Heterogeneous Beliefs and Time Variation in the Level of Prices

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Abstract

I look at a standard endowment complete market economy with “keeping up with the Joneses” preferences, where agents hold different beliefs about the conditional mean of the aggregate consumption growth. The heterogeneity in beliefs gives rise to two asset pricing elements: (i) a time-varying discount factor and (ii) additional endogenous risk. Even though the additional risk is not substantial to explain the entire average realized equity premium the time-varying discount factor is potentially able to contribute in explaining the joint evolution of the dividend yield and the risk-free rate. In particular the phenomenal increase in prices during the 1990’s can be explained by a convergence in beliefs which probably occurred due to a decreased consumption risk.

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1 Introduction

In a recent paper Lettau, Ludvigson, and Wachter (2008) propose that the unprecedented increase in asset prices during the last decade of the twentieth century has been due to a structural shift in macroeconomic risk. A decrease in macroeconomic risk causes a decrease in equity premium and if the risk-free rate stays unaltered then the prices might increase substantially. Even though there is evidence\footnote{Jagannathan, McGrattan, and Scherbina (2000), Fama and French (2002) and references therein.} that the equity premium has declined over the past years, the argument proposed presupposes that there is an established link between asset prices and macroeconomic fundamentals.

However, so far the asset pricing literature has not been successful in drawing such a direct link between macroeconomic risk and asset prices. The main problem as realized by Mehra and Prescott (1985), Kocherlakota (1996), Hansen and Jagannathan (1991) and others, is that consumption growth which is identified as the main aggregate asset pricing factor, is very smooth. In effect an asset whose risk depends on the covariance of its dividend process with the aggregate consumption should not be required to offer a high risk premium to be held by rational investors and its returns should not be very volatile. This is in obvious contradiction with the observed behavior of the markets. The real risk-free rate is much less volatile than the dividend yield of the market index and the average return on the market in excess of the risk-free rate has been well above the predicted value of any rational asset pricing model.

In this paper we provide an alternative explanation, based on heterogeneity of beliefs, about not only the run up in prices but also to a significant extent the joint evolution of the risk-free rate and the market dividend yield without the need to systematically explain the high equity premium in the sense of a high expected excess return. Fama and French (2002) in fact reach the conclusion “... that the average stock return of the last half-century is a lot higher than expected”. Similar evidence has been found by Soderlind (2008). Therefore, we pursue the explanation that the US market was a lucky market and that the factor responsible for this realization was belief heterogeneity. In particular, we argue that the stock market returns were high and the phenomenal increase in the price dividend ratio especially during the 1990’s was due to a significant convergence in beliefs.

We first show theoretically that heterogeneity in beliefs causes asset prices to be discounted. Similarly Jouini and Napp (2006a, 2006b and 2007) show in a more general setting that when heterogeneous beliefs are aggregated an additional time-varying discount factor arises. A brief economic reasoning goes as follows: When heterogeneity increases agents’ investments diverge resulting in higher individual consumption volatilities. If now agents are sufficiently
risk-averse\(^2\), the higher individual consumption risk makes agents to require a higher interest-rate and this decreases stock prices. The interest rate is also affected by the increased volatility through higher precautionary savings but not enough to cause an overall decrease in the interest rates. The closed form expression derived later looks at the relative importance between the two. The increased volatility in individual consumption also affects the equity premium. The main finding however is that the effect on equity premium, even though significant, is not enough to predict an excess return of more than 2%. The way the equity premium is affected in this economy is different than in the model of David (2008) where agents are assumed to be less-risk averse than a myopic agent.

The main contribution of this paper is to quantify this discounting effect and show that it could have possibly been the driver behind the history of the price-dividend ratio. The additional assumption required is that agents believe that this heterogeneity is persistent which means that the effect on long-lived securities is much greater than the risk-free rate. But despite this persistence, it is possible that during the 1990’s the expectations about the economy converged significantly. As people were gaining confidence that their beliefs were becoming more aligned with time they started bidding up the prices up to the point where this convergence of beliefs stopped. This convergence in beliefs might have been due to a downward shift in macroeconomic risk which then explains the high correlation between consumption risk and the price-dividend ratio as shown by Lettau et al. (2008). We provide similar evidence that the price-earnings ratio is strongly negatively correlated with the conditional volatility of aggregate consumption growth. In order to explain this we assume that the level of heterogeneity is an increasing function of this conditional volatility. For the increase in prices this assumptions is not needed. Through this assumption, however, we connect the run-up in prices with the decrease in macroeconomic risk.

Many theoretical studies tried to explain the high equity premium and most of those have concentrated on individual preferences. One of the most successful theories so far, advocated by Campbell and Cochrane (1999), conjectures that the element responsible for the observed asset pricing behavior is a mean-reverting counter-cyclical price of risk.\(^3\) Agents become more risk averse during recessions and therefore require a higher reward for bearing the aggregate risk. The mean-reversion and the counter-cyclicality are required to explain certain empirical patterns, specifically that excess returns look counter-cyclical and long-run excess returns appear to be predicted by the price dividend ratio. The problem with this theory is that such a varying price of risk has not yet been identified. For example Chan and Kogan (2002) show that a counter-cyclical risk-aversion arises in an economy with risk-aversion heterogeneity but

\(^{2}\)In this paper we consider power utility and sufficiently risk-averse in this case means more risk-averse than logarithmic utility as also shown by Varian (1985).

\(^{3}\)Another model with similar asset pricing mechanics is that of Constantinides and Duffie (1996) in which the role of the counter cyclical risk is played by the cross-sectional volatility of permanent uninsurable income-shocks.
Xiouros and Zapatero (2007) show that the level and the variation produced using a realistic set of parameters is by an order of magnitude less than the variability required by Campbell and Cochrane (1999).

The variation in the level of prices due to variations in belief heterogeneity also touches upon the issue of predictability of excess returns which is lately being questioned as for example by Ang and Bekaert (2007) or Boudoukh, Richardson, and Whitelaw (2008). The argument laid out in this paper offers a likely explanation for why the predictability sought for example by Cochrane (2008) has not been established. The model presented in this paper shows that the level of heterogeneity of beliefs in the economy can be a major factor in moving the dividend yield. The dividend yield decomposition of Campbell and Shiller (1988) states that the variation of the price dividend ratio can come either from variation in expectations of future returns or future dividends. Since the dividend yield does not appear to predict dividend growth the weight naturally falls on the expected returns. However, if the price dividend ratio increases due to a perceived nearly permanent shift in belief heterogeneity this increase happens because future prices increased without significant change either in the expected excess return or expected future dividends. In effect it would appear as if the structural relationship between prices and dividends changes. Further, this relation was not possible to be found if despite the general expectations about belief heterogeneity, beliefs continued to converge.

The idea that asset prices decrease with the level of belief heterogeneity in the economy is opposite to the prevailing opinion in the asset pricing literature. Miller (1977) with a static setting in mind argues that when there is divergence of opinions and short selling is not allowed, the price of an asset is likely to reflect the most optimistic agents. Harrison and Kreps (1978) go a step further and show that in a dynamic setting with risk-neutral agents the price in equilibrium will include a speculative component. The rationale behind it is that agents may be willing to pay more than their shadow valuation because there is a positive probability that in the future they will be able to sell the security to agents with higher shadow valuation than their own. Scheinkman and Xiong (2003, 2006) build on the same idea that with short-selling constraints and limited supply of shares the assets have an embedded option of reselling it at a higher future price. The value of this option then increases with belief heterogeneity. This result however is due to the risk-neutrality assumption and it holds as long as agents are less risk-averse than a myopic agent. However, it is believed that economic agents are more risk-averse than a logarithmic utility agent and further there is evidence by Doukas, Kim, and Pantzalis (2006) that stocks with higher dispersion in beliefs are priced at a discount. We also offer evidence here that the heterogeneity in professional macroeconomic forecasters predictions

4We use the term shadow value for an agent to mean the equilibrium price in an economy where the agent was the only agent in the economy.

5Diether, Malloy, and Scherbina (2002) provide the opposite evidence that the expected stock returns are higher for stocks with high dispersion of analysts beliefs.
has decreased over time which is consistent with the predictions of our model. Gallmeyer and Hollifield (2008) find that even with short-selling constraints the effect of belief dispersion on asset prices depends on the intertemporal elasticity of substitution.

The asset pricing literature on belief heterogeneity is quite rich. This is simply because it is quite evident either directly or indirectly through differences in investment strategies and trading that with respect to the conditional probability distribution of prices different investors hold different beliefs and this heterogeneity does not vanish. However it is still unclear as to the extend in which heterogeneity of beliefs matter for prices and in what way. In complete market models Detemple and Murthy (1994), Zapatero (1998) and Basak (2000, 2005) show that divergence of opinions puts downward pressure on the risk-free rate which in turn implies a higher risk premium. The possible effect identified is the higher volatility in individual consumption growth processes. Jouini and Napp (2006a) resort to pessimism and doubt in order to have an impact on the equity premium. In my analysis I find that the additional volatility does not have a significant impact on the equity premium. David (2008) in a similar frictionless economy is able to generate a significant effect on the equity premium by assuming a low risk-aversion which results to aggressive investment and significant volatility increase. Other studies that look at asset pricing implications of heterogeneous beliefs include Abel (1990b), Wang (1993), Basak (2005) and Buraschi and Jiltsov (2006).

The paper is organized as follows. Section 2 presents a general structure of a complete market economy with heterogeneous beliefs. In section 3 we derive the equilibrium conditions and in section 4 we look at the pricing implications at a theoretical level. We close the economy with specific assumptions about beliefs in section 5. In section 6 we calibrate the economy and look at the quantitative impact of belief heterogeneity. Section 7 concludes.

2 A General Economy with Heterogeneous Beliefs

Modeling belief heterogeneity has always been hard. First we do not really understand the origin of this kind of heterogeneity and second we have not found a way to model it without incorporating some form of irrational behavior so that belief heterogeneity does not vanish.

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6One way to approach the heterogeneity of beliefs is to take the different demand functions of different groups of investors as given or alternatively consider myopic agents as in Ciarella, Dieci, and Gardini (2006) or Brock and Hommes (1998)


8Pessimism refers to the underestimation of the good states of nature and doubt in overestimating the volatility.

9Kurz and Motolesi (2001) show that endogenous uncertainty arising from belief heterogeneity in conjunction with correlated beliefs between optimists and pessimists is able to reconcile a high equity premium with a low risk-free rate while explaining also the high stock price volatility.
through learning. If agents share the same information about the economy and its fundamental structure that drives the asset prices, then rational agents will obviously have their opinions converge since they will soon learn whatever is possible to learn about the underlying structure and state of the economy and their priors will vanish. One possible explanation of the fact that beliefs do not seem to converge is that there is a significant mass of irrational, or boundedly rational agents that manage to survive in the economy and hence do not have the incentive to learn. A second explanation has to do with lack of information. If the structure of the underlying risk in the economy evolves faster than the release of relevant information then it is possible that the beliefs of agents can be both rational and still divergent as shown by Kurz (1994a) and Kurz (1994a). However, this only gives a mathematical justification to belief heterogeneity.

