Disagreement, Habit and the Dynamic Relation Between Volume and Prices

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Abstract

Trading volume is known to be strongly related to asset prices; volume is usually high when returns are high and during periods of high return volatility. This article develops a model economy that is able to account for the empirically observed joint dynamics of volume and prices while at the same time yielding realistic price processes with the observed equity premium, return volatility, “leverage” effect and other well established price characteristics. The model economy features two risk-averse agents with opinion differences about economic fundamentals and external habit preferences. The key innovative elements are the introduction of an intermediate period where agents receive information, the time varying disagreement with respect to the interpretation of this information and the assumed asset structure with which agents construct their optimal investment strategies. In accounting for several key characteristics of the financial markets, the paper generates predictions about how speculative trading and hedging demands vary over time and the relation between the type of assets traded and the positive correlation between volume and returns.

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1 Introduction

Explaining asset prices and trading volume is essential to understanding the workings and the efficiency of the financial markets. In addition, prices and volume have been found both on the aggregate and in the cross section to be strongly related to each other; in particular there is a contemporaneous correlation between volume and return volatility and also high volume is associated with high returns.\(^1\) Karpoff (1987) reviews the early empirical and theoretical literature on these two empirical regularities.\(^2\) The asset pricing literature, however, has been unable so far to understand this relation mainly because the fundamental economic factors that drive either prices or the extraordinarily high trading volume are at best unclear. Furthermore, to the author’s knowledge, there has not been an attempt to explain both in a unified framework. This paper presents a new dynamic asset pricing model of heterogeneous agents that is able to jointly account for the observed price dynamics and the relation between prices and volume. There are several key elements to this model but the main point of departure is that the primary factors behind prices and trading are the time-varying risk-aversion and the disagreement about the interpretation of common information relating to economic fundamentals, respectively.

Trading can be attributed to time varying hedging needs and to speculation. Reconciling, however, the extraordinary trading volume with time varying hedging needs requires a leap of faith. Speculation is a more promising explanation, but it implies that investors hold different beliefs about prices. The difference in beliefs arises either because of different information sets and frictions that do not allow the information aggregation or because investors hold different opinions about the common information.\(^3\) This paper follows the second route. Kandel and Pearson (1995) provide evidence that analysts disagree on the interpretation of public information and also that trading volume is higher during public announcements.\(^4\) This evidence suggests that one source for both the heterogeneity of beliefs and trading is indeed disagreement.\(^5\) The paper attempts to answer how disagreement affects trading, how it affects prices, in what way disagreement connects prices and trading and finally show certain conditions under which the empirical relation between volume and prices can be generated.

The first implication of disagreement is that it is positively related to trading. Varian (1989) in an

\(^{1}\)Interestingly the two observed patterns are reflected in the adages “it takes volume to move prices” and “volume is heavy in bull markets and light in bear markets”.

\(^{2}\)Further studies on the empirical volume-volatility relation are those of Schwert (1989) and Gallant et al. (1992). Other empirical studies that analyze the dynamic relation between trading and returns for either the cross-section or the aggregate stock market include Campbell et al. (1993), Llorente et al. (2002), Chordia et al. (2007) and Griffin et al. (2007) while Lo and Wang (2000) offers a further list of references. Further evidence is provided by Hong and Stein (2007) that show the striking correlation between annual changes to the level of the market and changes in the turnover of the NYSE.

\(^{3}\)As Cochrane (2005) puts it financial markets are ‘at bottom markets of information (or, some might say, opinion)’.

\(^{4}\)Similar evidence provided by Chae (2005) shows that abnormal volume increases significantly upon earnings announcements.

\(^{5}\)Disagreement in general refers to the interpretation of information which in turn generates heterogeneity in beliefs. In particular disagreement affects the conditional volatility of the difference in beliefs (or expectations). To avoid confusion in this model when disagreement will be defined it will be being referred to as disagreement intensity.
Arrow-Debreu setting shows that higher difference in beliefs leads to more trading due to greater wealth reallocation. Harris and Raviv (1993) and Scheinkman and Xiong (2003) connect trading with the number of changes in the relative beliefs. In general speculative positions change to the extent that the relative beliefs of investors change and the faster or bigger the changes are, the more trading there is. In a dynamic setting the rate at which the relative beliefs change depends on the level of disagreement about new information. Therefore, changes in the level of disagreement implies changes in the amount of speculative turnover as also shown in Banerjee and Kremer (2010). These results indicate that trading due to disagreement is possibly related to a number of factors which are: (i) the level of wealth dispersion in the case where speculative positions are wealth dependent, (ii) the level of belief heterogeneity which affects wealth reallocation and (iii) the level of disagreement that affects how relative beliefs change. The trading implications of all these factors are studied in the model we present. The related question is then how these factors affect prices and in what way they connect prices and trading.

Typically, the trading models of disagreement make simplistic assumptions that result in prices whose variation is due to changing expectations. Two problems arise because of this: First belief heterogeneity has additional pricing implications as provided by the relevant literature. For example risk-aversion and belief heterogeneity cause prices to be discounted and prices not only depend on the individual beliefs but also on the wealth distribution. Secondly, the asset pricing literature has moved away from the hypothesis that prices vary primarily due to changing expectations but they rather seem to vary due to changing risk-characteristics like the level of risk-aversion or the amount of risk. Consequently, in order to understand how disagreement connects prices and trading its effects have to be studied under realistic assumptions and within a model that is able to provide realistic price processes. The model we present achieves this by assuming power utility preferences with external habit that result in a time-varying risk-aversion.

The trading and pricing implications of the preference assumption are in turn very interesting. For example, we will show that disagreement in the presence of habit preferences creates a significant negative relation between belief heterogeneity and the equity premium. Also the time-varying risk aversion affects trading significantly for speculative and hedging purposes. Effectively, speculative trading decreases and hedging trading increases with risk-aversion which creates a negative relation between return volatility and the speculative part of trading. Belief heterogeneity itself, which is determined by both wealth dispersion and the difference in beliefs, generates a positive relation between trading and volatility but the marginal effect of belief heterogeneity on volatility is small.

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6Several studies have examined the pricing implications of belief heterogeneity a few of which are Abel (1990), Basak (2005), Detemple and Murthy (1994), Zapatero (1998), Jouini and Napp (2006b) and David (2008).

7See for example Jouini and Napp (2006a), Gallmeyer and Hollifield (2008) and Dumas et al. (2009). Interestingly, the bubble pricing component found by Scheinkman and Xiong (2003) depends on the assumption of risk-neutrality rather than the assumption of short-sale constraints.

8See for example Campbell and Cochrane (1999).

9It is common to also call speculative trading as informational trading even though informational trading typically refers to the type of speculative trading arising from informational asymmetry rather than disagreement. Hedging trading, is also referred to as liquidity or non-informational trading.
The strong relation between volume and volatility is generated in this model by allowing disagreement to be time-varying. The significance of this assumption to the volume-volatility relation has been shown very recently by Banerjee and Kremer (2010). However, it will be shown that this assumption alone does not suffice due to the negative correlation between volume and volatility created by risk-aversion. Eventually, we will assume that disagreement is positively correlated with risk-aversion. Even though the assumption is ad hoc in this model the possibility that disagreement is related to the state of the economy, for example due to psychological factors, is at least interesting.

The other important characteristic of the price-volume relation is the positive correlation between returns and volume. We point out that this does not refer to the relation between the time-varying trend of volume and the level of prices but the positive correlation between the volume innovations and the changes to the level of prices. Despite its striking statistical significance the theoretical literature has largely neglected this fact. The reason is not surprising because a dynamic stationary equilibrium with trading does not create this asymmetric behavior in turnover. For example, turnover is proportional to the change in the level of belief heterogeneity irrespective of whether it is an increase or a decrease. The model is able to generate this correlation by introducing two new features. The first is an intermediate period where agents receive information about the changing economic fundamentals and the second is that the asset structure that dynamically completes the markets resembles one with a set of call options. The resulting behavior is that every period the relatively optimistic agent sells the market security to buy options and brings additional trading volume if the good state of nature realizes and the return is high. Otherwise, the previously optimistic agent needs to realize her losses and the resulting trading volume is much smaller. Jennings et al. (1981) attributed the return-volume relation to the asymmetric costs of long and short positions with a conjecture that as a result prices are less responsive to bad information with less trading. Karpoff (1987) says that this explanation, even though not perfectly rational, is consistent with the fact that a return-volume relation is absent in the futures market data. The explanation that we develop in this paper will not imply any kind of irrationality and will not assume market incompleteness. However, the explanation may be related to the additional costs in shorting stocks since it is possible that investors because of these asymmetric costs choose to complete the markets with call options as will be assumed in the model.

The rest of the paper is structured as follows: Section 2 reviews some of the theories related to trading. The model is presented in Section 3 and the equilibrium is derived in 4. The equilibrium without time-varying disagreement is analyzed qualitatively in Section 5 and the model is calibrated in Section 6. The calibrated model is used to build insight about the relative importance of all the elements and the general behavior of the model. Section 7 introduces time-variability in disagreement and the model is again calibrated and analyzed. Section 8 concludes the study.

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10 The pricing significance of disagreement, a separate effect to the difference in beliefs, has been realized by Dumas et al. (2009) even though in their model it is not modeled as a separate factor.

11 Hong and Stein (2007) show that the correlation in the annual data is around 0.5. Similar data will be used for the model calibration.
2 Theories Related to Trading

Theory under rational expectations views asset prices as reflecting all available information which is uniformly interpreted by everyone and any information whether private or public is reflected instantaneously in asset prices. As a result Milgrom and Stokey (1982) show that no trading is required to incorporate any new information.\footnote{Blume et al. (2006) show that the no-trade theorem fails to hold when markets are incomplete since new arrival of information offers new opportunities for risk-sharing.} A similar result is obtained by Judd et al. (2003) that show that in an economy with preference heterogeneity and global completeness agents do not trade beyond the first period.\footnote{Berrada et al. (2007) show that in a continuous-time framework with dynamic completeness, preference heterogeneity results in continuous trading coming from continuously changing hedging needs.} The insight gained from the no-trade theorems have spurred the literature that tries to explain trading. The first stream of research views trading as driven by time-varying liquidity (or hedging) needs that arise due to some form of time-varying uncertainty. In the noise-trading literature the time-varying uncertainty is related to informational noise in which case private information also generates informational trading since the positions are not fully revealing.\footnote{See for example Grossman and Stiglitz (1980), Hellwig (1980), Diamond and Verrecchia (1981), Kyle (1985), Admati and Pfeiderer (1988), Grrundy and McNichols (1989), Foster and Viswanathan (1990), Foster and Viswanathan (1993), Long et al. (1990), Kim and Verrecchia (991a), Kim and Verrecchia (991b), Shalen (1993) and Wang (1994). A more extensive discussion of this literature can be found in Harris and Raviv (1993).} The noise-trading literature has investigated extensively this line of thought and has generated certain insights as to how rational traders behave under informational uncertainty. However, the main shortcoming is that the resulting patterns of volume and prices depend heavily on the exogenous noise process and the magnitude of trading due to liquidity needs.

