is unique. To this end it is sufficient to show that if (25) is satisfied with equality for $\tau = 1$, then an decrease in α implies that the right-hand side of (25) is striclty greater than the left-hand side for $\tau = 1$.

In what follows we will work with α_2 rather than α, the fraction of high-type firms. We want to show that an increase in α_2 for a given k, and given that $\tau = 1$ implies that searching for unemployed workers become strictly more profitable than searching for employed workers. (from 25)

$$q_{12}(\lambda_2 - \lambda_1) < q_{02}(\lambda_2 - \lambda_0) \tag{29}$$

for α_2' marginally greater than α_2^*.

First note that an increase in α increases λ_1. Suppose λ_0 decreases. From p_{01} (26) it follows that decreases. From (26) it also follows that

$$\lambda_2 - \lambda_0 = \frac{y_2 - r\lambda_0}{r + s + \delta}$$

which thus increases. From (24) and the fact that p_{01} decreases, it follows that p_{02} decreases. From the matching function it follows that that q_{02} increases. Thus the right-hand side of 29 increases. An increase in α_2 increases p_2, and from (**??**) it follows that $(\lambda_2 - \lambda_1)$ decreases and q_{12} decreases. Thus the left-hand side of 29 decreases. Hence we are done in this case.

Suppose then that λ_0 is increasing in α_2 (which indeed seems likely). In what follows we rescale the model by setting $z = 0$. Clearly this can be done without loss of generality, as the maxmization problem is unchanged if all flows z, y_1, y_2 are reduced equally much. It follows that we can write

$$\lambda_0 = \frac{p_{01}}{r + p_{01}}\lambda_1$$

Thus, from (26)

$$\lambda_0(1 - \frac{s+\delta}{r+s+\delta})\frac{r+p_{01}}{p_{01}} = \frac{y_1 + p_{12}(\lambda_2 - \lambda_1)}{r+s+\delta}$$

Taking elasticities wrt α_2 gives

$$el\lambda_0 + X < elp_{12} + el(\lambda_2 - \lambda_1)$$

where $X = el\frac{r+p_{01}}{p_{01}} > 0$. An increase in α deIt follows that

$$el\lambda_0 < elp_{12} + el(\lambda_2 - \lambda_1)$$

From (26) it follows that $r\lambda_0 = p_{02}(\lambda_2 - \lambda_0)$. Taking elasticities and using the above equation give

$$elp_{02} + el(\lambda_2 - \lambda_0) < elp_{12} + el(\lambda_2 - \lambda_1)$$

or

$$elp_{02} < elp_{12} + el(\lambda_2 - \lambda_1) - el(\lambda_2 - \lambda_0) \tag{30}$$

As $\frac{\delta\lambda_1}{\delta\alpha_2} > \frac{\delta\lambda_0}{\delta\alpha_2}$ and $\lambda_2 - \lambda_1 < \lambda_2 - \lambda_0$ it follows that $0 > el(\lambda_2 - \lambda_0) > el(\lambda_2 - \lambda_1)$. From (30) it thus follows that $elp_{02} < elp_{12}$ and thus that $elq_{02} > elq_{12}$. Together this implies that (29) is satisfied.

10.5 Proof that expected profit is decreasing in k

10.6

Consider an increase in k. Suppose firs that the market structure is unchanged. The first step then is to show that $\lambda_j - \lambda_i$ is decreasing in k for all active submarkets $_{ij}$. Her tror jeg man må se på realøkonomien. Det er ikke så lett. For a given k, write $W = W(k)$. By construction, $W'(k) = A$. We want to show that $W''(k) < 0$. Suppose not. Compare two values k^1 and k^2, $k^1 < k^2$. Note that A denote the social value of increasing k, and by definition the optimal solution maximizes the social value of entry. For a given

10.7 Proof of lemma 5.

Consider a firm of type j that searches for workers employed at level i having an NPV wage of $\widetilde{M}_i$. Furthermore, assume that the workers in that firm searches for jobs in firms of type k which pay W_{jk} and which they obtain at rate p_{jk} (still we supress the dependence of k in the expressions below). Since the agents are risk neutral and use the same discount factor, the timing of the payment to the worker is irrelevant, and for notational convenicence we assume that the worker is paid the entire NPV wage W_{ij} upfront. The net present value of profit of a firm with initial labor stock N_0 can be written as (we index state variables by j and choice variables and the adjungated variable by ij)