In my model the heterogeneity of beliefs is assumed to be a function of the uncertainty in the economy and in turn the uncertainty is assumed to be a function of the level of macroeconomic risk. Therefore, instead of modeling directly the beliefs of the agents, I try to look at the pricing implications given that there is divergence of opinions in the market. However, this setup proxies for heterogeneity in the interpretation of new information in the economy as for example in Harris and Raviv (1993) and Kandel and Pearson (1995). Every period there are changes to the fundamentals of the economy but available information are not fully revealing and therefore agents may interpret them differently. Therefore the assumption that I will make later is that the more uncertainty there is the more disagreement there is among agents about the interpretation of new information. For now however, I start with a general economy where beliefs are heterogeneous. I first conduct a qualitative analysis of the pricing effects of belief heterogeneity and then I derive the equilibrium conditions that will be used to close the economy with further assumptions.

2.1 Uncertainty, Agents and Beliefs

I consider a probability space \((Ω, \mathcal{F}, P)\) in which the uncertainty of the economy lies. Any given element of the set \(Ω\) describes an entire history for the economy. Consider also the set of natural numbers \(\mathbb{N} = \{0, 1, \ldots\}\).

**Assumption 1.** Time is discrete and infinite and is denoted by \(t \in \mathbb{N}\). The aggregate uncertainty is driven by the vector \(s = (g, \epsilon) \in S\) of independent random processes. \(g\) denotes the natural logarithm of the growth process of the aggregate endowment \(Y (y = \log(Y))\) of a single perishable good. \(\epsilon\) is a random variable that affects the beliefs of the agents in the economy. The

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10In private information economies prices are fully revealing as shown by Milgrom and Stokey (1982) unless I introduce some form of incompleteness as in Shalen (1993) and Wang (1993).

11Kogan, Ross, Wang, and Westerfield (2007) provide a model with only final period consumption where irrational agents have a probability of survival.
history of \( s \) which is denoted by \( s^t = \{s_1, \ldots, s_t\} \) is observable and the associated information is denoted by the filtration \( \{\mathcal{F}_t, t \in \mathbb{N}\} \).

All agents in the economy have the public information and none of them have any private information. The true probability distribution \( \mathbb{P} \) is not known and therefore agents need to form their own beliefs about the generating process of the economic variables. The additional uncertainty \( \epsilon \) is only used for completeness to justify the divergence of opinions and does not affect the results. I think of this innovation as information other than the history of aggregate endowments that potentially affects the future distribution of aggregate endowment growth. For example it could contain information about macroeconomic indices, technological shocks, changes in monetary and fiscal policies, international monetary and political crises, oil shocks etc. Hence, as I stated already I assume that the belief heterogeneity in the economy is sustained due to the difference in interpretation of publicly available information.

I model beliefs in a quite general way. For this I need to define the set \( \mathcal{M}_1^+ \) of strictly positive martingales defined on the probability space, adapted to the filtration \( \{\mathcal{F}_t, t \in \mathbb{N}\} \) and with initial value one. The following corollary allows us to define beliefs.

**Corollary 1.** Given a martingale \( \xi \in \mathcal{M}_1^+ \) there exist a probability measure \( \mathbb{Q} \) equivalent to \( \mathbb{P} \) such that for all \( 0 \leq s < t < \infty \),

\[
\frac{\xi_t}{\xi_s} = \frac{d\mathbb{Q}(\cdot|\mathcal{F}_s)}{d\mathbb{P}(\cdot|\mathcal{F}_s)} \bigg|_{\mathcal{F}_s}
\]

It is clear from the above corollary that beliefs can be represented by a martingale process of the set \( \mathcal{M}_1^+ \). The ratio \( \xi_{t+k}/\xi_t \) would therefore represent certain conditional beliefs held at some time \( t \) about the path of the economy over the next \( k \) periods. Learning would then be represented by a process \( \xi \) of which the variation of this ratio decreases with time \( t \) irrespective of the horizon \( k \) and vanishes eventually. Convergence on the other hand between two different beliefs would mean that these ratios become with time more “correlated” with each other, until they become identical. Hence, \( \epsilon \) is a process that will be used to prevent this convergence.

This is a non-standard way to define beliefs since in order to reconstruct a set of beliefs I need to know the process \( \xi \) under the objective probability measure \( \mathbb{P} \) which I have assumed that I do not know. The first reason is technical and it has to do with ensuring that the probability measures considered are equivalent to the true probability measure as well as equivalent to each other. In equilibrium this guarantees that for all finite times there is zero probability of an agent loosing all his or her wealth. The second reason is that such a definition renders the analysis much easier since I isolate the effect of heterogeneity. Two different martingale processes indicate the relation between two sets of beliefs which is what I need to analyze the
effect of heterogeneity without worrying about the actual beliefs. Finally, the whole analysis
is correct under any equivalent probability measure that I decide to use.

A belief process $\xi$ being adapted to the filtration $\mathcal{F}_t$ implies that these beliefs are held both
with respect to the aggregate endowment growth in the economy as well as the process $\epsilon$. This
not only allows for heterogeneity in beliefs about the growth of the economy but also in beliefs
about the level of belief heterogeneity in the future. Even though this is a realistic assumption
later on I restrict to cases where agents agree on $\epsilon$.

**Assumption 2.** *There is a continuum of agents of mass one that is represented by the set
$I = [0, 1]$. Belief heterogeneity is described by a function,\
$$\xi : I \rightarrow \mathcal{M}_1^+$$
from the set of agents to the set of beliefs.*

In the case where all agents have homogeneous beliefs then $\xi(i)$ is the same process for all
$i \in I$ and in the case where everyone has the correct beliefs then $\xi(i)$ is the constant process
for all agents. Given therefore a process $\xi^i$ the beliefs of agent $i$ are given by $d\mathbb{P}^i = \xi^i d\mathbb{P}$ for
all possible future periods.\(^{12}\)

### 2.2 Complete Financial Markets

The financial structure of the economy is not explicitly modeled because first I assume that
markets are dynamically complete and second I am only interested to look at the prices of
certain financial securities while I am not interested in portfolio holdings. In particular I am
interested in the joint evolution of the price of the market security and the risk-free rate.

**Assumption 3.** *There exists a continuum of both short-lived and infinitely lived assets that
span the local uncertainty of the economy. The aggregate endowment is paid as dividends by
the market security.*

The above assumption about the aggregate dividend growth process of the market was made for
simplicity and in order to be able to compare my results with other studies that make the same
assumption. Any other assumption can be made as long as no additional aggregate uncertainty
is introduced. For example it is straight-forward to modify the model when I assume that the
aggregate dividend growth is linear in the aggregate endowment growth plus an additional
independent shock. Such an assumption would potentially decrease the correlation of the
market returns with consumption growth. This is however not needed in this model as the
variation in the heterogeneity of beliefs that is independent of consumption growth will decrease
this correlation.

\(^{12}\)From the rest of the paper I will use the notation $\xi^i$ to denote $\xi(i)$. 

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Market completeness and equilibrium no arbitrage prices imply that given the objective probability measure $P$ there exists a unique $\mathcal{F}_t$-adapted pricing kernel process $(p_t, t \geq 0)$, given a $p_0$, that prices all assets. $p_t$ is collinear with the marginal utility of consumption at time $t$ of a hypothetical agent with the correct beliefs. Hence, for the market portfolio that pays the aggregate dividend, the ex-dividend equilibrium price is given by,

$$P_t = \mathbb{E}_t \left[ \sum_{k \in \mathbb{N}} \delta^k p_{t+k} Y_{t+k} \right]$$

(1)

where $\mathbb{E}_t[\cdot]$ denotes the expectation operator under the objective probability measure, conditional on the information set $\mathcal{F}_t$. $\delta$ is a preference parameter and it denotes the common subjective discount factor.\(^{13}\)

Market completeness and equilibrium also imply that for any equivalent probability measure $P^i$, believed by an agent $i$, there exists a unique (up to a factor) pricing kernel process $(p^i_t, t \geq 0)$ by which the agent prices the assets. This is the same as saying that in equilibrium all agents agree on the prices of the Arrow-Debreu securities at time 0. An agent’s price of an Arrow-Debreu security that pays one unit of consumption at some future date $s^i$ is given by $p(s^i) dP(s^i).^{14}$ Therefore in equilibrium it has to be that,

$$p^i_t = p_t \left( \xi^i_t \right)^{-1}, \quad \forall (i, t) \in \mathcal{I} \times \mathbb{N}.$$ 

(2)

Since $p^i_t$ is (up to a factor) the marginal utility of consumption at time $t$ of agent $i$ in equilibrium, relation (2) implies that the marginal rates of substitution between any two states are different for two agents that have different beliefs. One might think that this is a result of some kind of incompleteness in the market due to the heterogeneity in beliefs. However, in an incomplete market setting agents do not agree on all Arrow-Debreu security prices while here agents do agree on all prices as it is assumed from (2).

Even though the marginal rates of substitution vary cross-sectionally, the allocation is Pareto optimal when considering the subjective probability measures of the agents. This implies that the social planner considers that fact the agents have heterogeneous beliefs. However, the allocation is not Pareto optimal under the objective probability measure but this is only due to the assumption that there is lack of information in the market and the available information is not used similarly by all agents. Lack of information refers to the assumption that agents do

\footnote{In order for prices to have their fundamental values I assume the transversality condition \[ \lim_{t \to \infty} \mathbb{E} [\delta^{i} p_{t} P_{t}] = 0 \] holds.} \footnote{Note that any process that is adapted to the filtration $\mathcal{F}_t$ can be written as a function of the history $s^i$.}
not know the exact process that drives the aggregate risk. Hence, the allocation would cease to be Pareto optimal in case complete information was revealed to the economy but only due to the fact that the initial incompleteness of information was assumed to give rise to differences in beliefs. Otherwise, if agents held the same beliefs, even in the case of incomplete information, the allocation would still be Pareto optimal under complete information and given some wealth reallocation. The wealth reallocation would be needed because under different probability measures asset prices are different. Of course if agents were allocated a fixed proportion of the aggregate endowment then there would not be any need for wealth reallocation.

2.3 Preferences and Endowments

The preferences assumed are standard time and state separable preferences with external habit similar to that of Abel (1990a).\(^\text{15}\)

**Assumption 4.** Agents are utility maximizers with time and state separable preferences over consumption processes as follows:

\[
U_t(c, X) = \mathbb{E}_0^t \left[ \sum_{t \in \mathbb{N}} \delta^t u(c_t, X_t) \right],
\]

where

\[
u(c, X) = \frac{c^{1-\gamma}}{1-\gamma} X^{\gamma - \eta}.
\]

and \(\eta \leq \gamma\) and \(\gamma > 0\). \(X\) denotes the common external habit. The log of the external habit is formed according to

\[
x_{t+1} = \lambda x_t + (1 - \lambda) y_t, \quad t \geq 0,
\]

where \(\lambda_x \in (0,1)\).