A second way of generating trade is by relaxing the assumption of the uniform interpretation of information.\footnote{Examples of this literature are Varian (1989), Harris and Raviv (1993), Kandel and Pearson (1995), Hong and Stein (2003), Scheinkman and Xiong (2003), Cao and Yang (2009) and Banerjee and Kremer (2010).} There are several conceptual advantages to this investigative route: Firstly, there is ample evidence, both empirical and anecdotal, that indeed economic agents may interpret information differently. Secondly, one needs to assume a certain structure for the disagreement process, that is the type of bounded rationality that creates it, and this structure endogenously generates both trading and price effects. Thirdly, evidence shows that information release generates trade which in the context of disagreement is an outcome of both convergence and divergence of beliefs, where convergence and divergence are endogenously generated by the assumed structure of disagreement. The noise-trading literature on the other hand views public information release as decreasing the informational uncertainty generated by an exogenous process and can generate trade only in the presence of private information as shown by He and Wang (1995). Finally, unlike in several noise-trading models, speculative trading due to disagreement does not require the existence of hedging trading since it is caused by the differences in beliefs. In fact in our model hedging trading arises from the additional price volatility generated by the speculative trades.

The types of theoretical arguments found in the literature concerning the relation between prices and volume depend on the reason behind trading. The noise-trading literature typically establishes
a mechanical connection between volume and prices driven by the exogenous noise process. Blume et al. (1994) relates volume with absolute price changes because volume itself is associated with higher precision in the information revealed and therefore bigger price adjustment. Wang (1994) explains this relation in a model with asymmetric information where the informed investors enjoy private investment opportunities. As a result the uninformed investors are unable to extract all the information from prices and the drop in price required by them to buy the stock is related to the magnitude of informational asymmetry. One related matter is that these models as well as a few others in this literature imply that volume itself contains additional information which in turn implies a causal relation in the direction of volume to prices for which the evidence is at best mixed.\footnote{Papers with implications about the informational content of volume include Grundy and McNichols (1989), Kim and Verrecchia (1991a) and Barron and Karpoff (2004). Studies that look at the volume-price causal relation are Hiemstra and Jones (1994), Chen et al. (2001), Lee and Rui (2002) and Chuang et al. (2009) among others.}

The literature on opinion differences generates a different connection between volume and prices. Harris and Raviv (1993) and Scheinkman and Xiong (2003) propose similar arguments about the positive correlation between volume and volatility using models with risk-neutral agents and short-selling constraints. In both models the most optimistic agent holds the stock and the stock changes hands when the beliefs ‘cross’. Consequently more ‘crossings’ imply more trading and higher volatility. Even though not modeled explicitly in these papers the argument put forth hints on a time varying disagreement, what we will be referring to as disagreement intensity. Banerjee and Kremer (2010) also assume explicitly a time-varying disagreement and draw the link between volume and volatility. Interestingly, a similar type of argument was proposed in the microstructure literature with the mixture of distribution hypothesis by which there is a distinction between event time and calendar time.\footnote{See for example Harris (1986) and Harris (1987).} This hypothesis states that if conditional on new information there is trading and a price change then volatility and volume will be related if the number of information releases in a given time interval is random. The time-varying disagreement essentially provides an economic basis for this hypothesis because disagreement can be time varying either because the amount of information released in a given period varies or because the impact of information varies; either way the effect is the same. However, we will shown that this argument does not easily go through without additional assumptions in an asset pricing model with more realistic pricing predictions.

One shortcoming that is common to both schools of thought is that most theories were developed within partial equilibrium frameworks. Consequently, they neglect individual wealth concerns; the noise-trading literature resorts to exogenous liquidity shocks whereas the disagreement literature leaves out liquidity trading all together. The portfolio choice literature for example has been developed on the realized need of constantly rebalancing the hedging positions as a response to the changing investment opportunity set. It becomes clear, therefore, that in order to generate an insight on prices and volume a framework is needed in which all the different sides, that is speculative trades, hedging trades and prices are endogenously determined. The model we present next was...
developed with this in mind.

3 The Model

The model is of an infinitely lived endowment economy with a single consumption good where two risk-averse agents disagree on the time-varying conditional mean of aggregate endowment growth. The beliefs of the two agents constantly change and do not converge because many periods they disagree on the interpretation of new information that relates to changes in economic fundamentals.

3.1 Aggregate Endowment

Time is discrete and infinite, \((t = 0, 1, 2, \ldots)\). Production is not modeled while we assume that the aggregate endowment \(Y\) grows every period according to:

\[
\ln \left( \frac{Y_{t+1}}{Y_t} \right) = \mu + \sigma_{\epsilon_t} Y_t, \tag{1}
\]

where the shock \(\epsilon_y\) is distributed according to:

\[
\epsilon_{y_{t+1}} = \begin{cases} 
+1, & \text{w. probability } \pi_t, \\
-1, & \text{o/w. }
\end{cases} \tag{2}
\]

The probability of the good endowment state \((\epsilon_y = +1)\) is time varying and follows an autoregressive process:

\[
\pi_{t+1} = \phi_\pi \pi_t + \frac{1 - \phi_\pi}{2} (1 + \epsilon_{\pi_{t+1}}), \tag{3}
\]

where \(\phi_\pi \in (0, 1)\) and

\[
\epsilon_{\pi_{t+1}} = \begin{cases} 
+1, & \text{w. probability } 1/2, \\
-1, & \text{o/w. }
\end{cases} \tag{4}
\]

Under these assumptions we have that \(\pi\) is always in the set \((0, 1)\), with unconditional mean of \(1/2\) and unconditional volatility of \(0.5 \sqrt{\frac{1 - \phi_\pi}{1 + \phi_\pi}}\). The aggregate endowment growth has unconditional moments of \(\mu\) and \(\sigma\), and first lag autocorrelation of \(\phi_\pi \frac{1 - \phi_\pi}{1 + \phi_\pi}\), which is always positive and small. The volatility of the conditional mean of aggregate endowment growth is equal to \(\sigma \frac{1 - \phi_\pi}{1 + \phi_\pi}\) and hence the uncertainty about the conditional mean is decreasing in \(\phi_\pi\). Below it will be assumed that the conditional mean of endowment growth is unobservable by the agents of this economy and it will be the basis for generating disagreement.


### 3.2 Information and Beliefs

There are two agents in this model, \( i = 1, 2 \), that have incomplete and common information about the state of the economy. Every period they observe the aggregate endowment growth but do not observe the probability \( \pi \) or the shocks to the probability, \( \epsilon^\pi \). However, with a lag of half a period at \( t + 0.5 \), agents receive two signals \( s_{t+0.5} := (s^1_{t+0.5}, s^2_{t+0.5}) \) that carry information about \( \epsilon^\pi_t \). The two signals take values of either +1 or -1 just like \( \epsilon^\pi \). This assumption of having information revealed in the middle of every period is central to this model in generating the positive correlation between price changes and volume. Independently, however, the assumption creates a realistic setting where incomplete information is revealed in addition to hard macroeconomic data.

The two agents in order to form their beliefs about the future of the economy they first need to form beliefs about the current state of the economy and in particular the value of \( \pi_t \). Since they do not observe it they use first the signals \( s_{t+0.5} \) in order to form their conditional beliefs about \( \epsilon^\pi_t \). Then, the conditional beliefs on \( \epsilon^\pi_t \) together with the prior beliefs on \( \pi_t \) are used in order to form the beliefs on \( \pi_t \). We assume that agents disagree in the way they interpret the signals. Specifically, agents have the following beliefs about the informativeness of the two signals:

\[
\mathbb{P}_i(s^1, s^2 | \epsilon^\pi) = \left( \frac{1}{2} + \rho \right) \frac{1}{2} \mathbb{1}(s^i = \epsilon^\pi) + \left( \frac{1}{2} - \rho \right) \frac{1}{2} \mathbb{1}(s^j = \epsilon^\pi),
\]

where \( \rho \in [-\frac{1}{2}, \frac{1}{2}] \), \( i, j = 1, 2 \) and \( j \neq i \). \( \mathbb{1}(A) \) is the indicator function that takes the value of one if the statement \( A \) is true and zero otherwise.\(^{18}\) \( \mathbb{P}_i \) denotes the probability measure that reflects the beliefs of agent \( i \). The way assumption (5) is constructed states that each agent forms beliefs about the joint realization of the signals conditional on the realization of \( \epsilon^\pi \). Further, as long as \( \rho \) is different than zero the two agents have different conditional beliefs and because of this interpret the new information differently. For example when \( \rho \) is positive the first agent’s beliefs are more affected by the first signal whereas the beliefs of the second agent are more affected by the second signal.\(^{19}\) In this sense in the example of \( \rho \) being positive the first agent believes that the first signal carries more information whereas the second agent believes the opposite.

Assumption (5) further states that both agents place zero probability that both signals are wrong and both believe that there is 1/2 probability that both signals are correct. The two agents differ in their interpretation of the new information when the two signals are conflicting and the quantity

\[^{18}\text{Another way of presenting the same assumption is as follows:}\]

\[
\begin{array}{c|cc}
 s^1 = s^2 = \epsilon^\pi & \mathbb{P}_1 & \mathbb{P}_2 \\
 s^1 = \epsilon^\pi \neq s^2 & \frac{1}{4} + \rho/2 & \frac{1}{4} - \rho/2 \\
 s^1 \neq \epsilon^\pi = s^2 & \frac{1}{4} - \rho/2 & \frac{1}{4} + \rho/2 \\
 s^1 \neq s^2 \neq \epsilon^\pi & 0 & 0 \\
\end{array}
\]

\[^{19}\text{This type of disagreement is sometimes attributed to overconfidence similar to Daniel et al. (1998) or Gervais and Odean (2001). However, it is envisioned here that disagreement possibly stems from the fact that agents are uncertain about the underlying structure of the economy in which case several models might be statistically indistinguishable.}\]
\( \rho \) has a value other than zero. In the example mentioned above when \( \rho \) is positive the first agent places a higher probability than does the second agent on the event that the first signal is correct and the second signal is wrong. Therefore, in such a case if the signals are conflicting the first agent will place more weight on the first signal whereas the second agent will do the opposite. This is the assumption that perpetuates the heterogeneity of beliefs in this economy.