$$\begin{aligned}
\Pi_{ij} &= \int_0^\infty [N_j[y_j + sM_0 + p_{jk}W_{jk}] - \frac{\widetilde{v}_{ij}N_j}{2} - N_j\widetilde{v}_{ij}W_{ij}q(W_{ij}))]e^{-(r+\delta)t}dt \\
s.t.N_j(0) &= N_0 \\
\dot{N}_j &= \widetilde{v}_{ij}q(W_{ij})N_j - (s + p_{jk})N_j
\end{aligned}$$

where $\widetilde{v}_{ij} = v_{ij}/N_j$ and where $q(W_{ij}) \equiv q(p_i(W_{ij}, M_i)$, The Hamiltonian reads

$$H = N_j[y_j + sM_0 + p_{jk}W_{jk} - N_j\widetilde{v}_{ij}W_{ij}q(p(W_{ij})))] + \lambda_{ij}[\widetilde{v}_{ij}q(W_{ij})N_j - (\delta + p_{jk})N_j].$$

First order conditions for W reads (after some manipulation)

$$\begin{aligned}
W_{ij} &= (\lambda_{ij} - M_i)\beta \\
\widetilde{v}_{ij} &= (\lambda_{ij} - M_i)(1-\beta)q \\
(r+\delta)\lambda_{ij} &= y_j + sM_0 + p_{jk}(W_{jk} - \lambda_{ij}) + [(\lambda - W)\widetilde{v}^* q(W^*) - \frac{\widetilde{v}_{ij}}{2c}]
\end{aligned}$$

Which gives us the conditiions in the lemma (with $\widetilde{M_{ij}}$ substituted in for λ_{ij}). (Have to say something about the max, that is postphoned).

10.8 Proof of proposition 10

In order to show that the problem is well defined it is sufficient to show that $\widetilde{M_{ij}}$ is bounded for all $\widetilde{v_{ij}}$ and all W_{ij}. We will show that this is always the case for sufficiently high search costs, i.e., for sufficiently low values of c. By assumption, $(r+s)M_i > y_i + s\widetilde{M}_0$, hence $q_{ij}(W_{ij})$ is finite for any finite W_{ij}. Define

$$\overline{W}_j = \frac{y_j + \max_{k>j} p_{jk}W_{jk} + (s+\delta)\widetilde{M}_0}{r+s+\delta}$$

and define $\overline{q}_j = q_{ij}(\overline{W}_j)$. Note that $\overline{W}_j$ strictly exceeds the NPV of the income a worker generates, hence by paying $\overline{W}_j$ to all the workers the firm surely obtains a negative profit. Rewrite (16) to

$$\begin{aligned}
\widetilde{M_{ij}} &= \frac{y_j + \max_k p_{jk}(W_{jk} - \widetilde{M_{ij}}) + (s+\delta)\widetilde{M_0} - \widetilde{v}^*_{ij}q^*_{ij}(1-\beta)\widetilde{M_i} - \widetilde{v}^{*2}_{ij}/2c]}{r+s+\delta - \widetilde{v}^*_{ij}q^*_{ij}(1-\beta)} \\
&< \frac{y_j + \max_k p_{jk}(W_{jk} - \widetilde{M_{ij}}) + (s+\delta)\widetilde{M_0} - \widetilde{v}^*_{ij}\overline{q}_j(1-\beta)\widetilde{M_i}) - \widetilde{v}^{*2}_{ij}/2c}{r+s+\delta - \widetilde{v}^*_{ij}\overline{q}(1-\beta)}
\end{aligned}$$

In the second expression we have substituted in $\overline{q}_j > q_{ij}$. For sufficiently high values of $\widetilde{v}_{ij}$, the denominator is negative. Define $\widetilde{v}^0$ as the value of $\widetilde{v}_{ij}$ that makes the denominator equal to zero. It follows that

$$\widetilde{v}^0 = \frac{r+s+\delta}{(1-\beta)\overline{q}}$$

It is sufficient to show that the nominator is negative for $v = \widetilde{v}^0$, i.e., that

$$y_j + \max_k p_{jk}(W_{jk} - \widetilde{M}_{ij}) + (s+\delta)\widetilde{M}_0 - \widetilde{v}^0\overline{q}(1-\beta)\widetilde{M}_i) - (\widetilde{v}^0)^2/2c < 0$$

which is trivially satisfied for sufficiently low values of c.