The habit process imposes an externality on all agents of the economy because it affects their marginal utilities. The marginal utility of consumption is given by,

\[
u_c(c, X) = c^{-\gamma} X^{\gamma - \eta}.
\]

Essentially agent preferences are over their consumption surpluses over the common habit. Simply for interpretation purposes I assumed that \(\eta < \gamma\) and therefore the derivative of the

\(^{15}\)Habit formation preferences have been extensively explored in the literature in various forms. Significant contributions include Ryder and Heal (1973), Sundaresan (1989), Constantinides (1990), Detemple and Zapatero (1991), (Gali, 1994) and Hindy, Huang, and Zhu (1997).
marginal utility with respect to the level of the habit,

$$\frac{\partial u_c(c, X)}{\partial X} = (\gamma - \eta)c^{-\gamma}X^{\gamma-\eta-1},$$

is positive. Therefore a higher level of habit for a given consumption level induces agents to consume more and invest less.

The preference assumption of external habit was made for several reasons. First external habit produces potentially enough time-variability in discount rates and hence potentially enough variability in price-dividend ratios in the absence of belief heterogeneity. When heterogeneous beliefs are introduced the variability of the price-dividend ratio can match the variability found in the data. In a homogeneous beliefs setting the observed persistence in the price-dividend ratio can easily be generated in a model with external habit as shown for example in Campbell and Cochrane (1999).

Secondly, in an economy where the expectations about the future vary, CRRA preferences without habit produce a counter intuitive prediction, namely that more optimistic expectations are associated with lower prices. This is an obvious outcome of the fact that in power utility preferences the intertemporal elasticity of substitution is determined by the risk aversion parameter which is normally higher than one. The higher is the risk-aversion the lower is the elasticity of intertemporal substitution since agents have a higher propensity to smooth consumption inter-temporally. So when agents are optimistic about the future, in an attempt to smooth consumption inter-temporally they want to borrow more and this increases interest rates and decreases prices. However, with the inclusion of a persistent habit this relation can reverse if the habit parameter $\gamma - \eta$ is sufficiently high and the habit does not follow a close to unit root process.

In order to see the mechanism behind such an effect I need to examine what happens to the current interest rate when habit is introduced. A high consumption growth today increases consumption over habit today as well as tomorrow because habit is not fast in adjusting to increased consumption. In this argument we can think of tomorrow’s consumption growth as fixed. An increase in consumption over habit decreases the price of consumption today. However, the decrease in today’s price of consumption is bigger than tomorrow’s decrease because habit does increase by $1 - \lambda_x$. Therefore $(1 - \lambda_x)$ gives the wedge by which today’s price of consumption increases in relation to tomorrow’s increase. This effect decreases interest rates and the decrease is higher the smaller $\lambda_x$ is. Since, now an increase in consumption growth today decreases the interest rate through habit then a higher expected consumption growth will decrease the expectation of tomorrow’s interest rate. Now, if habit is sufficiently persistent then optimism about consumption growth will decrease the expectation of not only
tomorrow’s interest rate but also the expectation of many interest rates in the future. This causes current prices of long-lived securities like stock prices to decrease.

**Assumption 5.** Each agent is endowed with initial holdings of the dividend generating financial assets. This translates to proportion of the aggregate endowment initially allocated to each agent that is represented by

\[ \theta_0 : \mathcal{I} \to [0, 1], \]

such that,

\[ \int_{\mathcal{I}} \theta_0(di) = 1. \]

For a given pricing kernel process \( p^i_t \) believed by an agent \( i \) and given market completeness, an agent’s sequence of budget constraints collapses to a single inter-temporal budget constraint,

\[ E^i \left[ \sum_{t \geq 0} \delta^t c^i_t p^i_t \right] \leq E^i \left[ \sum_{t \geq 0} \delta^t \theta_0(di) Y_t p^i_t \right]. \]

Using (2) I can rewrite the budget constraint as,

\[ E \left[ \sum_{t \geq 0} \delta^t \left( c^i_t - \theta_0(di) Y_t \right) p_t \xi^i_t \right] \leq 0 \quad (5) \]

after switching to the objective probability measure.

### 3 Complete Market Equilibrium with Heterogeneous Beliefs

Given the assumptions I can derive the optimal consumption process for each agent and once I define the equilibrium concept I can derive the equilibrium conditions. By equilibrium conditions I mean the endogenous aggregate state vector of the economy and its law of motion that are needed in order to determine prices of financial securities.

Since the economy grows, the analysis is simplified if I express the relevant variables in terms of the aggregate endowment. I define therefore \( \alpha_t(di) := c^i_t/Y_t \) to denote the consumption proportion and \( \omega_t := \log(Y_t/X_t) \) to be the log of the endowment habit ratio, henceforth referred to as *endowment surplus*. \( \omega \) is a state variable and it follows the process,

\[ \omega_{t+1} = \lambda_\omega \omega_t + g_{t+1}. \quad (6) \]

Below I derive the optimal consumption proportion process for each agent. For ease of notation and since the pricing kernel process is unique up to a scaling factor, I fix \( p_0 = \exp(-\eta x_0 - \gamma \omega_0) \).
Lemma 1. Given the price process \( p_t \), an agent \( i \) that maximizes (3) subject to the budget constraint (5) has an optimal consumption process that is adapted to \( \mathcal{F}_t \) and satisfies the following first order condition,

\[
\alpha_t(di) = \alpha_0(di) \left( \frac{p_t}{\xi_t^i} \right)^{-1/\gamma} \exp \left( -\frac{\eta p_t}{\gamma} - \omega_t \right).
\]  

(7)

\( \alpha_0(di) \) is the initial consumption proportion and is optimally chosen according to

\[
\alpha_0(di) = \frac{\mathbb{E}_0 \left[ \sum_{t \in \mathbb{N}} \delta_t p_t Y_t(\xi_t^i)^{-1} \theta_0(di) \right]}{\mathbb{E}_0 \left[ \sum_{t \in \mathbb{N}} \delta_t p_t Y_t(\xi_t^i)^{-1} (p_t/\xi_t^i)^{-1/\gamma} \exp \left( -\frac{\eta p_t}{\gamma} - \omega_t \right) \right]}
\]  

(8)

Let us now define the concept of equilibrium which is standard.

Definition 1. An equilibrium is a set of \( \mathcal{F}_t \)-adapted processes for the pricing kernel \( p_t \), \( t \in \mathbb{N} \) and the consumption allocations \( \alpha \equiv \{ (\alpha_t(di), t \in \mathbb{N}) \}_i \) such that:

(i) given the pricing kernel process \( p_t \), for each agent \( i \in \mathcal{I} \), \( \alpha(di) \) maximizes his utility (3) given his budget set (5), and

(ii) in every period the consumption good market clears,

\[
\int_{i \in \mathcal{I}} \alpha_t(di) = 1, \quad \forall t \in \mathbb{N}
\]  

(9)

With this concept of equilibrium I am assuming that all agents have structural knowledge in the sense that they can associate observed prices with the exogenous state of the economy and agents only differ in their beliefs about the probabilities of the draws of nature. This concept is different than the concept of Rational Belief Equilibria of Kurz (1994a) and Kurz (1994b) by which agents do not have structural knowledge and their beliefs are different with respect to the conditional distribution of prices.

Note from the market clearing condition of equilibrium that in any period \( t \) the consumption proportion function \( \alpha_t \) is a probability distribution over the set of agents \( \mathcal{I} \). For the rest of the study I will denote any integral with respect to the consumption distribution across agents in some period \( t \) with \( \mathcal{E}_t \).\(^{16}\) Since the beliefs \( \xi_t \) is a function over the set of agents then \( \alpha_t \) is a distribution over the beliefs in some period \( t \). Further, since both \( \xi_t \) and \( \alpha_t \) are \( \mathcal{F}_t \)-adapted it means that,

(i) \( \alpha_t \) and the primitive function \( \xi_t \), that describe the distribution across beliefs in period \( t \) and how these beliefs will evolve, along with

\(^{16}\)For example the market clearing condition in some period \( t \) can be trivially written as \( \mathcal{E}_t 1 = 1 \).
(ii) $\omega_t$ the surplus ratio and its law of motion,

carry all the relevant information about the economy. In fact the surplus ratio and the consumption distribution will be the economy’s state vector as will be made clear once I characterize the equilibrium.

Before I characterize the equilibrium I will introduce some variables that relate to the aggregation of beliefs. Aggregation results regarding beliefs date back to Rubinstein (1975), Verrecchia (1979) and Varian (1985). More recent results are those of Calvet, Grandmont, and Lemaire (2004), Jouini and Napp (2006a) and Jouini and Napp (2007). This aggregation will provide us with further insight about the heterogeneity of beliefs and will help us analyze the price effects. I will also show that the economy is equivalent to a representative agent economy with a certain beliefs process and a modified habit process.

**Definition 2.** Let $\alpha_0$ be the equilibrium consumption allocation in the initial period. The **aggregated beliefs** process is defined as

$$\tilde{\xi}_t := \left(\mathcal{E}_0 \left(\xi_i^t\right)^{1/\gamma}\right)^\gamma.$$  

The **belief compensator** is defined as

$$b_t := -\log\mathbb{E}_t \left[\frac{\tilde{\xi}_{t+1}}{\tilde{\xi}_t}\right],$$

where $b_0 = 0$ and the **consensus beliefs** process is defined sequentially by

$$\xi_{t+1}^r := e^{b_t} \frac{\tilde{\xi}_{t+1}}{\tilde{\xi}_t} \xi_t^r,$$

where $\xi_0^r = 1$.

Let us now clarify what these processes are and in order to do this I use the next corollary. For convenience I introduce the notation $\xi_{t,t+1} := \xi_{t+1}/\xi_t$, to represent the one period conditional beliefs in period $t$. I remind that for an agent $i$ and a state $s$, $\xi_{t,t+1}(s)$ is the ratio of his conditional probability about the state $s$ over the true probability.