The two signals \( s \) are meant to represent different information about the economy, for example two macroeconomic indicators, but do not reveal the exact state of the economy.\(^{20}\) Unconditionally the two signals are believed by both agents to be independent with each signal carrying \( 1/2 \) probability of being correct, that is having the same value as \( \epsilon^\pi \). This means that the two agents have uniform beliefs about the realization of the two signals and in particular each realization has a probability of \( 1/4 \). The bounded rationality imposed is that the two agents do not uniformly update their beliefs about the true generating process of the signals.

\( \rho \) determines that disagreement between the two agents in the interpretation of new information and the higher \( \rho \) is in absolute value the higher is the disagreement when the signals are conflicting as shown in the following remark.\(^{21}\) The following remark gives a clear interpretation of \( \rho \):

**Remark 1.** The disagreement about the informativeness of a signal \( s^i \) is given by,

\[
|\mathbb{P}^1(s^i|\epsilon^\pi) - \mathbb{P}^2(s^i|\epsilon^\pi)| = |\rho|.
\]

where \( i \in \{1, 2\} \).

We will be referring to the absolute value of \( \rho \) as disagreement intensity which for now it is assumed to be constant. Later it will be assumed to be stochastic which will be central in producing the positive relation between volume and return volatility. Remark 1 shows that if \( \rho \) is not zero then each agent thinks that one signal is more informative than the other, whereas if \( \rho \) is zero, that is no disagreement, then both agents consider both signals equally informative and they interpret the new information in the same way. The following lemma makes this point even clearer.

**Lemma 1.** For each agent \( i \in \{1, 2\} \) the expectation of \( \epsilon^\pi \) given the signals \( s \) is given by:

\[
\mathbb{E}^i(\epsilon^\pi|s^1, s^2) = \left( \frac{1}{2} + \rho \right)s^i + \left( \frac{1}{2} - \rho \right)s^j, \quad j \neq i.
\]

Lemma 1 states that an agent’s beliefs about the shock \( \epsilon^\pi \) is a weighted average of the signals received. However, because of assumption (5) the two agents put the opposite weights on the two pieces of information and the difference in those weights is determined by the disagreement intensity.

The common information available to the two agents is the history of the shocks \( \epsilon := (\epsilon^y) \) observed at the beginning of every period and the history of the two signals \( s \) observed in the middle of every

\(^{20}\)To be more precise the signals \( s \) should be regarded as changes to macroeconomic indicators.

\(^{21}\)Scheinkman and Xiong (2003) consider a similar assumption where each agent believes that only one signal is informative as a proxy for overconfidence.
period. Later when the disagreement intensity will be assumed to be stochastic $\epsilon$ will also include the innovations to $\rho$. The information set at the beginning of a period $t$ is denoted with $\mathcal{F}_t$ and at the middle of a period with $\mathcal{F}_{t+0.5}$. Implicit from this assumption is that agents know the structure and the parameters of the aggregate endowment process and the disagreement intensity $\rho$ but they do not update their beliefs about the signal generating process.

Each agent enters a period $t$ with certain beliefs about $\pi_{t-1}$ that is based on the information up to that period and the way the agent interpreted that information. Given those beliefs an agent’s expectations about the previous value of $\pi$ is denoted as follows:

$$\pi_t^i := \mathbb{E}^i (\pi_{t-1}|\mathcal{F}_t).$$

Once agents observe the new signals at time $t+0.5$ they form their beliefs about the new state of $\pi$ according to,

$$\pi_{t+0.5}^i := \mathbb{E}^i (\pi_t|\mathcal{F}_{t+0.5}) = \phi^i_\pi \pi_t^i + \frac{1 - \phi^i_\pi}{2} \left[1 + \mathbb{E}^i(\epsilon^i_t|\mathcal{F}_{t+0.5})\right],$$

where the conditional expectation of the shock $\epsilon^i$ is given in Lemma 1.

The two agents in forming their expectations about the future realizations of the aggregate endowment growth they only require their current predicted value, for example $\pi_{t+0.5}^i$, because their expectations about the future are independent of the uncertainty they have about $\pi$ as stated in the next lemma.

**Lemma 2.** Agents form expectations according to the following beliefs about the realization of the next period’s shock,

$$\mathbb{P}^i(s_{t+0.5}, \epsilon_{t+1}^y|\mathcal{F}_t) = \frac{\mathbb{P}^i(\epsilon_{t+1}^y|\mathcal{F}_{t+0.5})}{4} \mathbb{P}(\epsilon_{t+1}^y \epsilon_{t+1}^y|\mathcal{F}_{t+0.5}),$$

where

$$\mathbb{P}^i(\epsilon_{t+1}^y|\mathcal{F}_{t+0.5}) = \pi_{t+0.5}^i \mathbbm{1}\{\epsilon_{t+1}^y = 1\} + (1 - \pi_{t+0.5}^i) \mathbbm{1}\{\epsilon_{t+1}^y = -1\}$$

and $\pi_{t+0.5}^i$ is a function of $\pi_t^i$, $s_{t+0.5}$ and $\rho$ as given by Lemma 1 and equation (7).

The uncertainty about $\pi$ only matters when agents update their beliefs once they observe the realization of the aggregate endowment growth. For reasons of computational simplicity and since it is of secondary importance, we assume that the uncertainty is constant over time. Specifically, we assume that the beliefs of the two agents after observing the two signals is given by:

$$\pi_t|\mathcal{F}_{t+0.5} \sim \text{Beta} \left[\pi_{t+0.5}^i \frac{K}{1 - K}, (1 - \pi_{t+0.5}^i) \frac{K}{1 - K}\right],$$

(8)
where $\kappa \in (0, 1)$. Hence, once the realization of the aggregate endowment growth is observed agents update their beliefs according to,

$$\pi_{t+1}^i = \kappa \pi_{t+0.5}^i + (1 - \kappa)I \{\varepsilon_{t+1}^i = +1\}. \quad (9)$$

Parameter $\kappa$ turns out to be unimportant for the main results of the paper.

### 3.3 Consumption and Preferences

Agents in this economy consume every half period and $C_i$ denotes the consumption process of agent $i$. They have power utility preferences over streams of consumption in surplus of an external habit similar to Sundaresan (1989), Constantinides (1990), Detemple and Zapatero (1991), Campbell and Cochrane (1999) and others:

$$U_i(C_i, X) = \mathbb{E}_t^i \left\{ \sum_{t=0}^{\infty} \delta^t \left[ u(C_i^t, X_t) + \sqrt{\delta} \cdot u(C_i^{t+0.5}, X_t) \right] \Big| \mathcal{F}_0 \right\}, \quad (10)$$

where $\delta \in (0, 1)$ is the subjective discount factor. The subperiod utility is given by

$$u(C, X) = \frac{(C - \beta X)^{1-\gamma} - 1}{1 - \gamma}, \quad (11)$$

where $\gamma > 0$ is the utility curvature parameter and $X$ is the external habit common to both agents. The coefficient of relative risk aversion is state dependent and is equal to $\gamma C/(C - \beta X/2)$. Habit preferences are known to be able to generate realistic price processes. For example, it has been shown that this type of preferences is able to reconcile the high equity premium and the high return volatility with a low curvature parameter and low real aggregate risk. The external habit can be shut down by setting $\beta$ to zero in which case the preferences simplify to the standard power utility preferences.

External habit is very important in generating realistic characteristics for prices and therefore it is essential in our analysis of their relation with turnover. We have already noted that the price-turnover predictions of a theoretical model might not be robust to a setting with more realistic price characteristics and in particular with realistic return volatility. This in fact turns out to be the case with the volume-volatility relation. We will show later that heterogeneity in beliefs does generate a positive relation between return volatility and volume but this relation vanishes with the introduction of external habit. For this reason another factor which is the time-varying disagreement intensity will be needed to generate theoretically this prediction.

In order to keep things simple and comparable to the case where consumption takes place only once every period, we assume that $X$ is constant throughout a given period. As it is typical in the literature, we assume that the external habit is a weighted average of past aggregate consumption,
as for example in Ryder and Heal (1973), and it is given by:

\[ x_{t+1} = \phi_x x_t + (1 - \phi_x) y_t, \]  

(12)

where \( x \) and \( y \) are the natural logarithms of the corresponding variables. In the assumed preferences \( X \) is divided by two since consumption is divided over two instances within a period whereas it changes every whole period. Even though it is typical with this form of habit preferences to assume that habit is internal, in order to simplify the model we rather assume it to be external as in the “keeping up with the Joneses” preferences of Gali (1994).

Agents are able to consume in the middle of the period out of what they decide to store in the beginning of the period. We further assume that agents can store without costs the consumption good between time \( t \) and \( t + 0.5 \), but not between \( t + 0.5 \) and \( t + 1 \). The assumption that agents consume every half period does not change the results from the case where they consume every whole period, but this assumption makes the exposition simpler. In equilibrium it turns out that each agent consumes what she stores even though the two agents are able to trade both the consumption good and the financial assets in the middle of every period.

3.4 Financial Markets

Between time \( t \) and \( t + 0.5 \) and between time \( t + 0.5 \) and \( t + 1 \) there are four and two possible states of nature respectively, that the two agents can observe. Between the beginning of a period and the intermediate period the four states are generated by the two signals \( s \) and between the intermediate and the end of a period the states are generated by \( \epsilon := (\epsilon^y). \) Agents trade every half period on equal number of independent assets as the states of nature so that markets are always dynamically complete.

Since markets are complete the market structure does not have any effect either on equilibrium allocations or on equilibrium prices. However, the asset structure does have an effect on the trading volume and its relation to price changes. The first asset that is always available for trading is an infinitely lived stock that pays the aggregate endowment and is in unit net supply. The stock is referred to as the market or the market security, its price is denoted with \( P \) and its price dividend ratio is denoted with \( \tilde{P} = P/Y \). The assets that dynamically complete the market are half-period zero net supply contingent claims that pay the aggregate endowment of the period only in one of the immediate future states.