Then we turn to uniqueness. Note that if W_{ij} and v_{ij} satisfies (17) and (18), then this is a local maximum for $\widetilde{M}_{ij}$. Suppose W'_{ij} and v'_{ij} constitute a local but not a global maximum for joint income, and let $\widetilde{M}'_{ij}$ denote the corresponding joint income. Then it follows that for $\widetilde{M}'_{ij}$, the first order conditions (17) and (18) have at least two solutions. However, given $\widetilde{M}'_{ij}$ the firm's maximization problem is exactly as in the previous section

11 Computation of the General Equilibrium

To solve the model one needs to set following 10 parameters: r, s, δ, y_1 and y_2, z, β c, k, α. In addition, the matching function we use is cobb douglas with share parameter β and with constant A.

The procedure to compute the equilibrium is as follows. First, the procedure tries to solve for the model with three submarkets. If this fails the procedure switches to the pure job ladder equilibrium. The solution is basically computed in four steps. The first steps (step i) solves for the asset equations in the general model, the second steps (step ii) computes τ, the proportion of good firms that hire directly from the unemployement pool and the final steps solves for the stock. Step three (step iii) is reached only if the proportion of firms that hires directly from the unemployed is less than one. In case this proportion τ is greater than one, the procedure goes to the step four (step iv) and solves for the pure job ladder equilibrium.

11.1 Step i): Solving for the Asset values in the general model

The procedure starts from assigning an arbitrary initial guess value of $M_1 = M_1^{'}$ and $rM_0 = rM_0^{'}$. Given the initial guess, one can compute recursively $M_2^{'}, p_{01}^{'}, p_{02}^{'}, p_{12}^{'}$

$$
\begin{aligned}
M_2' &= \frac{y_2 + (r+\delta)M_0'}{r+\delta+s}; \qquad \text{using } M_2 = \frac{y_2 + (r+\delta)M_0}{r+\delta+s} \\
p_{12}' &= \frac{(r+\delta+s)M_1' - y_1 + rM_0'}{\beta(M_2' - M_1')} \qquad \text{using } M_1 = \frac{y_1 + (s+\delta)M_0 + p_{12}\beta(M_2 - M_1)}{r+\delta+s} \\
p_{01}' &= \frac{rM_0' - z}{\beta(M_1' - M_0')} \qquad \text{using } rM_0 = z + \beta p_{01}(M_1 - M_0) \\
p_{02}' &= \frac{rM_0' - z}{\beta(M_2' - M_0')} \qquad \text{using } rM_0 = z + \beta p_{02}(M_2 - M_0)
\end{aligned}
$$

Given these values we define the function $d(M_1', M_0')$ as the difference in profits across high type firms so that

$$
\begin{aligned}
d(\Pi) &= \Pi_{12}() - \Pi_{02}() \\
d &= \frac{[(M_2' - M_0')(1-\beta)p_{12}']^2}{2} - \frac{[(M_2' - M_1')(1-\beta)p_{02}'^{\,-\frac{\beta}{1-\beta}}]^2}{2}
\end{aligned}
$$

For given value of M_0', the procedure updates the value of M_1' so that

$$M_1'' = M_1' - \lambda d(\Pi)$$

where $\lambda > 0$ is an adjustment parameter. In other, words we reduce the value M_1'' as long as $d()$ is positive. Given M_1'' and holding fixed M_0' update $M_2''', p_{01}'', p_{02}'', p_{12}''$ using M_2'' and proceed further until

$$d(\Pi) \simeq 0$$

Given M'' expected profits at entry are

$$dE\Pi = \alpha\Pi_{01} + \Pi_{12} - k$$

and update the value of M_0' so that

$$M_0'' = M_0' + \lambda_1 dE\Pi$$

Given M_0'', update the asset values and redo the procedure for finding $d(\Pi) \simeq 0$, and calculating M_0'''. The equilibrium in the first step is obtained for a couple M_1^* and M_0^* so that

$$
\begin{aligned}
d(\Pi) &\simeq 0 \\
dE\Pi(\Pi) &\simeq 0
\end{aligned}
$$

11.2 Step ii): Obtaining the fraction of firms τ that hire directly from the employed

The first step of the model has solved for M_1, rM_0 $M, p_{01}, p_{12}, p_{02}$. The rest of the equations are obtained from

$$
\begin{aligned}
(p_{01})^{\frac{1}{1-\beta}} &= \frac{(1-\alpha)kv_1(0)}{k_{01}n_0} \\
(p_{12})^{\frac{1}{1-\beta}} &= \frac{\tau\alpha kv_2(1)^{1-\beta}}{n_1} \\
(p_{02})^{\frac{1}{1-\beta}} &= \frac{(1-\tau)\alpha kv_2(0)}{(1-k_{01})n_0}
\end{aligned}
$$

and the flows conditions

$$
\begin{aligned}
p_{02}k_{02}n_0 + p_{12}k_{01}n_1 &= (\delta+s)n_2 \\
p_{01}k_{01}n_0 &= (\delta+s+p_{12})n_1 \\
n_0+n_1+n_2 &= 1 \\
k_{01}+k_{02} &= 1
\end{aligned}
$$