**Corollary 2.** In equilibrium the **aggregated beliefs** process satisfies the following relation,

$$\tilde{\xi}_{t,t+1} = \left(\mathcal{E}_t \left(\xi_{t,t+1}^i\right)^{1/\gamma}\right)^\gamma.$$
the aggregated beliefs is not a martingale process because due to Lyapunov’s inequality, when
\( \gamma > 1 \)

\[
\tilde{\xi}_{t+1} \leq \mathcal{E}_t(\xi_{t+1})
\]

and therefore since \( \mathbb{E}_t(\xi_{t+1}) = 1 \) for all agents, \( \mathbb{E}_t(\tilde{\xi}_{t+1}) \leq 1 \). Using the belief compensator process \( e^b \), I re-scale the aggregated beliefs to construct the consensus beliefs. As I will see later on, the belief compensator will have the greatest impact on asset prices and will be my major additional pricing element. I postpone the discussion about its economic meaning for later on until I look at the equilibrium prices. For now I only note that in the absence of belief heterogeneity or in the case where \( \gamma = 1 \), \( b_t \) is identically zero and therefore the aggregated beliefs are also the consensus beliefs. Further I note that \( b_t \) varies across time when the level of heterogeneity varies. So far I have been using the term “level of heterogeneity” loosely. For any given future state \( s \) the variability of \( \xi_{t+1}(s) \) across agents (using the consumption distribution \( \alpha_t \) for weights) is the level of heterogeneity for that state. The higher the variability the lower is \( \tilde{\xi}_{t+1}(s) \) when \( \gamma \) is greater than one. Therefore for a given \( \gamma > 1 \), \( b_t \) is a transformation of the average of \( \tilde{\xi} \) over all future states and hence itself can be used as a measure of the level of heterogeneity. Finally, the consensus beliefs \( \xi^r \) is obtained once I divide the aggregated beliefs by their mean \( e^b \). Since \( e^{-b} \) is related to the average heterogeneity across states then \( \xi_{t+1}(s) \) is the relation of the level of heterogeneity in state \( s \) to the average heterogeneity. A value higher than one means that for the given state \( s \) agents agree more than they agree on average. \( \xi_{t+1}(s) \) also increases when agents believe that the probability of the particular state is higher than the true.

Now I can characterize the equilibrium. For existence of time-homogeneous Markov equilibria in settings of heterogeneous agents one can refer to Duffie, Geanakoplos, Mas-Colell, and McLennan (1994).

**Corollary 3.** Let \( \alpha_0 \) be the equilibrium consumption allocation in the initial period and let \( p_0 = \exp(-\eta x_0 - \gamma \omega_0) \). Then in equilibrium the pricing kernel is sequentially determined by

\[
p_{t+1} = p_t \tilde{\xi}_{t+1} \exp \left[ -\eta(1 - \lambda_t)\omega_t - \gamma(\omega_{t+1} - \omega_t) \right],
\]

and the equilibrium consumption allocations are described by,

\[
\alpha_{t+1}(d) = \alpha_t(d) \left( \frac{\xi_{t+1}}{\tilde{\xi}_{t+1}} \right)^{\frac{1}{\gamma}}, \quad \forall i \in I.
\]

The equilibrium consumption process (15) is quite intuitive. An agent increases her consumption in states that she believes to be more probable when compared to the aggregated beliefs. The optimal consumption relation also indicates that the individuals’ consumption processes
are more volatile when the beliefs are more disperse, while they are less volatile when the risk-aversion is higher. When agents are more risk-averse they are more cautious to invest according to their beliefs because this implies higher variability across their consumption plan. David (2008) exploits the impact of the risk-aversion on the volatility of consumption and hence on the marginal utilities, by choosing a risk-aversion between 0 and 1. Such a low risk-aversion parameter generates substantial variability in the pricing kernel but not necessarily a higher equity premium as I will show later.

The law of motion of consumption is independent of the consumption surplus and this is simply due to the fact that the habit is common to everyone and the preference parameters are the same. The consumption surplus however, does affect prices as I observe from (14) and it depends clearly on \( \frac{\gamma}{\eta} \).\(^{17}\) The rate of change in the pricing kernel (which is the price of the Arrow-Debreu security that pays a unit of consumption next period at the given state divided by the probability of the state) depends on the consumption surplus, the consumption distribution of beliefs and the exogenous state. Therefore for a given economy the equilibrium state vector is \( z := (\omega, \alpha) \). Further note that the equilibrium is path dependent because beliefs are path dependent. The law of motion for the economy’s state vector is denoted by,

\[
zs_{t+1} = L(z_t, s_{t+1}).
\]  

(16)

I remind that the conditional beliefs across all agents \( \xi_{t,t+1}^i \) are a function of the exogenous state \( s^t \) by assumption. This means that if I know the path of the economy up to and including time \( t \) I know the conditional beliefs about next period of every agent. The equilibrium is then described by the initial consumption allocation and the law of motion \( L \). The law of motion is given by the primitive belief function \( \xi \), the exogenous process (6) and the endogenous process (15). The initial consumption allocation pins down the equilibrium.

**Proposition 1.** Let \( \mathcal{D}(\mathcal{I}) \) be the set of distribution functions over the set \( \mathcal{I} \). Let also \( p_t = p(a, s^t) \), \( Y_t = Y(Y_0, s^t) \), \( x_t = x(x_0, s^t) \) and \( \xi_t^i = \xi(i, s^t) \). Define the functional \( F: \mathcal{D}(\mathcal{I}) \to \mathbb{R}^{\mathcal{I}} \), where \(| \cdot |\) denotes the cardinality of a set, by

\[
F(a, j) = \int_0^j \frac{E_0 \left[ \sum_{t \in \mathbb{N}} \delta^t p_t Y_t(\xi_t^i)^{\gamma} \right]}{E_0 \left[ \sum_{t \in \mathbb{N}} \delta^t p_t Y_t(\xi_t^i)^{\gamma} (p_t/\xi_t^i)^{1/\gamma} \exp \left( -\frac{\eta x_t}{\gamma} - \omega_t \right) \right]} \theta_0(di), \quad \forall i \in \mathcal{I}.
\]

Then the equilibrium is determined by the solution to the following functional equation,

\[
F(a) = a,
\]

\(^{17}\)The term \(-\eta(1 - \lambda_s)\omega_t - \gamma(\omega_{t+1} - \omega_t)\) can be re-written as \(-\gamma g_{t+1} + (1 - \lambda_s)(\gamma - \eta)\omega_t\).
4 Equilibrium Asset Prices

The heart of every asset pricing model is its stochastic discount factor (SDF) since with it we can price all assets in the economy. The price of an asset $j$ that pays a (risky) stream of future cash flows $D_j$ is given by,

$$P_{j,t} = E[M_{t,t+1}(P_{j,t+1} + D_{j,t})|\mathcal{F}_t]$$

(17)

where the equilibrium SDF is given by,

$$M_{t,t+1} := \frac{\delta^{p_{t+1}}}{p_t}.$$  

(18)

The price of a risk-free bond that pays a unit of consumption next period is denoted by $Q_t$ and $P_t$ gives the price of the market security. Once I have derived the equilibrium pricing kernel it is straightforward to express the SDF under the objective probability measure. I have already noted that in equilibrium it is given by some function $M(z_t, s_{t+1})$ of the endogenous state vector and the exogenous shocks. From equation (14) and the definition of the consensus beliefs the equilibrium SDF is given by

$$M_{t,t+1} = \delta \xi_{t,t+1}^r \exp\left[-\gamma g_{t+1} + (1 - \lambda x)(\gamma - \eta)\omega_t - b_t\right].$$

(19)

where I have used that $x_{t+1} - x_t = (1 - \lambda x)\omega_t$. The expression of the stochastic discount factor simplifies to its usual form when all agents in the economy have the correct beliefs since both the consensus belief process as well as the belief compensator vanish from the expression. The expression also accommodates an economy with homogeneous but possibly erroneous beliefs in which case only the belief compensator $b$ vanishes. If I would like to associate such a homogeneous beliefs economy with a heterogeneous beliefs economy I would have to derive the additional variation coming from $b$ from a modified habit process. The following corollary is straightforward:

**Corollary 4.** The heterogeneous agent economy can be represented as a homogeneous agent economy with a modified habit process $x^r$ such that,

$$x^r_{t+1} - x^r_t = (1 - \lambda x)\omega_t - \frac{b_t}{(\gamma - \eta)},$$

and beliefs given by $\xi^r$.

The importance of corollary 4 is not in deriving an equivalent representative agent economy but in noting the additional elements that appear when we introduce heterogeneous beliefs.
We can see that an economy with heterogeneous beliefs introduces two new elements. The first one is a time varying discount factor \( b \) which reflects the level of belief heterogeneity in the economy over the next period at a given point in time. This element results in discounted prices when compared to a homogeneous beliefs economy. The second element is the additional risk factor as given by \( \xi^r \) and in principle can affect the risk-premium, but let us examine these two new elements in turn and their possible connection. The only thing I need to mention in order to be able to connect the two effects is that a higher value for \( \gamma \) increases the discount factor \( b \) and the variability of \( \xi^r \).

### 4.1 Discounted Prices

The following corollary formalizes the concept of discounted prices.

**Corollary 5.** Consider an asset \( j \) that pays a future stream of cash flows. The shadow price for an agent \( i \), denoted by \( P^i_j \), is the equilibrium price of the asset in a homogeneous beliefs economy with beliefs \( \xi^i \), in which case the SDF is given by,

\[
M^i_{t,t+1} = \delta \xi^i_{t,t+1} \exp \left[-\gamma g_{t+1} + (1 - \lambda_x)(\gamma - \eta)\omega_t\right].
\]

Then, when \( \gamma > 1 \) and when beliefs are heterogeneous, the equilibrium asset prices are strictly less than the consumption weighted average of individual shadow prices,

\[
P_{j,t} < \mathcal{E}_t P^i_{j,t}.
\]

The main result of this study is to show that quantitatively this discounting in prices due to heterogeneity is very significant and therefore variations in the level of heterogeneity causes prices to vary across time. In fact when heterogeneity is sufficiently high the equilibrium price can be less than the shadow price of the most pessimistic agent. Prices become discounted because investors are afraid that their different investment behaviors might lead them to states with very low levels of wealth. These are under their assessment low probability events but they are still afraid of them because they would be hit really hard by those states. Agents are afraid of their individual bad states more than they value their good states due to being more risk averse than a myopic agent (\( \gamma > 1 \)).

This result has a clear implication about the economy. Note first that the belief compensator affects all prices at the same time regardless of their riskiness because it reflects the reluctance of the investors to transfer wealth to the next period. During periods of great divergence of beliefs we should observe low prices while prices should increase when beliefs are aligned. As I will show with the calibration of the economy that this can explain the behavior of the aggregate price-dividend ratio that seems to be non-stationary. It appears that since the second
world war the price-dividend ratio has a slight upward trend and this could be a result of a slow convergence of beliefs. With the same token I can explain the dramatic increase in the price-dividend ratio during the 1990’s if we accept that the beliefs of the investors for some reason were aligned. The reason seems to be that the level of the macroeconomic risk decreased significantly and this probably lead to a decrease in the level of heterogeneity as I will show later on. This explanation does not require that beliefs converged to the true probability distribution but that the majority of the investors were having the same expectations about the prospects of and riskiness in the economy and they were aware of this. Consequently, they were not afraid that they would end up in a state with a very low level of wealth and they were bidding up the prices as they were gaining the confidence that such an event was not possible. The hit of the recession lead to a significant decrease in prices probably because the recession caused the beliefs to diverge.

I have so far talked about the effect of heterogeneity on asset prices as a whole but not about their relative valuation. In particular I am interested in the relative valuation of the market security and the price of the risk-free bond. This depends on the divergence of beliefs not only with respect to the next period but also with respect to all future periods, because the market security is infinitely lived and pays dividends every period. When changes in belief heterogeneity are persistent then long-lived securities like the market security will be more sensitive than short-lived securities and therefore variations in the price-dividend ratio can be explained without excessive variation in the risk-free rate.