The benchmark asset structure is such that there do not exist contingent claims that pay-off in the state \( s_{t+0.5} = (-1, -1) \) between \( t \) and \( t + 0.5 \) and in the state \( \epsilon^y_{t+1} = -1 \) between \( t + 0.5 \) and \( t + 1 \). The payoff matrices are denoted with \( R \). A given column gives the payoff of the corresponding asset in the four immediate future states and a given row gives the payoff of all the assets in the future states.

\( ^{22}\)Later with the introduction of another source of uncertainty that will drive \( \rho \) there will also be four states of nature in the second subperiod and \( \epsilon = (\epsilon^y, \epsilon^\rho) \).
corresponding future state and is denoted with $R$. The payoff-dividend matrix of the benchmark asset structure is as follows:

$$
\tilde{R}_t = \begin{pmatrix}
\tilde{P}_{t+0.5}(s_{t+0.5} = (+1, +1)) & 1 & 0 & 0 \\
\tilde{P}_{t+0.5}(s_{t+0.5} = (+1, -1)) & 0 & 1 & 0 \\
\tilde{P}_{t+0.5}(s_{t+0.5} = (-1, +1)) & 0 & 0 & 1 \\
\tilde{P}_{t+0.5}(s_{t+0.5} = (-1, -1)) & 0 & 0 & 0
\end{pmatrix}, \quad \tilde{R}_{t+0.5} = \begin{pmatrix}
\tilde{P}_{t+1}(\epsilon_{t+1} = +1) & +1 & 1 \\
\tilde{P}_{t+1}(\epsilon_{t+1} = -1) & +1 & 0
\end{pmatrix}.
$$

The time subscript on the payoff matrix indicates the time when the set of assets are available for trade which is when the payoff matrix is known. The benchmark asset structure will be responsible for generating the positive correlation between the market security returns and the volume of trade. Other asset structures will be discussed also.

## 4 Financial Equilibrium

This section presents the financial equilibrium conditions, prices and holdings. The equilibrium is then expressed in a recursive form with respect to the appropriate state vector. The recursive characterization of prices allows the qualitative analysis in terms of the main factors that determine asset prices in equilibrium. The recursive expression of the equilibrium also allows for the characterization of the optimal holdings of the two agents.

### 4.1 Equilibrium Conditions

We consider first the optimization problem of each agent. Each agent every half period decides on her portfolio holdings that we denote with the vector $\bar{\theta}_i^t$. Further, every whole period each agent decides how much consumption good to store between time $t$ and $t + 0.5$, that we denote with $b_i^t$.

Agents can store the consumption good without cost from the beginning of every period until the intermediate period which implies that the half period interest rate at the beginning of every period will be zero. In the intermediate period agents are allowed to trade both the consumption good and financial assets. Let further $\mathbf{P}$ denote the vector of asset prices. The optimization problem of agent $i$ can then be written as follows:

$$
\max_{(\bar{\theta}^i, b^i)} U_i(C^i, X) \\
\text{s.t.} \quad C^i_t + b^i_t + \bar{\theta}^i_t \mathbf{P}_t \leq \bar{\theta}^i_{t-0.5} R_t \\
C^i_{t+0.5} + \bar{\theta}^i_{t+0.5} \mathbf{P}_{t+0.5} \leq \bar{\theta}^i_t R_{t+0.5} + b^i_t
$$

(\text{O})

Let us now define the notion of equilibrium.

**Definition 1.** Let $\mathcal{T} = \{0, 1, 2, \ldots\}$ and $\mathcal{X} = \{0, 0.5, 1, 1.5, \ldots\}$. A financial market equilibrium is a process of portfolio holdings $\{(\bar{\theta}^1_t, \bar{\theta}^2_t); t \in \mathcal{X}\}$, a process of consumption storages $\{(b^1_t, b^2_t); t \in \mathcal{T}\}$ and a process of asset prices $\{\mathbf{P}_t; t \in \mathcal{X}\}$ such that:
(i) Processes $\bar{\theta}^i$ and $b^i$ are optimal for each agent $i$, given optimization problem $(O)$ and the process of asset prices $P$;

(ii) Given optimal holdings $(\bar{\theta}^1, \bar{\theta}^2)$ financial markets clear for all $t \in \Sigma$. That is all holdings of the market security sum to one and all holdings of the contingent claims sum to zero.

Speculative bubbles and Ponzi schemes are excluded from this equilibrium even though we have not assumed explicitly the required conditions. A financial equilibrium is well known to generically deliver equilibria with efficient allocations to the extent that markets are always complete in which case the equilibrium is equivalent to a social planner equilibrium with stochastic weights.

Given a set of prices the optimization problem yields the following usual Euler conditions with respect to the holdings of an asset $j$:

\begin{align*}
P^j_t &= \sqrt{\delta} \mathbb{E}^i_t \left[ \left( \frac{2C^i_{t+0.5} - \beta X_t}{2C^i_t - \beta X_t} \right)^{-\gamma} R^j_{t+0.5} \right] \\
P^j_{t+0.5} &= \sqrt{\delta} \mathbb{E}^i_{t+0.5} \left[ \left( \frac{2C^i_{t+1} - \beta X_{t+1}}{2C^i_{t+0.5} - \beta X_{t+1}} \right) R^j_{t+1} \right],
\end{align*}

where $\mathbb{E}^i_t$ is shorthand for $\mathbb{E}^i(\cdot | \mathcal{F}_t)$ (and similarly $\mathbb{P}^i_t := \mathbb{P}^i(\cdot | \mathcal{F}_t)$). The above Euler conditions hold at all times, for all assets and for all agents. We note that the expectations are different for the two agents when they hold different beliefs about the probabilities of the future states. The expectations, however, in equation (14) are the same for both agents since the two agents have uniform beliefs about the signals $s$. This together with the market completeness implies that the marginal rates of substitutions between any whole period $t$ and $t + 0.5$ are the same for the two agents. Therefore, it means that the two agents have the same growth in their consumption surpluses between $t$ and $t + 0.5$. The consumption surplus of an agent refers to her consumption net of her habit. Summing up across the two agents gives the following equilibrium condition:

\begin{align*}
\frac{2C^i_{t+0.5} - \beta X_t}{2C^i_t - \beta X_t} &= \frac{C_{t+0.5} - \beta X_t}{C_t - \beta X_t},
\end{align*}

where $C$ denotes the total consumption of the two agents in a given subperiod.

The next first order condition is obtained out of choosing $b$ which is the consumption to be stored between $t$ and $t + 0.5$:

\begin{align*}
1 &= \sqrt{\delta} \mathbb{E}^i_t \left[ \left( \frac{2C^i_{t+0.5} - \beta X_t}{2C^i_t - \beta X_t} \right)^{-\gamma} \right].
\end{align*}

Conditions (16) and (17) and the market equilibrium condition $C_{t+0.5} = b^1_t + b^2_t$ imply that the marginal rate of substitution between $t$ and $t + 0.5$ is the same across states and equal to one.\(^{23}\)

\(^{23}\)The equilibrium condition $C_{t+0.5} = b^1_t + b^2_t$ is implied by the financial market clearing condition and the fact that at equilibrium the budget constraints of the two agents are satisfied with equality.
This in turn implies that,

\[ P_t^j = \mathbb{E}_t \left[ R_{t+0.5}^j \right], \tag{18} \]

for all assets \( j \) and for all whole periods \( t \). Equation (18) is a natural outcome of the fact that there is no real (consumption) risk in the economy between any time \( t \) and \( t + 0.5 \) and therefore no asset carries any risk-premium. Real risk would be produced if agents held different beliefs about the set of signals \( s \). In such a case the agents would choose to introduce endogenous consumption variation across states because of their differences in beliefs.\(^{24}\) Despite the complexity of such a setting it would be interesting to examine the variation of and the relation between prices and trading in such a case. This is left however for future research.

The equilibrium marginal rate of substitution between \( t \) and \( t + 0.5 \), which is independent of the realization of the signals \( s_{t+0.5} \), implies the following relation for the consumption surplus of each agent in period \( t \):

\[ C^i_t - \beta \frac{X_t}{2} = \frac{C^i_{t+0.5} + C^i_t - \beta X_t}{1 + \delta^{1/2\gamma}}. \tag{19} \]

The above relation shows that the assumed setting is equivalent to a setting where consumption takes place only every whole period and \( C^i_{t+0.5} \) is the agent’s consumption in a given period. It is also clear how the total consumption surplus for a given period, which is equal to \( Y_t - 2\beta X_t \), is divided between the two agents. It is natural in this setting to define a new variable, which is assumed to be stationary in equilibrium, and it is each agent’s share of the total consumption surplus:\(^{25}\)

\[ \alpha^i_t := \frac{C^i_t + C^i_{t+0.5} - \beta X_t}{Y_t - 2\beta X_t}. \tag{20} \]

It might seem peculiar to define the variable \( \alpha^i_t \) in terms of the consumption at \( t + 0.5 \) but it is obvious from the equilibrium condition (19) that \( C^i_{t+0.5} \) is \( \mathcal{F}_t \)-measurable, that is it is known at time \( t \). It is noted that \( \alpha^i_t \) becomes the consumption share in the absence of habit \( (\beta = 0) \).

One more variable needs to be defined that will be used as a state variable for the economy and is the logarithm of the habit normalized by the level of the aggregate endowment:

\[ \omega_t := \ln \left( \frac{X_t}{Y_t} \right). \tag{21} \]

It is necessary to normalize the level of the habit since it is non-stationary in this growing economy while \( \omega \) is, and it will be referred to as the habit-endowment ratio. Given its definition and the law

\(^{24}\)Pricing implications of extraneous risk have been examined by Basak (2000).

\(^{25}\)The stationarity of this variable, which is a proxy of the wealth held by an agent, is an outcome of the assumption that every agent on average is equally wrong with respect to the true probabilities of the states of nature. This means that no agent has superior beliefs so that in time she ends up with the entire wealth of the economy.
of motion of \( x \) assumed in (12) the law of motion of the habit-endowment ratio is given by:

\[
\omega_{t+1} = \phi_x \omega_t - (y_{t+1} - y_t).
\]  

(22)

From the above results and definitions the following lemma is derived.