Since $\frac{n_1}{k_{01}n_0} = \frac{p_{01}}{\delta+s+p_{12}}$ dividing the equation for $\theta_{01} = (p_{01})^{\frac{1}{1-\beta}}$ and $\theta_{12} = (p_{12})^{\frac{1}{1-\beta}}$ one obtains immediately and expression for τ as

$$
\tau^* = \frac{\theta_{12}}{\theta_{01}} \frac{\alpha v_1(0)}{(1-\alpha)v_2(1)} \frac{p_{01}}{\delta+s+p_{12}}
$$

where $v_i = c(M_i - M_0)_i q(p_i)$ $i = 1,2$. If $\tau^* < 1$ the equilibrium with all submarket is consistent and steps iii can be completed. Conversely, if $\tau^* > 1$ the routine solves for the pure job ladder equilibrium.

11.3 Step iii): Obtaining stocks in the general model

Assume $k = k'$ and $k_{01} = k'_{01}$ and obtain recursively

$$
\begin{aligned}
n_0^{'} &= \frac{\delta+s}{\delta+s+p_{01}k'_{01}+p_{12}(1-k'_{01})} \\
n_1^{'} &= \frac{p_{01}k_{01}n_0}{\delta+s+p_{12}} \\
n_2^{'} &= 1-n_0^{'}-n_1^{'}
\end{aligned}
$$

Given these values obtain the function dk as

$$
dk = (1-k_{01})\theta(p_{02})n_0 - (1-\tau)(1-\alpha)k'v_2(0)
$$

and update

$$k'' = k' + \lambda dk$$

Continue the procedure as long as k'' is such that

$$dk \simeq 0$$

With the completion of step iii the general equilibrium is fully solved.

Given k'' obtain the function dk

$$dk = k - \frac{n_1 \theta_{12}(p_{12})}{\tau * (1-\alpha) * v_2(1)}$$

and update the value of k' so that

$$k'' = k' - \lambda_1 dk$$

Given k'' , update the stocks and redo the procedure for finding $d(k) \simeq 0$, and calculating k'''. The equilibrium in the first step is obtained for a couple k^* and k^* so that

$$\begin{aligned} d(k) &\simeq 0 \\ dE\Pi(k) &\simeq 0 \end{aligned}$$

12 Step iv. Solve for the pure job ladder equilibrium

The step iv is reached only if the routine finds a value of $\tau > 1$ in step ii. The procedure starts from an arbitrary initial guess value of $M_1 = M_1^{'}$ and $rM_0 = rM_0^{'}$. Given the initial guess, it computes recursively $M_2^{'}, p_{01}^{\prime}, p_{02}^{\prime}, p_{12}^{\prime}$

$$\begin{aligned}
M_2^{'} &= \frac{y_2 + (r+\delta)M_0^{'}}{r+\delta+s}; \qquad \text{using } M_2 = \frac{y_2 + (r+\delta)M_0}{r+\delta+s} \\
p_{12}^{\prime} &= \frac{(r+\delta+s)M_1^{'} - y_1 + rM_0^{'}}{\beta(M_2^{\prime} - M_1^{\prime})} \qquad \text{using } M_1 = \frac{y_1 + (s+\delta)M_0 + p_{12}\beta(M_2 - M_1)}{r+\delta+s} \\
p_{01}^{\prime} &= \frac{rM_0^{\prime} - z}{\beta(M_1^{\prime} - M_0^{\prime})} \qquad \text{using } rM_0 = z + \beta p_{01}(M_1 - M_0) \\
n_0^{\prime} &= \frac{\delta + s}{\delta + s + p_{01}^{\prime}} \\
n_1^{\prime} &= \frac{p_{01}^{\prime}(\delta + s)}{(\delta + s + p_{01}^{\prime})(\delta + s + p_{12}^{\prime})} \\
n_2^{\prime} &= 1 - n_1^{\prime} - u_1^{\prime} \\
k^{\prime} &= \frac{n_1^{\prime}\theta_2(p_{12}^{\prime})}{(1-\alpha)v_2(1)}
\end{aligned}$$

13 References

Bartelsman, Eric J. and Mark Doms (2000). Understanding productivity: Lessons from longitudinal

microdata. *Journal of Economic Literature* 38, no. 3: 569: 594

Bartelsman, Eric; Scarpetta, Stefano and Schivardi, Fabiano, (2005) "Comparative Analysis of Firm Demographics and Survival: Micro-level Evidence for the OECD countries " *Industrial and Corporate Change*, Vol. 14, pp. 365-391, 2005.