4.2 The Equity Premium

The expected excess return on the market over the risk-free rate, which I denote with $R^e$, can be affected by the heterogeneity of beliefs in a number of ways. From (17) I derive the standard expression of equilibrium excess returns, $\mathbb{E} [MR^e] = 0$, and expanding it using the equilibrium expression of the SDF I have the following:

$$\mathbb{E} (R^e) = -\frac{C (e^{-\gamma g}, R^e)}{\mathbb{E} (e^{-\gamma g})} - \frac{C (\xi^r, e^{-\gamma g} R^e)}{\mathbb{E} (e^{-\gamma g})}$$

where $C$ denotes the covariance under the objective probability measure. The first part is the standard expression which apart from the possible additional variability in returns it is not affected. I therefore examine the second part. I remind that $\xi^r$ is a random variable with average value of one. It has higher values for states that on average agents believe that their probability is higher than the true. It also has higher values for states that agents agree more than average. The equity premium can increase with belief heterogeneity through the second part due to the following reasons:

(i) $\xi^r$ can be positively correlated with consumption growth $g$. This could happen either
because agents are optimistic or because agents agree more about the probabilities of the good states and disagree more about the bad states. None of these reasons can be realistically assumed to cause equity premia to be consistently higher than a homogeneous economy.

(ii) $\xi^r$ can be negatively correlated with $R^e$. The optimism argument applies equally here. I do expect however for $\xi^r$ to be negatively correlated with future prices because it is probable that states about which there is high disagreement ($\xi^r$ is low) are also states in which the belief heterogeneity about the future is high (and therefore $P$ low). In the reduced form model presented in the next section I do have this effect.

The risk aversion parameter $\gamma$ plays a double role with respect to the equity premium. When agents are less risk averse their investment behavior is more radical and the variability of the additional risk-factor $\xi^r$ increases. Therefore, if everything else stays the same the equity premium should increase. The second effect is related to the second point above. With higher risk-aversion prices become more discounted with heterogeneity and therefore the effect of the correlation between $\xi^r$ and $R^e$ increases.

5 The Economy

In this study I depart from the usual approach of modeling belief heterogeneity by which the beliefs of the different agents are assumed explicitly. If I assumed a particular generating process of which its structure is known by the agents and that they are rational and update their beliefs in a Bayesian fashion then their beliefs will very soon converge. This is however not what we observe in reality. There is an inherent difficulty in sustaining belief heterogeneity in a model while not departing from rationality. The issue is not that agents are not rational, but in reality the generating process is much more complex than can be assumed in a model and hence the available information is not able to exclude many “sets” of beliefs to be held by investors. Additionally, investors’ information set is not only the history of consumption growth but their beliefs are also influenced by other types of information like technological shocks, wars, the price of oil, anything that can affect aggregate production in an economy and in a possibly non-stationary way.

Therefore, instead I assume that agents in equilibrium hold different beliefs and this divergence does not vanish in time. Now that I have derived the general equilibrium conditions I can model directly the equilibrium law of motion for the divergence of beliefs. The following assumptions are about the function $\xi$ but do not explicitly characterize it. These assumptions will refer to the conditional distribution of $g$ held by agents at any period $t$. Let $f_t(g)$ denote the density
function of one period consumption growth believed by agent $i$ at time $t$ and let $f_i(g)$ denote the true and unknown density function.

**Assumption 6.** Agents disagree only about the probability distribution of consumption growth and know the true dynamics of $\epsilon$,

$$\frac{\xi^i(s^{t+1})}{\xi^i(s^t)} = \xi(i, g_{t+1}, s^t)$$

where $s^{t+1} = (s_{t+1}, s^t)$.

Then I have that,

$$\xi(i, g_{t+1}, s^t) = \frac{f^i_i(g_{t+1})}{f_i(g_{t+1})}.$$  

Note that the time subscript denotes the dependence of the beliefs on the observed history $s^t$ but the opinions differ only with respect to the consumption growth. This assumption is made for simplicity since if I assumed otherwise the equilibrium state vector would become too large too allow us to approximate the price functions accurately. It would have been realistic to assume otherwise since if beliefs change over time then beliefs about the evolution of beliefs might differ as well. After all this is what Keynes implied when he talked about beauty contests. Such an additional assumption would introduce further riskiness in the economy.

Let us now continue with the assumptions about $f^i_i$ and its evolution.

**Assumption 7.** In any period $t \in \mathbb{N}$ each agent believes that the aggregate consumption growth for the next period is conditionally normally distributed

$$g_{t+1} \mid \mathcal{F}_t \sim N(\mu^i_t, \sigma^2_t)$$

where $(\mu^i_t, \sigma_t)$ are $\mathcal{F}_t$-measurable. Agents are certain about $\sigma_t$ but are uncertain about the conditional mean,

$$\mathbb{P}^i : \mu_t \mid \mathcal{F}_t \sim N(\mu^i_t, u^2_t)$$

Note that I do not assume any particular process for consumption growth and therefore do not care about $f_t$. For pricing it is irrelevant anyway since only beliefs matter. Later on when I need to express unobserved quantities like the equity premium I will use the probability measure of the consensus beliefs. Further note that agents are assumed to disagree only about the mean of aggregate consumption growth. It would be interesting to see what is the effect of heterogeneity of beliefs about the second moment but I leave this for future research.

The variance term $u^2$ denotes the uncertainty in the economy about the conditional mean and it is common to all agents. This uncertainty adds further risk to the economy since agents
discount the future according to the following probability distribution,

\[ \mathbb{P}^i : g_{t+1} \mid \mathcal{F}_t \sim N(\mu^i_t, \sigma^2_t + u^2_t). \]

With assumption 7 I have reduced the belief heterogeneity to just differences in opinions about the conditional mean of macroeconomic risk. Hence, I can aggregate agents according to their types and define a new probability measure,

\[ \tilde{\alpha}_t(\mu) = \mathcal{E}_t \mathbb{1}\{\mu^i_t = \mu\}. \quad (20) \]

\( \mathbb{1}\{A\} \) denotes the indicator function that takes the value of 1 in the subset \( A \) of the set of agents and 0 otherwise. With this notation I can now define the cross-sectional distribution of beliefs.

**Assumption 8.** The initial consumption distribution \( \tilde{\alpha}_0 \) of beliefs about means across the set of agents is given by

\[ \tilde{\alpha}_0 : \mu \sim N(\mu_0, \nu_0). \]

With the above assumption I parameterize belief heterogeneity in some period \( t \) with \( \nu_t \) the consumption weighted volatility of the beliefs about the mean. Homogeneity of beliefs implies that \( \nu_0 = 0 \). With this assumption I can characterize the consensus beliefs for time 0 or any other period where the distribution of beliefs is normally distributed.

**Lemma 2.** The consensus beliefs about consumption growth is normally distributed,

\[ N\left(\mu_t, \sigma^2_t + u^2_t + \frac{\nu^2_t}{\gamma}\right). \]

The belief compensator is given by,

\[ b_t = (\gamma - 1) \log \left( \sqrt{1 + \frac{\nu^2_t}{\gamma(\sigma^2_t + u^2_t)}} \right). \]

From this lemma we see very clearly the two pricing effects of belief heterogeneity and what they depend on. The belief compensator depends first linearly on the cautiousness of the agents as parameterized by \( (\gamma - 1) \). There is also a counter effect that appears in the term \( \nu^2_t / \gamma(\sigma^2_t + u^2_t) \) which represents the fact that when agents are more cautious they speculate less and they have less to fear about their beliefs being wrong. This explains further why the additional risk in the economy is inversely proportional to \( \gamma \). Finally the divergence of beliefs naturally increases both the discounting term as well as the risk of the economy.

In such a setting where the consensus beliefs about aggregate consumption growth is normally distributed I can express the interest rate \( r^f = \log(Q) \) in closed form. This will be the
equilibrium interest rate since I will assume that in equilibrium the beliefs continue to have this form in all periods.

**Corollary 6.** The equilibrium interest rate is given by,

\[
r_t^f = - \log(\delta) + \gamma \mu_t - (1 - \lambda_\gamma)(\gamma - \eta)\omega_t + (\gamma - 1) \log \left( \frac{\nu_t^2}{\gamma(\sigma_t^2 + u_t^2)} \right) - \frac{1}{2} \gamma^2 \left( \sigma_t^2 + u_t^2 + \frac{\nu_t^2}{\gamma} \right).
\]

First I note that the standard expression about the interest rate is obtained in the case of homogeneity and no uncertainty, \( \nu = u = 0 \). Then I note that the effect of the heterogeneity variable \( \nu \) depends on the beliefs about the volatility \( \sigma \). This is natural since in the case where \( \nu \) is very low in comparison to \( \sigma \) then agents have a considerable disagreement only in states that are far away from the mean. Essentially, what describes the level of heterogeneity in the economy is the ratio \( \nu/\sigma \).

Before I complete the set of assumptions about the beliefs let us look at a specific simple economy where the law of motion is derived explicitly in equilibrium. In this economy there is no extra shock \( \epsilon \) to the beliefs but are in fact dogmatic.

**Example 1.** Agents have dogmatic beliefs, i.e. \( \mu_i^t = \mu^t \forall t \in \mathbb{N} \). Agents have common beliefs about \( \sigma \) which can be time varying. Then in equilibrium the distribution of beliefs about the mean continues to be normal,

\[
\alpha_t : \mu^i \sim N(\mu_t, \nu_t)
\]

with time varying moments,

\[
\mu_{t+1} = \frac{\nu_{t+1}^2}{\nu_t^2} \mu_t + \frac{\nu_{t+1}^2}{\gamma \sigma_t^2} g_{t+1},
\]

\[
\frac{1}{\nu_{t+1}^2} = \frac{1}{\nu_t^2} + \frac{1}{\gamma \sigma_t^2},
\]

\( \forall t \in \mathbb{N} \).

It is quite interesting to first note that the dogmatic agents are represented in equilibrium by a consensus consumer that behaves as a bayesian updating agent. This represents the fact that in equilibrium agents that have beliefs that are far away from the average long-term consumption growth increasingly loose their wealth. Eventually the agents that happen to have the correct beliefs about the long-term mean are the only ones who survive. The rate of convergence depends on the risk-aversion parameter and it is faster the lower is \( \gamma \). This is because then
agents are less cautious and therefore those agents with the wrong beliefs loose their wealth faster.

Further if all agents are perfectly rational in the sense that they update their beliefs in the same way then the convergence will be even faster. However, the reality is that economic agents do hold different beliefs and these differences do not vanish. For this reason I assume that the additional information \( \epsilon \) is interpreted differently across agents and does not allow beliefs to converge. However, I depart from the usual approach whereby we would need to model the beliefs of every agent and I assume instead the law of motion of the distribution of beliefs in equilibrium which I do next.