**Lemma 3.** In equilibrium the market security satisfies the following Euler equation for each agent:

\[
\bar{P}_\tau = \delta \mathbb{E}_\tau \left[ \left( \frac{Y_{t+1}}{Y_t} \right)^{1-\gamma} \left( \frac{1 - 2\beta \exp(\omega_{t+1})}{1 - 2\beta \exp(\omega_t)} \right)^{-\gamma} \left( \frac{\alpha_{t+1}^i}{\alpha_t^i} \right)^{-\gamma} \left( \bar{P}_{t+1} + 1 \right) \right], \quad \tau = t, t + 0.5.
\]

Lemma 3 shows that the price of the market security satisfies the usual condition and the price changes in the middle of a period because of the change in beliefs. The intermediate period beliefs even though uncertain at time \( t \) do not generate any risk premium for the reason already mentioned which is the fact that the two agents are perfectly rational with respect to the way the intermediate beliefs are formed.

The stochastic discount factor (or marginal rate of substitution) in Lemma 3 is comprised of three parts.\(^{26}\) The first part is the standard stochastic discount factor of a homogeneous agent economy with standard power utility preferences. The second part is introduced because of the external habit and it vanishes when the parameter \( \beta \) is set to zero. The last part arises when agents have heterogeneous consumption processes due to a source of uncertainty that is not the fundamental, which in this model comes from the heterogeneity of beliefs.

### 4.2 Recursive Characterization of Prices and Holdings

The financial equilibrium can be written in a recursive form with a state vector that fully characterizes the equilibrium quantities. Lemma 3 indicates that the price-dividend ratio of the market security is a function of the individual beliefs, the habit-endowment ratio and the consumption surplus distribution. Let the share of the consumption surplus of the first agent be denoted with \( \alpha \) which is sufficient to describe the consumption surplus distribution in equilibrium. The beliefs and their evolution are characterized by the individual quantities \( \pi^i \) and the disagreement intensity \( \rho \).\(^{27}\) It remains to verify that the state vector \( z := (\alpha, \pi^1, \pi^2, \omega) \) fully characterizes the financial equilibrium.

\(^{26}\)Note that the stochastic discount factor for agent \( i \) is equal to

\[
\left( \frac{Y_{t+1}}{Y_t} \right)^{-\gamma} \left( \frac{1 - 2\beta \exp(\omega_{t+1})}{1 - 2\beta \exp(\omega_t)} \right)^{-\gamma} \left( \frac{\alpha_{t+1}^i}{\alpha_t^i} \right)^{-\gamma}
\]

since the pricing condition of Lemma 3 is written for the price of the market security normalized by the aggregate endowment, the price dividend ratio.

\(^{27}\)The disagreement intensity was defined to be the absolute value of \( \rho \) but as long as it is constant \( \rho \) will also be referred to as disagreement intensity.
The pricing condition of Lemma 3 implies that the marginal rates of substitution multiplied by the probability of the corresponding state is equalized between the two agents. This together with Lemma 2 yield the next result.

**Lemma 4.** The law of motion for the consumption share of the first agent is given by

\[ \alpha' = \alpha \left( \frac{\mathbb{P}^{1}(\epsilon^{y}|z, s)}{q(\epsilon^{y}|z, s)} \right)^{1/\gamma} \]

where the prime indicates next period’s value and quantity \( q \) is defined according to

\[ q(\epsilon^{y}|z, s) = \left[ \alpha \mathbb{P}^{1}(\epsilon^{y}|z, s)^{1/\gamma} + (1 - \alpha) \mathbb{P}^{2}(\epsilon^{y}|z, s)^{1/\gamma} \right]^\gamma. \]

Quantity \( q \) is the generalized weighted average of the individual beliefs with exponent the curvature parameter \( \gamma \) and weights the individual shares of consumption surplus and therefore the above lemma states that the consumption surplus share of an agent changes to the extent that the agents’ beliefs are different for the particular future state.

Lemma 4 together with the law of motion of the individual beliefs \( \pi^{i} \) and the law of motion of the habit-endowment ratio constitute the law of motion of the state vector \( z \) which can be generally represented by \( z' = L(z, s, \epsilon) \). As a result \( \mathbb{P} \) and other equilibrium quantities can now be expressed as functions of the state vector. Lemma 3 expresses the equilibrium market security price in terms of the individual expectations. It is convenient to express it in terms of a single probability measure. A natural choice is to use the consumption surplus weighted average of the probability measures of the two agents which can be considered as an unbiased estimator of the true probability measure. For this reason the following quantity is defined:

\[ \xi(\epsilon^{y}|z, s) := \frac{q(\epsilon^{y}|z, s)}{\alpha \mathbb{P}^{1}(\epsilon^{y}|z, s) + (1 - \alpha) \mathbb{P}^{2}(\epsilon^{y}|z, s)}. \]  

(23)

Quantity \( \xi \) is the ratio of the consumption surplus weighted average of individual probabilities over their generalized weighted average with exponent \( \gamma \). The market security price dividend ratio can now be expressed as follows:

\[ \tilde{P}(z) = \delta \mathbb{E}^{*} \left[ \xi(\epsilon^{y}|z, s) e^{(1-\gamma)(\epsilon'-y)} \left( \frac{1 - 2\beta e^{\omega'}}{1 - 2\beta e^{\omega}} \right)^{-\gamma} \left( \tilde{P}(z') + 1 \right) \right] \]

(24)

where the star indicates the probability measure

\[ \mathbb{P}^{*}(\epsilon|z, s) := \alpha \mathbb{P}^{1}(\epsilon|z, s) + (1 - \alpha) \mathbb{P}^{2}(\epsilon|z, s). \]

(25)

In the homogeneous agent economy, in which case \( \xi \) is identically one, the prices vary due to the variation in the habit-endowment ratio and the variation in the expectation of the aggregate consumption growth. Quantity \( \xi \) and how it is affected by the differences in beliefs as well as how
it affects the dynamic behavior of prices will be analyzed in the next section.

The two agents need to trade in order to finance their optimal consumption within a period and their new optimal holdings. We denote the equilibrium financial wealth of the first agent standardized by the period’s aggregate consumption with \( \widetilde{W} \). From the equilibrium pricing and allocation conditions already derived we obtain the following result:

**Lemma 5.** The equilibrium financial wealth for the first agent in state \( z \) satisfies the following condition

\[
\widetilde{W}(z) = c(z) + \delta \mathbb{E}^\pi \left[ \xi(z, s) e^{(1-\gamma)(y'-y)} \left( \frac{1-2\beta e^\omega}{1-2\beta e^\omega} \right)^{-\gamma} \widetilde{W}(z') \right],
\]

and once the signals \( s \) are observed it is given by

\[
\widetilde{W}(z, s) = \delta \mathbb{E}^\pi \left[ \xi(z, s) e^{(1-\gamma)(y'-y)} \left( \frac{1-2\beta e^\omega}{1-2\beta e^\omega} \right)^{-\gamma} \widetilde{W}(z') \right].
\]

The whole period financial wealth is required to finance both the new optimal holdings, the consumption and the storage, that is the total consumption of the period where \( c(z) = \alpha + \beta e^\omega (1-2\alpha) \) is the consumption share. The half period financial wealth on the other hand is only required to finance the new holdings since the consumption storage is what is used for the middle of the period consumption. This accounts for the fact that in Lemma 5 only the whole period financial wealth includes consumption.

The equilibrium wealth function of the first agent \( \widetilde{W} \), together with the equilibrium price function of the market security \( P \), are sufficient to give the equilibrium holdings of the two agents for the asset structure considered. Let \( \bar{\theta}(z) \) denote the holdings of the first agent in state \( z \), therefore,

\[
\bar{\theta}(z) = \bar{R}(z)^{-1} \widetilde{W}(z), \tag{26}
\]

where \( \widetilde{W}(z) \) denotes the vector of the first agent’s wealth for the four states of nature following state \( z \), that is states \((z, s)\) for the four different realizations of the signals \( s \). The holdings of the first agent in a state \((z, s)\) are obtained in a similar fashion and the holdings of the second agent are obtained from market clearing.

### 4.3 Computation of Equilibrium

The model presented in this paper is very tractable in the sense that equilibrium prices and allocations can be computed from a single function which is \( \widetilde{W} \), the equilibrium financial wealth of the first agent standardized by the aggregate consumption of the corresponding period. Since there is no labor income in this economy the equilibrium price of the market security is determined by aggregating the wealth of the two agents plus the aggregate endowment of the period. Hence, the
price-dividend ratio of the market security is given by

\[
\hat{P}(z) = \hat{W}(z) + \hat{W}(\hat{z}) - 1,
\]

(27)

where \( \hat{z} := (1 - \alpha, \pi_2, \pi_1, \omega) \). Due to the symmetrical property of the model with respect to the two agents the equilibrium wealth of the second agent is given by the function \( \hat{W} \) after changing the consumption surplus share and switching the beliefs. Once the equilibrium price is obtained the holdings of the two agents can be computed for the asset structure considered in (30).

The only thing needed therefore in order to compute the equilibrium quantities is to be able to compute the wealth function \( \hat{W} \) for each state \( z \). The unknown function has no general closed form expression and therefore it needs to be approximated. We approximate it with a complete Chebyshev polynomial of order \( n \).\(^{28}\) We estimated the values of the polynomial parameters with a projection method applied on the first dynamic functional equation presented in Lemma 5.\(^{29}\) \( \hat{W}(z, s) \) is computed using the approximated function \( \hat{W}(z) \) and the second equation of Lemma 5.

### 5 The Dynamic Behavior of Prices and Trading

In this section the joint dynamic behavior of prices and trading volume will be examined qualitatively. In particular we will focus our analysis on the dynamic relation between the turnover of the market security and the price of the market security as well as the risk-free rate. For this reason we will derive approximate expressions for the continuously compounded risk-free rate \( r^f \), the excess return on the market security over the risk free rate \( r^m - r^f \), and the turnover of the market security over a single period \( T \). The approximations are derived in Appendix B.

The factors that affect both prices and volume are (i) the \( \mathbb{P}^\pi \)-probability of the high growth state \( \pi(z) \), (ii) the differences in beliefs denoted with \( \Delta(z) := \pi_1 - \pi_2 \), (iii) the dispersion in consumption surplus denoted with \( h(z) := \alpha(1 - \alpha) \), which can be regarded as a proxy for the wealth distribution, and (iv) the habit-consumption surplus ratio \( \omega \). Let also \( \mu(z) \) and \( \sigma(z) \) denote the conditional mean and conditional volatility of the log endowment growth.