Baily, Martin Neil, Charles Hulten and David Campbell. (1992) "Productivity Dynamics in Manufacturing Establishments," *Brookings Papers on Economic Activity*: Microeconomics, 1992, 187-249.

Belzil, Christian, (2000). "Job Creation and Job Destruction, Worker Reallocation, and Wages," *Journal of Labor Economics*, University of Chicago Press, vol. 18(2), pages 183-203,

Bertola and Cabalero, (1994); Cross Sectional Efficiency and Labour Hoarding in a Matching Model of Unemployment, *Review of Economic Studies*, Vol 61(3) 435-457

Bertola, G. and Garibaldi, P. (2001), "Wages and the Size of Firms in Dynamic Matching Models." *Review of Economic Dynamics* 4 (2001): 335-368

Davis, Steven J., and John Haltiwanger (1999) "Gross Job Flows," in Ashenfelter and Card.(eds.), *Handbook of Labor Economics*, North-Holland, 1999, 2711-2805.

Delacroix, Alan and Shi, Shouyong (2006), "Directed Search On the Job and the Wage Ladder", *International Economic Review* 2006, vol 47, 651-699

Hopenhyn, Hugo. (1992) "Entry, Exit, and Firm Dynamics in Long Run Equilibrium," *Econometrica*, 60(5), 1127-50.

Hopenhyn, Hugo, and Richard Rogerson. (1993) "Job Turnover and Policy Evaluation: A General Equilibrium Approach," *Journal of Political Economy*, 101(5), 915-38.

Melitz (2003) The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity *Econometrica* 71

Nobuhiro Kiyotaki & Ricardo Lagos, (2006). "A model of job and worker flows," Staff Report 358, *Federal Reserve Bank of Minneapolis*

Foster Lucia; Haltiwanger, John; and Chad Syverson (2007). "Reallocation, Firm Turnover and Efficiency: Selection on Productivity or Profitability?" *University of Maryland.*

Klette TJ and S Kortum (2004), " Innovating Firms and Aggregate Innovation", *Journal of Political Economy*, 112, 986–1018

Lane, Julia; Haltiwanger, John and Spletzer, James (1999) "Productivity Differences Across Employers: The Role of Employer Size, Age, and Human Capital," *American Economic Review*, 94-98.

Restuccia Diego , and Richard Rogerson (2007): Policy distortions and aggregate productivity with heterogenous plants. *NBER Working Paper* 13018

Moen, Espen (1997) "Competitive Search Equilibrium". *Journal of Political Economy* 1997, vol 105, 385- 411

Moen Espen, and Åsa Rosen (2004) "Does poaching distort training?" *Review of Economic Studies*, 2004 vol. 71, 1143-1162.

Lentz, Rasmus, and Mortensen, Dale, (2005) "Productivity Growth and Worker Reallocation" *International Economic Review*, Vol. 46, No. 3, pp. 731-749,

Lentz, Rasmus, and Mortensen, Dale (2006), "An Empirical Model of Growth through Product Innovation". Unpublished paper.

Lentz, Rasmus, and Mortensen, Dale (2007), "Labor Market Friction, Firm Hetrogeneity, and Productivity". Unpublished paper.

Mortensen Dale., and Pissarides, Christopher Albert 1994, Job Creation and Job Destruction in the Theory of Unemployment, *Review of Economic Studies* Vol 61(3) 397-415

Oi, Walter and Idosn, Todd (1999) Firm Size and Wages, in *Handobook of Labor Economics*, Vol 3. O. Ashenfleter and D. Card eds, 2165-2214, Elsevier Science, Amsterdam

Pissarides (1994), Search Unemployment with On-the-job-search, *Review of Economic Studies*, Vol 61(3) 457-494

Shi (2006)

Shimer (2006), On-the-job Search and Strategic Bargaining, *European Economic Review*, 50(4): 811-830.

Yashiv, Eran (2000a) The Determinants of Equilibrium Unemployment, *American Economic Review*, Vol. 90 No. 5, 1297-1322

Yashiv, Eran (2000b) Hiring as Investment Behavior, *Review of Economic Dynamics*, Vol. 3, 486-522.