**Assumption 9.** The average beliefs about the mean and volatility of consumption growth evolve according to,

\[
\begin{align*}
\mu_{t+1} &= \lambda_\mu \mu_t + (1 - \lambda_\mu) g_{t+1}, \\
\sigma_{t+1}^2 &= \lambda_\sigma \sigma_t^2 + (1 - \lambda_\sigma)(\mu_t - g_{t+1})^2.
\end{align*}
\]

where \( \lambda_j \in (0, 1), j \in \{\mu, \sigma\} \). Uncertainty and belief heterogeneity in equilibrium are given by,

\[
\begin{align*}
\nu_t &= \sigma_t^{1+\kappa}, \\
u_t &= \phi_u \sigma_t,
\end{align*}
\]

where \( \kappa \in (0, 1) \).

The autoregressive processes of \( \mu_t \) and \( \sigma_t \) reflect the adaptation of beliefs in the economy as a whole. The economy recognizes that the conditional moments of consumption growth change and the average beliefs follow these simple processes and only depend on \( g \) and the previous average beliefs. Even though individual beliefs are not modeled implicitly it is assumed that each agent \( i \) updates his beliefs about the current mean of consumption growth according to,

\[
\mathbb{E}^i(\mu_{t+1}\mid\mathcal{F}_{t+1}) = \lambda_\mu \mathbb{E}^i(\mu_t\mid\mathcal{F}_t) + (1 - \lambda_\mu) g_{t+1} + \varepsilon(\epsilon_{t+1}, i)
\]

The constant updating weight \( \lambda_\mu \) is consistent with Bayesian updating and the assumption that the uncertainty is a constant fraction of macroeconomic risk. In fact, this puts a restriction on the parameter \( \phi_u \) since,

\[
\lambda_\mu = \frac{\sigma_t^2}{\sigma_t^2 + u_t^2},
\]

and therefore \( \phi_u = \sqrt{1/\lambda_\mu - 1} \).

The individual error terms \( \varepsilon(\epsilon_{t+1}, i) \) is what keeps the heterogeneity variable \( \nu_t \) from going to zero. Every period the level of heterogeneity changes three times. First it decreases because
wealth is transferred to agents whose predictions were more aligned with the realization of $g$. It decreases even more because they observe $g$ and this makes them update their beliefs about the previous conditional mean. Finally, $\epsilon_{t+1}$ carries information about the change in the conditional mean and it is implicitly assumed that is interpreted differently and therefore causes $\nu$ to increase on average.

The assumption that $\nu_t$ is an increasing function of $\sigma_t$ is a natural one since it is realistic to assume that the uncertainty is increasing in the macroeconomic risk and in turn the level of heterogeneity is increasing in the uncertainty. The important assumption however for the results is that effective heterogeneity as it is expressed by the ratio $\nu_t/\sigma_t$ is an increasing function of $\sigma_t$. Through this assumption I will be able to generate a high correlation between the price level and macroeconomic risk.

With the last assumption the model is complete. The economy and its equilibrium are parameterized by the preference parameters $(\delta, \gamma, \eta)$ the autoregressive parameters $(\lambda_j, j \in \{x, \mu, \sigma\})$ and the parameters $(\phi_\nu, \kappa)$. The endogenous state vector is,

$$z = (\omega, \mu, \sigma)$$

and the law of motion is described by that of $\omega$ as given by (6) and by assumption 9. The equilibrium interest rate is then given by corollary 6.

6 Calibration

Theoretically, the introduction of heterogeneous beliefs affects both the level of prices as well as the equity premium. While the overall effect on the equity premium is unclear, the effect on the level of prices is straightforward, namely that when heterogeneity rises prices decrease. I have further argued that when changes in the level of heterogeneity are persistent, then the level of the stock market is more sensitive to those changes than the risk-free rate is. The calibration exercise that will be presented will show that quantitatively the important asset pricing element is the variations in the discounting factor and that the increase in prices during the last decade of the 20th century can be explained by a decrease in the level of belief heterogeneity.

For the calibration exercise I use quarterly data of consumption growth, the market index return and the real risk-free rate from the first quarter of 1947 until the last quarter of 2007. $t = 0$ corresponds to the start of the data period. The data that I use are first the real aggregate quarterly consumption growth of non-durables and services as obtained from NIPA tables. I then obtain the nominal 3-month T-bill yield from the Fama risk-free rates file of the CRSP monthly treasuries database. The real rate is obtained after deflating the nominal with the realized inflation, measured by the CPI index obtained from the Bureau of Labor Statistics.
The market price-dividend ratio is imputed from the quarterly CRSP market index returns including and excluding dividends.

### 6.1 Homogeneous Economy with Uncertainty

The first parameters that need to be chosen are the updating weights \( \lambda_\mu \) and \( \lambda_\sigma \) which reflect how the average beliefs about the conditional mean and conditional volatility of aggregate consumption growth are adapted. Since beliefs have not been modeled I have chosen to use the maximum likelihood estimates assuming the particular time-series model. In addition to these two parameters I also estimate the initial values for \( \mu \) and \( \sigma \) for the time period. The estimates are shown in table 1. The conditional volatility appears to be quite persistent with a value of 0.94 whereas the conditional mean shows significantly less persistence with a value of 0.76. The time series of these two conditional moments during the time period are shown in the top two panels of figure 2. The bottom panel of the same figure shows the consumption surplus variable \( \omega = y - x \), given the selected parameter value for \( \lambda_x = 0.95 \). The parameter \( \lambda_x \) was chosen in order to match the persistence in the price-dividend ratio in the date, which is 0.93. It is very interesting to note that the estimated persistence of the price-dividend ratio is very close to the estimated persistence of the conditional volatility of consumption growth.

The fitted time-series of the conditional volatility of consumption growth was plotted against the contemporaneous price-dividend ratio. The correlation between the two series is impressive and the sample linear correlation is \(-0.78\). This correlation however cannot be explained in a homogeneous economy with varying beliefs and uncertainty. I first calibrate such an economy by trying to match the evolution of the price-dividend ratio. Table 2 shows the parameter values for both the homogeneous and heterogeneous economy. The risk-aversion parameter \( \gamma \) was chosen to be 4, a realistic value for asset pricing. The subjective discount factor \( \delta \) and the habit parameter \( \eta \) were chosen to match the level and variability of the price-dividend ratio until the end of 1980’s. The initial consumption surplus \( \omega_0 \) was set at its long-term average value.

Figure 3 shows the data and the model implied time-series for the price-dividend ratio and the risk-free rate. I observe that the model is able to match the price-dividend ratio quite well until the end of the 1980’s. Then, despite the decrease of both the macroeconomic risk as well as the uncertainty the price-dividend ratio stays around the same levels. The risk-free rate implied by the model is significantly higher than the data since the model is unable to generate enough equity premium.

Table 3 compares the sample period price statistics of the data with the model implied statistics. First I note that the model is able to match the first lag autocorrelation of the price-dividend ratio at 0.93 but it is not able to match the fourth lag autocorrelation. The average risk-free
rate is much higher for the model while the variability is a little smaller. The volatility of the price-dividend ratio implied by the homogeneous economy is about half of the data sample volatility and this is attributed to the fact that the model is not able to explain the significant increase in prices after the end of 1980’s. Figure 4 shows two plots of the price-dividend ratio over different combinations of the state vector. In the left plot I fix the conditional mean at the data sample average and vary the other two parameters. I see clearly that the effect of the conditional volatility on the price-dividend ratio is very small contrary to what I see in the data. Neither the level of macroeconomic risk nor the level of uncertainty cause any significant variation in the stock price.

On the right plot of figure 4 I fix the conditional volatility to the sample average and vary the conditional mean and the consumption surplus. These two state variables almost equally affect the price-dividend ratio in a natural way. When the economy is more optimistic, prices increase as well as when the current consumption is high in relation to habit. To understand this behavior I need to see the effect of these two state variables on the risk-free rate as shown in the right plot of figure 5. When the consumption surplus is high agents want to transfer wealth to the next period which causes the risk-free rate to fall. The opposite happens when the conditional mean increases in which case agents want to borrow. However, when the conditional mean is high it means that the expected future consumption surplus is high and therefore the future interest rates are low. This causes the current stock price to increase since habit is persistent. The left plot of the same figure shows that the macroeconomic risk does not affect the interest rate much and for the same reason neither the price-dividend ratio. In fact I see that such a model of homogeneous beliefs even with the inclusion of uncertainty that is proportional to the level of the risk cannot explain the variation of the price-dividend ratio.

The conditional volatility does affect on the other hand the equity premium, as shown in the left plot of figure 11. Figure 11 shows the homogeneous economy’s expected excess return for different combinations of the state vector. As I see the expected excess return is not so much affected by the other state variables. In fact the endowment habit ratio has almost no effect whereas the conditional mean of consumption growth determines the level of the term premium through variations in the risk-free rate. The model implied time series of the conditional equity premium as shown in figure 6 starts from a value around 1.6% annually and steadily declines down to around 0.5% annualized and is driven by the decrease of the conditional volatility of consumption growth. The conditional volatility of returns on the other hand, which is shown in the lower panel of 6, is driven by all three variables.

6.2 Heterogeneous Beliefs Economy

In order to see the impact of belief heterogeneity I keep the same parameters as for the homogeneous economy except from the subjective discount factor. I chose the subjective discount
factor and the parameter $\kappa$, that determines the sensitivity of the level of heterogeneity in the economy to the macroeconomic risk, in order to match the time series mean and volatility of the price-dividend ratio. The subjective discount factor was calibrated at a value greater than one. This was necessary due to the fact that the rest of the parameters were taken from the homogeneous economy. Otherwise a stronger habit (by making $\eta$ more negative) would require a smaller $\delta$.

Table 3 shows the model implied time series statistics and it is noted that the mean and volatility of the price-dividend ratio were matched quite well. The first lag autocorrelation is slightly higher than in the data while the fourth lag autocorrelation is at 0.81 instead of 0.91. However, the autocorrelation shows an improvement compared to the homogeneous economy because now the autocorrelation is also affected by the autocorrelation of the macroeconomic risk. The model average of the risk-free rate is significantly higher than in the data, 1.06% as opposed to 0.25% but at the same time significantly lower than the one generated by the homogeneous economy which is 1.97%. This is a result of the impact on the equity premium as well as the increase in the price-dividend from 1990 and onwards.

The main result of the study is shown in the top panel of figure 8 and the left panel of figure 9. The model implied time series of the price-dividend ratio captures quite well the data since the sensitivity of the price-dividend ratio to a decrease in the consumption risk is sufficient to explain the phenomenal increase in the prices during the last decade of the 20th century. The left panel of figure 9 shows the model implied variation of the price-dividend ratio for a fixed value of $\mu$ and the complete range of values of the other two state variables. Comparing the variation induced by the conditional volatility of consumption growth as predicted by the model with the data as shown in figure 1 I observe that the model does a very good job replicating this behavior.