#### 5.1 Prices

The behavior of the market security price depends principally on the dynamic behavior of the one-period risk-free rate and the one-period price of risk. Starting from the case with no disagreement, that is \( \rho = 0 \), the one-period stochastic discount factor is given by

\[
M(z, z') = \delta \left( \frac{e^{\gamma y} - 2\beta e^{\phi \omega}}{1 - 2\beta e^{\phi \omega}} \right)^{-\gamma},
\]

\(^{28}\)For the results presented in this paper \( n \) was chosen to be between 8 and 12 depending on the size of the state vector.

\(^{29}\)The projection method that was used is a variant of the projection methods presented in Judd (1998).
which can be approximately expressed in a log-normal form where the risk-free rate is given by

\[
rf(z) \approx -\log(\delta) + \gamma(\omega)\mu(z) + (\gamma(\omega) - \gamma)(1 - \phi_x)\omega - \frac{1}{2} \gamma(\omega)^2 \sigma(z)^2 \left(1 + \frac{1}{\gamma} - \frac{1}{\gamma(\omega)}\right)
\] (28)

and the time-varying price of risk is approximately equal to \(\gamma(\omega)\sigma(z)\) where

\[
\gamma(\omega) := \frac{\gamma}{1 - 2\beta e^\omega}
\] (29)

denotes the time-varying risk-aversion. In the above approximated expressions the change in the risk-free rate and the risk-aversion in the middle of the period are not taken into consideration but this has a secondary effect and the qualitative analysis does not change.

When the parameter \(\beta\) is positive the main driving force for prices is the habit-endowment ratio, \(\omega\), and its effect on the market security price increases with its level of persistence, \(\phi_x\). When habit increases relative to aggregate consumption the risk-aversion of the economy increases as well as the need to smooth consumption inter-temporally. The risk-free rate as a result of the increased need to smooth consumption inter-temporally and the increased habit has the tendency to increase but as a result of the increased hedging demands it has the tendency to decrease. The net effect when the habit-endowment ratio increases depends on the parameters and the state but in general is positive.

The market price-dividend ratio decreases with \(\omega\) both due to the increase in the risk-free rate as well as the increase in the price of risk. As a result the market security return and the endowment growth become strongly positively correlated because \(\omega\) decreases with \(y' - y\) as shown in (22).

The conditional mean of endowment growth also has a potentially significant effect on the risk-free rate since any increase/decrease in \(\mu(z)\) increases/decreases the need to borrow against the future and the magnitude of the effect depends on \(\gamma(\omega)\). The price-dividend ratio on the other hand is affected by the changes in \(\mu(z)\) only to the extent that they are persistent. However, this is not the case and the variation in the conditional moments of endowment growth have almost no effect on the variation of the market security price.

The conditional volatility of the market security return is potentially highly stochastic depending on the habit parameter \(\beta\). The resulting effect is that when habit increases in relation to consumption the market security return volatility increases. This effect comes from both the risk-free rate and the price of risk through the stochastic relative risk-aversion \(\gamma(\omega)\). Due to its form the first derivative of the stochastic risk-aversion with respect to \(\omega\) is positive and proportional to \(\gamma(\omega)\). This means that when \(\omega\) increases both the risk-free rate as well as the price of risk become increasingly sensitive to any changes in \(\omega\) and hence the market security price becomes more volatile.

The introduction of belief heterogeneity brings two additional factors to the one period stochastic discount factor through \(\xi\), as defined in (23), which are the difference in beliefs \(\Delta(z) = \pi^1 - \pi^2\) and
a measure of the wealth dispersion which is given by $h(z) = \alpha(1 - \alpha)$.

The difference in beliefs changes every half period and in particular to

$$\Delta(z, s) = \phi_\pi \Delta(z) + (1 - \phi_\pi)\rho(s^1 - s^2)$$

when the signals $s$ are revealed and to $\kappa \Delta(z, s)$ when the endowment growth is realized. Therefore, $\kappa \phi_\pi$ determines the persistence of $\Delta$ and $\kappa(1 - \phi_\pi)\rho$ determines its conditional volatility. The variable $h$ inherits its properties from $\alpha$ whose law of motion is given in Lemma 4. From its law of motion it can be inferred that $\alpha$ changes to the extent that there are differences in beliefs and its rate of change depends on the curvature parameter $\gamma$. It is typically persistent and its volatility is increasing with $\Delta$ and with the inverse of $\gamma$.

In order to understand the pricing effect of belief heterogeneity we need to note that $\xi$ is shock dependent, that is, it depends on the realization of $\epsilon^y$ for which different beliefs are held. Therefore, we need to see how $\xi$ depends on the already mentioned factors as well as how it varies across states. It turns out that the effect of belief heterogeneity on prices depends heavily on the curvature parameter $\gamma$ as shown in the following remark.

**Remark 2.** Consider $\alpha \in (0, 1)$, $\pi \in (0, 1)$ and $\Delta \in (0, \min \left[ \frac{\pi}{\alpha}, \frac{1 - \pi}{1 - \alpha} \right])$ as the independent variables and let the beliefs of the two agents for the high growth state be $\pi_1 = \pi + h\Delta/\alpha$ and $\pi_2 = \pi - h\Delta/(1 - \alpha)$ where $h = \alpha(1 - \alpha)$. Then $\xi$ as defined in (23) has the following properties:

(i) $\xi \leq 1$,  
(ii) $\frac{\partial \xi}{\partial (h\Delta)} \leq 0$ and  
(iii) $\frac{\partial \xi}{\partial \pi} \geq 0$ when $\gamma \geq 1$.

The first two properties of the above remark affect only the equilibrium interest rate. The first property affects $\xi$ independently of the state whereas the second property in this model affects both states equally since there are only two states and $\Delta$ has the opposite sign for the two states. The third property affects the risk-premia because it introduces variation across the states.

As already noted, $\xi$ will take values below or above one depending on the curvature parameter $\gamma$ (property (i) of Remark 2). In the most typical case where the two agents have a curvature parameter higher than one prices become discounted in the sense that the higher the level of belief heterogeneity, as measured by $h\Delta$, the lower are the prices (property (ii) of Remark 2). The reason for this can be explained in the following way. There are two effects on prices when heterogeneity of beliefs is introduced. The first is that the state prices increase because agents push up the prices of the states for which they believe that their probability is high. The second effect is that by diverting their investment decisions agents increase the riskiness of their positions which decreases their demand for investment and pushes prices down. The two effects cancel each other out when $\gamma$ is equal to one. The riskiness effect is greater when the curvature parameter is higher than one.

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30The product $h(z)\Delta(z)$ can be considered to be an overall measure of the level of belief heterogeneity in the economy since the impact that an agent has on prices is wealth dependent.
In this case the price-dividend ratio decreases with $\Delta$ as well as the absolute value of $\rho$ because $\rho$ predicts a high value for $\Delta$ in the future. These effects reverse when $\gamma$ is less than one.\textsuperscript{31}

The third property of $\xi$ shown in Remark 2 has an effect on the price of risk. For the most typical case of $\gamma > 1$ the third property states that the lower the probability of a certain state is, the higher is the discounting of that particular state. As a result the heterogeneity of beliefs introduces a component to the price of risk that moves in the opposite direction to $\mu(z)$. It turns out, however, that this effect is quite small and in particular dwarfed by the overall effect of habit, at least for the model parameters considered.

The introduction of belief heterogeneity, therefore, has little effect on the price of risk but it has an effect on the interest rate. The additional element to the risk-free rate as approximated in (28), that shows the effect of belief heterogeneity, can be approximately given by the following expression:\textsuperscript{32}

$$-2h \frac{\sigma^2}{\sigma(z)^2} \frac{1 - \gamma}{\gamma} \mathbb{E} \left[ \Delta(z, s)^2 | z \right]$$

where $\mathbb{E} \left[ \Delta(z, s)^2 \right] = \left[ (\phi_{x} \Delta(z))^2 + 2(1 - \phi_{x})^2 \rho^2 \right]$. As already shown in Remark 2 the effect exists as long as $\gamma$ is different than one, it is positive when $\gamma > 1$ and negative when $\gamma < 1$. Further, the magnitude of the effect on the interest rate is increasing in the level of belief heterogeneity as given by $h\Delta$. Note that what matters is the half-period volatility of the difference in beliefs since in the middle of the period the two agents form new beliefs about $\epsilon$. Therefore, in the case of $\gamma > 1$ higher $\rho$ implies higher conditional volatility of $\Delta$ which in turn implies higher interest rate.

Belief heterogeneity, as already noted, affects the market security price through the effect on interest rates. Being a claim to a stream of future cash-flows the price of the market security is more sensitive to the wealth dispersion, as approximated by $h$, than to the difference in beliefs as given by $\Delta$, since $h$ is much more persistent. In fact $h$ has possibly strong effect on the level as well as the volatility of the market security price. For this reason belief heterogeneity has one more effect on the market security through the changes in $h$. Even though the price of risk is hardly affected by the heterogeneity in beliefs the risk-premium of the market security can be affected significantly. The reason is that the conditional correlation of the market security return with the stochastic discount factor changes and it typically decreases with the level of heterogeneity. This is because the correlation between changes in $h$ and endowment growth, that both affect the market security return, depends on the state and it can take any value. For example think of a case where the relatively pessimistic agent is also the more wealthy. This means that a high growth state will

\textsuperscript{31}Earlier research on the asset pricing effects of belief heterogeneity, for example Miller (1977), Harrison and Kreps (1978) and Scheinkman and Xiong (2003), proposed that belief heterogeneity causes a bubble effect on prices. It is shown here that the result obtained by these papers is due to assuming risk-neutrality and not an outcome of the short-selling constraint as conjectured. The short-selling constraint is needed when agents are risk-neutral to give an equilibrium otherwise positions become infinite.

\textsuperscript{32}Here we only show an approximate expression of the additional element arising from the introduction of belief heterogeneity.
coincide with an increase in $h$. As a result the partial effect of $h$ because of the high growth state on the market security return is negative.

5.2 Trading

The main focus of this paper is to examine in an economy with realistic price characteristics in which trading arises naturally how the need to trade varies with prices and how the realized volume of trade can be related with asset price changes. For this purpose three elements are employed in this economy: First the constant need to trade is generated by the difference in beliefs which generates the set of endogenous states of $(\alpha, \pi^1, \pi^2)$. Therefore, agents need to re-balance their portfolio primarily because they change their plan as to what states of nature they would like to increase or decrease their wealth (and consumption). The second is the introduction of an intermediate state where agents only receive information and the third element is the specific asset structure assumed.