Finally, looking at the predicted equity premium as shown in figure 12, the heterogeneity of beliefs as modeled in this study does have a significant impact on the expected excess return with values that reach above 2% annually. The main avenue through which the equity premium is affected is again related to the discounting effect. Due to the assumption of normality and in addition the assumption that agents disagree about the conditional mean of consumption growth, the largest disagreement among agents is on the tails of the distribution, which means concerning large absolute values of consumption growth. Now large future values of consumption growth either positive or negative will increase the level of heterogeneity in the next period and therefore decrease the future prices. Hence, there is a high negative correlation between the endogenous risk-factor $\xi^r$ and the future stock returns. I remind here that $\xi^r$ is low in states where there is a lot of disagreement. Therefore, the market is a bad hedge for the states were agents’ consumption is low. Note that even though different agents have different states of low consumption, the stock has a low return in all of those states.
The only drawback concerning the equity premium of this model is that it is increasing in the price-dividend ratio. This is mostly because of the particular function chosen that relates the level of heterogeneity of beliefs with the level of consumption risk. In particular it becomes more and more sensitive to changes in consumption risk as consumption risk decreases. The higher sensitivity increases in absolute value the negative correlation of the endogenous risk-factor with the market returns and hence the equity premium. The particular function was only chosen due to its simplicity and it is not crucial to the results of this study.

6.3 Evidence of Convergence in Beliefs

In order to see whether there is any evidence concerning the evolution of belief heterogeneity in the economy concerning future consumption growth I looked at the Survey of Professional Forecasters provided by the Federal Reserve Bank of Philadelphia. In one of the surveys the professional forecasters are asked to predict the next quarter real consumption growth. From this survey I compute the cross-sectional standard deviation of forecasts for every period provided. Data are available from the third quarter of 1981 until today.

This time series turned out to be quite volatile and for this reason I extracted the trend from this series using the Hodrick-Prescott filter. The trend was then plotted along with the time-series of $\nu$ of the calibrated heterogeneous agent economy. The plot is shown in figure 13. Even though the heterogeneity of forecasts by the professional forecasters is not considered to be representative of the heterogeneity of believes in the economy due to strategic behavior, the data does provide significant evidence that during this period beliefs about consumption growth converged significantly.

7 Conclusion

In this study I show that the driving force of heterogeneous beliefs is very important in shaping asset prices. For example I successfully show through my model economy that the phenomenal increase in the aggregate price-dividend ratio during the 1990’s can be explained by an alignment in the beliefs of the various investors. During this time the macroeconomic risk as given by the conditional volatility of consumption growth declined significantly causing in this way a decrease in the heterogeneity of beliefs in the economy. Hence, agents became confident that all investors had the same prospects in mind about the economy which implies that they did not fear individual bad states and this led to a significant increase in the level of prices.

The risk premium in my economy is also significantly increased due to an endogenous risk factor that arises due to the different investment behavior. This new risk factor is negatively correlated

\[ \text{See for example Ottaviani and Sørensen (2006) and references therein.} \]
with stock returns because when the heterogeneity of beliefs is positively autocorrelated then the stock pays poorly at states where the heterogeneity is high. In those states consumption is low for many agents and hence a bad hedge for their risk.

The link that connects consumption risk with asset price behavior has been particularly elusive, so much that significant doubts have been cast over the canonical asset pricing model. The results of this study however show that the quantitative behavior of asset prices predicted by the model is significantly improved once I introduce differences in beliefs. If we are therefore to understand asset markets we must understand how and why beliefs differ and how they evolve over time.
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31


A Proofs

Below I provide the proofs of the propositions, lemmas, and corollaries found in chapter 3. The preference assumption is that agents have time and state separable power utility preferences with external habit, and the running utility is given by

\[ u(c, X) = \frac{c^{1-\gamma}X^{\gamma-\eta} - 1}{1-\gamma}. \]

where \( c \) denotes consumption. I let \( \alpha(di) = c^i / Y \) be the consumption proportion of the marginal agent \( di \). Agents differ in their beliefs denoted with \( (\xi^i_t, \xi^i_0) = 1 \) which is the Radon-Nikodym process with respect to the objective measure. I also use the notation \( \omega = y - x \), where \( y \) and \( x \) are the natural logs of \( Y \) and \( X \) respectively. I denote with \( p_t \) the pricing kernel process under the objective probability measure and I set \( p_0 = \exp(-\gamma \omega_0 - \eta x_0) \). For an agent with beliefs \( \xi^i \) the pricing kernel is given by \( p^i_t = p_t / \xi^i_t \).

**Proof of Lemma 1.** Given the habit process \( X_t, t \geq 0 \), the pricing kernel process \( (p_t, t \geq 0) \), and the initial total wealth \( W_0 \), the optimization problem of an agent \( i \) with initial wealth allocation \( \theta(di) \) is given by,

\[
\max_{(c^i_t, t \geq 0)} \mathcal{L}_i = E_0^i \sum_{t \geq 0} \delta^t u(c^i_t, X_t) - \lambda_i \left[ E_0^i \sum_{t \geq 0} \delta^t p^i_t \left( c^i_t(\gamma) - \theta(di)Y_t \right) \right]
\]

Due to the preference assumptions, the optimal solution is interior, and is characterized by the first order condition,

\[
\left( \frac{c^i_t}{X_t} \right)^{-\gamma} = \lambda_i \frac{p^i_t}{\xi^i_t} X_t^\eta, \quad \forall t \geq 0.
\]

and the equality of the inter-temporal budget constraint (5). Given the initial value of \( p_0 \) I have that \( \alpha_0(di) = \lambda_i \) and hence I get the rearranged first order condition (7). I then substitute in the budget constraint, satisfied with equality, the optimal consumption share from (7), to arrive at the equilibrium value for \( \alpha_0(di) \).

**Proof of Corollary 2.** From the first order condition (7) at two periods \( t \geq 0 \) and \( t + k, k \geq 0 \), I have that

\[
\alpha_{t+k}(di) = \alpha_t(di) \left( \frac{\xi^i_{t+k}}{\xi^i_t} \right)^{1/\gamma} \left( \frac{p^i_{t+k}}{p^i_t} \right)^{-1/\gamma} \exp \left[ -\frac{\eta}{\gamma} (x_{t+k} - x_t) - (\omega_{t+k} - \omega_t) \right]. \tag{21}
\]
Integrating both sides with respect to the set of agents I get

\[ \mathcal{E}_t(\xi^i_{t,t+k})^{1/\gamma} = \frac{z_{t+k}}{z_t}, \]

where \( z_t = p_t^{-1/\gamma} \exp \left( -\frac{\gamma}{\gamma} x_t - \omega_t \right) \). The result is directly obtained from the derived general relation (22) since \( z_t = \xi^i_t \forall t \geq 0 \).

**Proof of Corollary 3.** Note that \( x_{t+1} - x_t = (1-\lambda x)\omega_t \) and substitute in (22) for the special case \( k = 1 \) to obtain (14). Equation (15) is obtained after I substitute \( \tilde{\xi}^{i/\gamma} = p_t^{1/\gamma} \exp \left( \frac{\gamma}{\gamma} x_t + \omega_t \right) \) in (21).

**Proof of Proposition 1.** I only need to recognize from the result,

\[ p_t = \tilde{\xi}_t \exp (\gamma x_t - \gamma \omega_t), \quad \forall t \geq 0 \]

which is given by (14), that \( p_t \) is a functional of the initial consumption distribution \( \alpha \) and the exogenous beliefs processes \( \xi^i \). Then \( F(\alpha, i) \) is a function of the initial distribution of consumption and exogenous processes \( y_t, x_t \) and \( (\xi^i_t, i \in I) \) that are given. Hence using lemma 1 I get \( F(\alpha, j) = \int_0^\gamma \alpha(d\xi) = \alpha(j) \).

**Proof of Corollary 5.** Consider period \( t \geq 0 \) and let \( (D^i_{t,t+k}, k \geq 1) \) be the stream of future cash-flows for which asset \( j \) is a claim. Then the shadow valuation of agent \( i \) is given by,

\[ P^i_{j,t} = \mathbb{E}_t \left[ \sum_{k \geq 1} M^i_{t,t+k} G^i_{j,t+k} \right] \]

where \( M^i_{t,t+k} = \prod_{k \geq 1} M^i_{t+k-1,t+k} \) and

\[ M^i_{t+k-1,t+k} = \delta \xi^i_{t+k-1,t+k} \exp \left[ -\gamma g_{t+k} + (1 - \lambda x)(\gamma - \eta) \omega_{t+k-1} \right] . \]

that is given by (19) in the special case of a homogeneous beliefs economy with beliefs \( \xi^i \). \( G^i_{t,t+k} = D^i_{t,t+k}/D^i_t \) is the cash-flow grow from period \( t \) to \( t + k \). Integrate equation (23) over the set of agent consumption distribution at time \( t \) and substitute in \( M \) the heterogeneous economy SDF:

\[ \mathcal{E}_t P^i_{j,t} = \mathbb{E}_t \left[ \sum_{k \geq 1} M_{t,t+k} G^i_{j,t+k} \frac{\xi^i_{t,t+k}}{\xi^i_{t,t+k}} \right] \]

But I already know that \( \mathcal{E}_t \xi^i_{t,t+k} \geq \tilde{\xi}_{t,t+k} \) with strict inequality when there exist heterogeneity of beliefs at least for one future period.
Proof of Lemma 2. Let the beliefs of agent $i$ about one period consumption growth to be,

$$\mathbb{P}^i : g_{t+1} \mid \mathcal{F}_t \sim N(\mu_i, \sigma_i^2 + u_i^2)$$

and the consumption distribution across beliefs to be,

$$\tilde{\alpha}_t : \mu \sim N(\mu, \nu_t^2)$$

Now let us aggregate beliefs and let $f$ denote the probability density while $\varphi$ denotes the normal probability density function:

$$\tilde{f}_t(g_{t+1}) = \left[ \int f_t(g_{t+1} | \mu)^{1/\gamma} \tilde{\alpha}(d\mu) \right]^{\gamma}$$

$$= \left( \frac{\sqrt{2\pi}(\gamma(\mu_i^2 + u_i^2))^{\gamma}}{2\pi(\mu_i^2 + u_i^2)} \right)^{\gamma}$$

$$= \left( 1 + \frac{\nu_t^2}{\gamma(\mu_i^2 + u_i^2)} \right)^{-1/\gamma} \varphi \left( g_{t+1} \mid \mu, \gamma(\mu_i^2 + u_i^2) + \nu_t^2 \right)$$

It is well known from Bayesian analysis that if $y \mid \theta \sim N(\theta, \sigma_1^2)$ and $\theta \sim N(x, \sigma_2^2)$ then $y \sim N(x, \sigma_1^2 + \sigma_2^2)$. Using this result I get,

$$\tilde{f}_t(g_{t+1}) = \frac{\sqrt{2\pi}(\gamma(\mu_i^2 + u_i^2))^{\gamma}}{2\pi(\mu_i^2 + u_i^2)} \varphi \left( g_{t+1} \mid \mu, \gamma(\mu_i^2 + u_i^2) + \nu_t^2 \right)$$

It is clear from the final expression that the first part is the belief compensator $\exp(-b_t)$ and the second term is the consensus beliefs $f^x$.