To a certain extent the stochastic properties of the volume of trade depend on the asset structure. For example consider an economy where two assets are required to dynamically complete the market every period and one of them is a stock. Then on a given equilibrium path the realized turnover on the stock will be different in the case where the second asset is a bond to a case where the market is completed with a call option or to a case where instead there exists a put option. However, apart from an effect on the magnitude of trading volume the asset structure alone cannot generate a specific correlation between price changes and volume. Trading volume in such a case is accompanied by both increases and decreases of the level of belief heterogeneity, either due to wealth reallocation or changes in the differences in beliefs, in which case it is not associated with either high or low market security returns.

In order to generate the positive correlation between price changes and turnover we introduced the intermediate period. As an assumption it is quite realistic since agents trade both on, what can be classified as, soft and hard information. Essentially, however, what the introduction of the intermediate period creates is a setting where there are rounds of trade in which agents invest based on their beliefs (or what can be called speculative trade) when new information arrives and then revert back to a hedging position where they hold the market in about the same proportion as their wealth share. Therefore, the types of assets that are available for (or are chosen by) the agents to trade will determine how the state of nature will affect the way the agents will switch from a speculative position back to a hedging position. This trading behavior will become clear below where the trading volume is analyzed for the assumed asset structure.

Every half period agents trade on the market security and half-period contingent claims that have positive payoff only in one state of nature as shown in (13). Since no redundant assets are assumed there is always one state for which only the market security has positive payoff and therefore, the investment on the market security is to generate the wealth of that state. Let in general $s^*$ and $\epsilon^*$ denote the states for which only the market security has positive payoff. In the assumed asset
structure (13) those states are $s^* = (-1, -1)$ and $\epsilon^* = -1$. Therefore, the optimal holding of the market security by the first agent, which is denoted with $\theta$, is given by

$$\theta(z) = \frac{\hat{W}(z, s^*)}{P(z, s^*)} \quad \text{and} \quad \theta(z, s) = \frac{\hat{W}(L(z, s, \epsilon^*))}{P(L(z, s, \epsilon^*)) + 1},$$

(30)

for the two different subperiods.

Let $\vartheta(z)$ denote the fraction of total wealth owned by the first agent in state $z$. In equilibrium the wealth share of the first agent satisfies the following relation:

$$\vartheta(z) = c(z) + \lambda(z)$$

where $c(z)$ is the consumption share of the first agent in the period starting with state $z$ and is equal to $\alpha(1 - 2\beta e^\omega) + \beta e^\omega$. The quantity $\lambda(z)$, which is defined in Appendix B, is the expectation of the wealth share change multiplied by the discounted price-dividend ratio and $\lambda(z, s)$ denotes the same expectation but after signals $s$ are observed. Quantity $\lambda$ is generally small and is non-zero to the extent that wealth share changes are correlated with discounted prices. In the case where $\beta = 0$ and $\gamma = 1$ the correlation is identically zero whereas for $\gamma$-values other than one it becomes non-zero and in certain middle of the period states it becomes significant. When $\beta$ is positive then this correlation increases because wealth share changes are correlated with the habit-endowment surplus ratio.

Given the definition of the wealth share the optimal holding of the market security in state $z$ can be expressed as follows:

$$\theta(z) = \vartheta(z) + \frac{\lambda(z, s^*)}{P(z, s^*)}.$$  
The optimal holding, therefore, is largely given by the wealth share of the agent and is little affected by the state $s^*$ which is the state in which only the market security has a positive payoff. At the beginning of every period agents agree on the probabilities of $s$ and as a result they take no speculative positions but hold the market security in about the same proportions as their wealth shares. When the signals $s$ arrive and agents form their new beliefs the optimal holding of the first agent becomes

$$\theta(z, s) = \vartheta(L(z, s, \epsilon^*)),$$

and her investment in the market security reflects how much of the total wealth the agent wants to have in state $\epsilon^*$. Agents want to change their wealth share from the previous period only to the extent that they have different beliefs and for this reason it can be called speculative trading. In the case where $\{e^y = -1\} \in \epsilon^*$ the turnover in the middle of a period can be approximated by the following relation (omitting the $\lambda$ terms):

$$T(z, s) \approx \frac{h(z)\Delta(z, s)}{\gamma(\omega)} \frac{f_1(z, s)}{1 - \pi(z, s)} + \beta(e^{\phi_{\omega - \mu + \sigma} - e^\omega}(1 - 2\alpha))$$
where $f_1(z, s)$ is an expression derived in Appendix B which fluctuates around one.\footnote{In the case where $\{e^y = -1\} \in e^*$ the difference in beliefs $\Delta(z, s)$ changes sign, the probability in the denominator becomes $\pi(z, s)$ and $\sigma$ changes sign.} The turnover in the middle of a period has two parts, the first, which is the speculative part, is proportional to belief heterogeneity $h(z)\Delta(z, s)$ and inversely proportional to risk-aversion $\gamma(\omega)$. The second part is related to hedging and it comes from the habit preferences and the heterogeneity in wealth. When agents are heterogeneous in their wealth they are also heterogeneous in their risk aversion. Consequently there is transfer of wealth from the more wealthy (and less risk-averse) to the less wealthy (and more risk-averse) when the habit-endowment ratio increases, and the opposite when it decreases. Hedging turnover increases both with $\omega$, that is when consumption is closer to habit, and with heterogeneity in wealth and therefore decreasing in $h$.

The positive correlation between price-changes and turnover is produced due to the second round of trading which is given by

$$T(z, s, \epsilon) = \left| \vartheta \left( L(z, s, \epsilon) \right) + \frac{\lambda \left( L(z, s, \epsilon), s^* \right)}{\bar{P}(L(z, s, \epsilon), s^*)} - \vartheta \left( L(z, s, e^*) \right) \right|.$$

It is clear that there is more turnover when the realized state $\epsilon$ is different than $e^*$. Since the investment of each agent in the market security in the middle of a period reflects the wealth share they will have if state $e^*$ realizes and since when $\epsilon$ realizes agents revert back to hedging positions where they hold the market according to their wealth shares, there is more trading when $e^*$ does not realize. Therefore, with the assumed asset structure in which $\{e^y = -1\} \in e^*$ and since the market security return is positively correlated with $e^y$ the second round of trade exhibits more turnover when the return is positive than when the return is negative. The magnitude of the turnover is once more proportional to belief heterogeneity and inversely proportional to risk-aversion.

The assumed asset structure is very relevant to an asset structure with a set of call options instead of contingent claims on the realization of $e^y$. This is because the market-security will be required to finance the wealth in the state for which there is no call option with positive payoff, that is for the lowest return and hence trading volume will be low with low returns and high with high returns. In fact an asset structure with call options would not even need the positive correlation between the aggregate consumption growth and the market security return to produce the return-volume relation as turnover would be directly related to what happens to the market security return.

5.3 The Dynamic Relation Between Prices and Trading

After having analyzed how the different factors in this model economy determine prices and turnover it remains to see how prices and turnover are dynamically related. The two stylized facts that are the focus of this paper are (i) the positive relation between volume and return volatility and (ii) the positive correlation between the market security return and volume. The discussion that follows looks at how the model so far behaves in relation to these stylized facts, starting with the second.
For the positive correlation between returns and turnover we have introduced certain features in
the theoretical model that enable it to generate the effect. These features are first the intermediate
period where agents receive information that they occasionally interpret differently and second
the special asset structure with which the two agents construct their speculative positions. We
have seen how these two features combined create the desirable effect, the argument however can
be made more general. Essentially what is required is first that occasionally investors receive
economy-related information that they interpret differently and because of that they deviate from
their hedging positions and then revert back to their hedging positions once the uncertainty for
which they received the information has been resolved. Secondly, in doing so investors for whatever
reason, for example to reduce costs related to trading, exchange such assets that the relatively
optimistic investors leverage up their position by selling the market and buying assets that pay well
if the market goes up, for example call options. As a result, more trading occurs if the market
indeed goes up in which case the previously optimistic agents increase their wealth and buy back
the market.

As for the volume-volatility relation the model has several factors that affect this relation in one
way or another. The first factor is that in this model there are two types of trading which are the
speculative due to the opinion differences and trading due to varying hedging demands coming from
the habit preferences and the heterogeneity in wealth. The first thing to note is that the two types
of turnover work in opposite directions, that is when the amount of the one increases the amount of
the other typically decreases. In particular, speculative trading increases when dispersion in wealth
(and the difference in beliefs) increases, that is when $h\Delta$ increases, whereas trading due to hedging
increases when wealth heterogeneity increases, that is when $h$ decreases. Also speculative turnover
decreases with $\omega$ because risk-aversion decreases whereas trading due to hedging increases with $\omega$
because risk-aversion and in particular heterogeneity in risk-aversion increases. The overall effect
depends, therefore, on the relative magnitude of these two trading activities and the effect of
$h$, $\Delta$ and $\omega$ on volatility.

It has been seen that the habit-endowment ratio is the main asset pricing factor in determining
the level of the market, return volatility, the equity premium and the risk-free rate. Effectively, the
market volatility, the interest rate and the equity premium are counter-cyclical as they go up with
$\omega$ and the market price-dividend ratio is pro-cyclical.\(^{34}\) Hence, speculative trading moves in the
opposite direction to volatility whereas trading due to hedging demands in the same direction as
volatility.

Belief heterogeneity, as given by $h\Delta$, is the second factor that affects return volatility. First, it needs
to be noted that $\Delta$ has a smaller effect on the market security price and hence on its return volatility
also due to its low persistence. The effect of $h$ on the other hand is significant but two-fold, one that

\(^{34}\)In the model economy the economic cycle is determined by the habit-endowment ratio where high values for $\omega$
mean “bad times” and low values mean “good times”. This definition is different than the macroeconomic definition
of an economic cycle which is determined by the consumption level in relation to its stochastic trend.
increases volatility and one that decreases it. Higher dispersion in wealth (lower $h$) also implies bigger wealth changes an effect that implies higher volatility. The second effect is through the persistence in $h$ which is time-varying and decreasing in $h$. When there is high wealth dispersion, $h$ has lower persistence which means that any change in $h$ affects less the market security price which in turn implies lower return volatility. The overall effect depends on the model parameters; for the calibrated model return volatility is in fact decreasing in $h$ but as it will be shown this relation reverses with a high $\rho$. A high value for $\rho$ means that the average level of belief heterogeneity is high. This in turn means that the persistence of $h$ decreases as well as its variation whereas its conditional volatility becomes more volatile and more important in affecting prices.