Proof of Corollary 6. The continuously compounded risk-free rate is simply given by $r_t^f = -\log \mathbb{E}_t (M_{t,t+1})$. Since the SDF is log-normally distributed by lemma 2 then,

$$r_t^f = -\mathbb{E}_t [\log M_{t,t+1}] - \frac{1}{2} \mathbb{V}_t [\log M_{t,t+1}]$$

where

$$\mathbb{E}_t [\log M_{t,t+1}] = \log \delta - \gamma \mu_t + (1 - \lambda_x)(\gamma - \eta)\omega_t - b_t$$

$$\mathbb{V}_t [\log M_{t,t+1}] = \gamma^2 \left( \sigma_t^2 + u_t^2 + \frac{\nu_t^2}{\gamma} \right),$$

and this completes the proof.

Example 1. In this example every agent has dogmatic beliefs, $\mu_i = \mu^1$ for all $t$, without un-
certainty, \( u_t = 0 \). Let us suppose that the consumption distribution over beliefs, \( \tilde{\alpha} \), at some period \( t \) is \( N(\mu_t, \nu_t^2) \). Re-expressing the law of motion for the consumption distribution (21) in terms of the convolution \( \tilde{\alpha} \) I get:

\[
\tilde{\alpha}_{t+1}(d\mu) = \tilde{\alpha}_t(d\mu) \left( \frac{\varphi(g_{t+1}|\mu, \sigma_t^2)}{f_t(g_{t+1})} \right)^{1/\gamma}
\]

Using the results obtained in lemma 2 I get:

\[
\tilde{\alpha}_{t+1}(d\mu) = \left( \frac{\sqrt{2\pi\gamma}\sigma_t}{\sigma_t} \right)^{1/\gamma} \frac{\varphi(g_{t+1}|\mu, \gamma^2\sigma_t^2)}{\left( \frac{\sqrt{2\pi\gamma}\sigma_t}{\sigma_t} \right)^{1/\gamma} \varphi(g_{t+1}|\mu, \gamma^2\sigma_t^2 + \nu_t^2)} \varphi(\mu|\mu_t, \nu_t^2) d\mu
\]

where

\[
\mu_{t+1} = \frac{\nu_{t+1}^2}{\nu_t^2} \mu_t + \frac{\nu_{t+1}^2}{\gamma\sigma_t^2} g_{t+1} \]

\[
\begin{align*}
\frac{1}{\nu_{t+1}^2} &= \frac{1}{\nu_t^2} + \frac{1}{\gamma\sigma_t^2}
\end{align*}
\]

which completes the example.

\[
\square
\]

### B Computational Approach

The complete computational approach that I employ to approximate the function of the price-dividend ratio in equilibrium is quite involved. Here I present an outline of the algorithm. In both homogeneous and heterogeneous beliefs economies the state vector is given by \( z = (\mu, \sigma, \omega) \) and the exogenous shock is the aggregate endowment growth whose natural logarithm is \( g \). What differs in these two economies is the law of motion which I denote in general with \( z = L(z, g) \) and the equilibrium stochastic discount factor (SDF) denoted with \( M \) under the consensus beliefs. The price-dividend ratio under the consensus beliefs is given by

\[
P(\mu, \nu, \omega) = \mathbb{E} \left[ M(\mu, \nu, \omega) e^{g} \left( P(z') + 1 \right) \right] | z|
\]

and let \( J(\mu, \nu, \omega) = M(\mu, \nu, \omega) e^{g} \). I first discretize the variable \( g \) by using 32 Hermite quadrature points of a standard normal variable and appropriately adjusting for the conditional mean and conditional standard deviation. Let \( G(z) \) be the set of 32 points and \( (w_j, j = 1 \ldots 32) \) be the Hermite quadrature weights.

The price dividend function is approximated by a complete product of Chebyshev polynomials of order \( d = 20 \) denoted with \( T_i \). Since the Chebyshev polynomials are over the interval
I set upper and lower bounds for all the state variables that are sufficiently broad. The approximate price dividend function is expressed as follows:

$$
\tilde{P}(z) = \sum_{0 \leq i_1 + i_2 + i_3 \leq d} \beta(i_1, i_2, i_3) T_{i_1} \left( 2 \frac{\mu - \mu}{\frac{\sigma}{\mu} - 1} \right) T_{i_2} \left( 2 \frac{\sigma - \sigma}{\frac{\omega}{\sigma} - 1} \right) T_{i_3} \left( 2 \frac{\omega - \omega}{\frac{-\omega}{\omega} - 1} \right)
$$

To determine the unknown coefficients $\beta$ I use the Galerkin projection method. With $n$ state variables an approximation of order $d$ implies a number $m = \frac{(d+n)!}{n!d!}$ of unknown $\beta$ coefficients. I define the residual to be the Euler equation error,

$$
R(z|\beta) = \tilde{P}(z) - \sum_{g \in G(z)} J(z, g) \tilde{P}(z') w(g) - \sum_{g \in G(z)} J(z, g) w(g),
$$

and then use the Galerkin projections,

$$
P_i(\beta) = \sum_{z \in Z} R(z|\beta) T_i(z).
$$

for all the Chebyshev polynomials used in the approximation. $Z$ is the set of grid points constructed from the zeros of the $n$ Chebyshev polynomials of order $(1 + d)$. Since the number of projections is the same as the number of unknown coefficients, the coefficients are determined by setting all the projections to zero and solving the resulting system of equations. The residual function is linear in the coefficients and therefore the system of equations is also linear. The approximation error is defined to be,

$$
err(\beta) = \sup_{z \in \mathbb{R}^n} \frac{|R(z|\beta)|}{\tilde{P}(z)},
$$

which is solved numerically.
Table 1: Aggregate consumption growth parameters

<table>
<thead>
<tr>
<th></th>
<th>μ₀ (%)</th>
<th>σ₀ (%)</th>
<th>λ_µ</th>
<th>λ_σ</th>
<th>log-likelihood</th>
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<tbody>
<tr>
<td>g</td>
<td>0.3316</td>
<td>0.7895</td>
<td>0.7551</td>
<td>0.9403</td>
<td>-970.1422</td>
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<tr>
<td>(0.5027)</td>
<td>(0.1825)</td>
<td>(0.0706)</td>
<td>(0.0166)</td>
<td>.</td>
<td></td>
</tr>
</tbody>
</table>

Maximum likelihood estimates of the process

\[ g_{t+1} \mid \mathcal{F}_t \sim N(\mu_t, \sigma_t) \]

where,

\[
\begin{align*}
\mu_{t+1} &= \lambda_\mu \mu_t + (1 - \lambda)g_{t+1}, \\
\sigma_{t+1}^2 &= \lambda_\sigma \sigma_t^2 + (1 - \lambda_\sigma)(g_{t+1} - \mu_t)^2.
\end{align*}
\]

The parameters were estimated using quarterly aggregate consumption data of non-durables and services from the first quarter of 1947 to the last quarter of 2007.

Table 2: Model parameters

<table>
<thead>
<tr>
<th></th>
<th>Homogeneous</th>
<th>Heterogeneous</th>
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<tbody>
<tr>
<td>δ</td>
<td>0.9475</td>
<td>1.087</td>
</tr>
<tr>
<td>γ</td>
<td>4.0</td>
<td>4.0</td>
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<tr>
<td>η</td>
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<td>-4.0</td>
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<tr>
<td>κ</td>
<td>0.07</td>
<td></td>
</tr>
<tr>
<td>λ_x</td>
<td>0.95</td>
<td>0.95</td>
</tr>
<tr>
<td>λ_µ</td>
<td>0.7551</td>
<td>0.7551</td>
</tr>
<tr>
<td>λ_σ</td>
<td>0.9403</td>
<td>0.9403</td>
</tr>
<tr>
<td>μ₀ (%)</td>
<td>0.3316</td>
<td>0.3316</td>
</tr>
<tr>
<td>σ₀ (%)</td>
<td>0.7895</td>
<td>0.7895</td>
</tr>
<tr>
<td>ω₀</td>
<td>0.4119</td>
<td>0.4119</td>
</tr>
</tbody>
</table>

The parameters \((\lambda_\mu, \lambda_\sigma, \mu_0, \sigma_0)\) were estimated using maximum likelihood as indicated in table 1. \(\lambda_x\) was chosen to match the persistence in the price-dividend ratio. \(\delta\) and \(\eta\) were chosen to match the level and variability of the price-dividend ratio until the end of the 1980’s.
Table 3: Quarterly Statistics (1947Q1-2008Q1)

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Homogeneous</th>
<th>Heterogeneous</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu(pd)$</td>
<td>4.8163</td>
<td>4.5617</td>
<td>4.7908</td>
</tr>
<tr>
<td>$\sigma(pd)$</td>
<td>0.4365</td>
<td>0.2296</td>
<td>0.4237</td>
</tr>
<tr>
<td>$\mu(r^f)%$</td>
<td>0.2450</td>
<td>1.9684</td>
<td>1.0633</td>
</tr>
<tr>
<td>$\sigma(r^f)%$</td>
<td>0.8651</td>
<td>0.6864</td>
<td>0.8043</td>
</tr>
<tr>
<td>$\rho(pd, r^f)$</td>
<td>0.1317</td>
<td>0.0661</td>
<td>-0.3828</td>
</tr>
<tr>
<td>$ACF_{pd}(1)$</td>
<td>0.9305</td>
<td>0.9348</td>
<td>0.9641</td>
</tr>
<tr>
<td>$ACF_{pd}(4)$</td>
<td>0.9136</td>
<td>0.6806</td>
<td>0.8120</td>
</tr>
</tbody>
</table>

The model implied statistics are generated using the observed history of consumption growth. $\mu(x)$ and $\sigma(x)$ denote the sample mean and standard deviation of variable $x$. $\rho(x, y)$ denotes the sample correlation and $ACF_x(n)$ is the autocorrelation of $x$ for lag $n$.

Figure 1: Scatter plot of the price-dividend ratio with the fitted conditional volatility of consumption growth. The correlation is $-0.78$. 
Figure 2: Time series of the state vector.

![Time series of state vector](image)

Figure 3: Homogeneous economy log price dividend ratio and risk free rate.

![Log price-dividend ratio and risk-free rate](image)
Figure 4: Homogeneous economy log price dividend ratio

Figure 5: Homogeneous economy risk-free rate
Figure 6: Homogeneous economy conditional mean and volatility of expected excess return

Figure 7: Heterogeneous economy conditional mean and volatility of expected excess return.
Figure 8: Heterogeneous economy log price dividend ratio and risk free rate.
Figure 9: Heterogeneous economy log price dividend ratio.

Figure 10: Heterogeneous economy risk-free rate.
Figure 11: Homogeneous economy expected excess return

Figure 12: Heterogeneous economy expected excess return.
Model time-series of cross-sectional volatility of beliefs, $\nu$ and the Hodrick-Prescott filter trend of the time-series of the cross sectional volatility of one quarter predictions on real consumption growth from the Survey of Professional Forecasters as provided by the Federal Reserve Bank of Philadelphia.