We will show later that for the calibrated model the overall relation between volume and volatility is at best weak. The model will be able to generate both price-volume relations and fit prices well when we introduce the stochastic disagreement intensity.

5.4 Other Asset Structures

It has become obvious that the asset-structure has an important effect on the dynamic behavior of turnover especially in this model with the introduction of the intermediate state. For example a simple change in the state $\epsilon^*$ will result in a negative correlation between market security return and turnover. If this is an important factor behind the stylized fact it means that, for whatever reason, the relatively optimistic investors are the ones who decrease their holdings of the market portfolio by investing for example in call options rather than the relatively pessimistic investors either shorting the market or taking speculative positions in put options. Such an explanation sounds quite plausible if there are increased transaction costs in betting against the market. It is important to note that this explanation does not hint on market incompleteness but that the markets are completed with assets that pay-off when the market security return is high.

Typically asset pricing models assume asset structures that include a risk-free asset instead of derivatives on the stock or other contingent claims. Such an assumption in this model would produce no correlation between asset returns and turnover since the optimal holding of the market security would be given by

$$\theta(z, s) = \frac{\tilde{W}(L(z, s, +1)) - \tilde{W}(L(z, s, -1))}{\tilde{P}(L(z, s, +1)) - \tilde{P}(L(z, s, -1))}$$

in the intermediate periods. As a result the resulting turnover at the end of every period would typically not depend on the endowment growth and therefore largely uncorrelated with the market security return. This is an empirical question but it is likely that agents in their speculative trading do not switch between bonds and stocks since the trading volume on bonds is much smaller than the trading volume in either stocks or derivatives.

Even though not shown explicitly in this paper in general the positive relation between belief
heterogeneity and trading volume is robust to the selection of the asset structure. The effect of belief heterogeneity, however, on the return volatility is not clear especially when volatility is mainly driven by other factors, as for example in this model by the habit-endowment ratio. Hence, it is not straightforward whether belief heterogeneity is able to produce a positive relation between volume and volatility. It will be shown that this relation can be produced without adversely affecting other price or volume characteristics of the model by introducing time-variability in the disagreement intensity.

6 Model Calibration

In order to analyze quantitatively the various effects that we presented and analyzed in the previous section, we calibrate the model to fit certain price and volume statistics. Both quarterly and annual data of turnover, market prices and consumption have been used. We obtained annual data on turnover from 1901 to 2003 from the NYSE Factbook. The annual data over the same period on per-capita consumption, market returns, the one-year risk-free rate and the market price-dividend ratio are from Robert Shiller’s website. Our quarterly market data of turnover, market returns and market price-dividend ratio are from CRSP spanning the period from the first quarter of 1927 to the last quarter of 2007. The turnover is the total turnover from the entire stock universe of CRSP and the market price-dividend ratio was obtained from the CRSP value-weighted returns with and without dividends. The 3-month risk free rate was obtained for the same period from the Federal Reserve Economic Data of the St. Louis Fed. Quarterly per-capita real consumption data comes from the NIPA tables that are available only from the first quarter of 1947. Throughout the entire data period the turnover data are highly non-stationary due to the fact that they exhibit time-varying trend. For this reason we detrended the turnover series by taking first differences.

No specific calibration procedure was followed in the sense of fitting exactly certain statistics due to the high computational demand of solving and simulating one instance of the model. We aimed to fit closely a number of quantities like the annual mean, volatility and autocorrelation of consumption growth, the average risk-free rate and the average excess return on the market, the mean, volatility and autocorrelation of the market price-dividend ratio, the volatility of the risk free rate and the volatility of the market security return, the average Sharpe-ratio, the correlation between the market security return and turnover as well as certain statistics related to the volume-volatility relation. Table 1 shows the calibrated parameters in which the first row refers to the constant \( \rho \) model. All tables and figures appear in Appendix C.

The data and the calibrated model statistics are shown in Table 2. The last two columns of the table refer to the calibrated model with stochastic disagreement intensity and will be discussed later. We obtained the model statistics by running 1000 simulations with 500 annual periods each. The column “Avg.” shows the simulation average and “Std.” the simulation standard deviation.
of each statistic. Mean, volatility and first-lag autocorrelation are denoted with $\mu$, $\sigma$ and $\rho_1$ respectively, $r^f$ denotes the risk-free rate, $r^m$ the market security return and $pd$ the log market price-dividend ratio. $\Delta x$ denotes the one-period change in a variable $x$ and $T$ denotes turnover. In order to analyze the calibrated model we plotted several equilibrium quantities in Figures 1 to 6 for different combinations of the state variables. In particular, we plotted the quantities against $\omega$, $\alpha$ that determines $h(z) = \alpha(1 - \alpha)$, $\pi = \pi^1 = \pi^2$ that determines $\mu(z)$ and combinations of $\pi^1$ and $\pi^2$ that determine $\Delta(z) = \pi^1 - \pi^2$.

The model overall is able to produce a very good fit to the data excluding the statistics related to the volume-volatility relation. The consumption growth statistics are well fitted except the correlation with the market security return which is close to perfect in the model. The model risk-free rate exhibits higher volatility than that of the quarterly data but lower than the annual data volatility whereas the mean is fitted closer to the quarterly value. The annual data risk-free rate exhibits higher mean and volatility because of the fact that it is an annual interest rate instead of a three month interest rate and the data goes back to 1903 which was a period of exceptionally volatile interest rates. Looking at Figure 2 we observe that for the calibrated model the risk-free rate is mainly determined by $\omega$ and to a lesser extent by $\mu(z)$ due to the fact that $\phi_\pi$ was calibrated at a very low level, 0.056 to fit the consumption growth autocorrelation of 0.052. Further, the risk-free rate is hardly affected by belief heterogeneity either due to $h$ or $\Delta$.

The model statistics of the log market price-dividend ratio, that is the mean, volatility and persistence are very well fitted. Its main determining factor, as seen in Figure 1, is the habit-endowment ratio through its persistence and the way it determines the risk-free rate and the equity premium as seen in Figure 3. The mean and volatility of the market security return and the equity premium are fitted equally well. The persistence parameter $\phi_x$ lends its persistence to the price-dividend ratio and makes it quite volatile. The subjective discount factor $\delta$ was chosen to fit the level of the price-dividend ratio whereas the curvature parameter $\gamma$ and the habit parameter $\beta$, that determine the risk-aversion, were chosen to fit the equity premium. Figure 1 also shows that belief heterogeneity, in particular $h$ because of its persistence, affects negatively the market security price whereas $\Delta$ and $\mu(z)$ have very small effects. The same holds for the market security equity premium shown in Figure 3 but this time $h$ has a noticeable effect because it decreases the correlation of the market security return with consumption growth.

Coming now to the statistics related to turnover, the model through choosing the parameter $\rho$ is able to fit well the correlation of turnover with the market security return, with the changes in the price-dividend ratio and with the excess return on the market. Also the model correlation of turnover with the risk-free rate is close to zero just like in the quarterly data whereas in the

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$^{35}$Standard errors can be obtained by dividing the standard deviation with $\sqrt{1000}$.

$^{36}$Note that it is enough to plot the figures for values of $\alpha$ between 0.5 and 1 since the model is symmetrical in $\alpha$. Wealth dispersion as measured by $h$ increases when $\alpha$ decreases from 1 to 0.5 whereas wealth heterogeneity increases in the opposite direction.
annual data it is negative. Another statistic that is not shown in Table 2 is the correlation of the price-dividend ratio with turnover which is also close to zero in both the annual data, 0.06, and the quarterly data, −0.03. This is not surprising since the turnover series was detrended through first differencing and hence it appears that no significant variation of its time-varying mean remains in the detrended series. Let the turnover series be decomposed into its time varying mean and its innovations with time-varying volatility:

\[ T_{t,t+1} = \mathbb{E}_t (T_{t,t+1}) + \sqrt{\mathbb{V}_t (T_{t,t+1})} \cdot \frac{T_{t,t+1} - \mathbb{E}_t (T_{t,t+1})}{\sqrt{\mathbb{V}_t (T_{t,t+1})}}, \]

where \( \mathbb{V} \) denotes variance. The price-dividend ratio and the risk-free rate being \( \mathcal{F}_t \)-measurable can only be related to the conditional expectation and the conditional variance of turnover and hence uncorrelated to the detrended series.

Since the turnover series was detrended there is no direct way of observing how the conditional mean of turnover relates to the level of prices. One indirect way to do it is to assume that the conditional volatility of turnover is proportional to its conditional mean and then see how the absolute turnover series is related to the log price-dividend ratio. The data show a strong negative correlation of −0.38 in the quarterly data and −0.45 in the annual data. This negative correlation is not surprising since return volatility is known to move in the opposite direction to prices while it is positively correlated to volume. The model on the other hand exhibits no correlation between turnover and the level of prices. The qualitative analysis of the model showed that there are two types of trading, speculative and hedging, and that there are two factors that affect this relation, \( \omega \) and \( h \). As the habit-endowment ratio is concerned speculative turnover is positively correlated and hedging turnover is negatively correlated with the price-dividend ratio. This is confirmed by Figure 5 where we see that when \( \alpha = 0.5 \) and therefore trading due to hedging is zero the expected one-period turnover decreases with \( \omega \). In the case where \( \alpha \) is close to 1 in which case most of the trading is due to hedging the expected turnover is increasing in \( \omega \). The second factor \( h \) generates the opposite relations between the level of the market and the two types of trading. When \( \omega \) is high in which case trading due to hedging is small the amount of trading increases with \( h \) and therefore speculative trading is negatively correlated with the price-dividend ratio. Trading due to hedging on the other hand decreases with \( h \) as we observe when \( \omega \) is high and speculative trading is low. Judging from the overall zero correlation between turnover and the price-dividend ratio all these effects appear to cancel each other out.

For the same reason the model is unable to produce an unconditional relation between volume and volatility. But first lets review the statistical evidence with respect to the volume-volatility relation. In order to uncover such a relationship we consider a number of different statistical relations. First, the absolute market security return and the absolute excess return are both positively correlated with turnover. However, this is due to the high unconditional means of these returns and the positive correlation between these returns and turnover. For this reason we also look at the correlations
between the absolute price-dividend ratio changes and the absolute demeaned returns with turnover. These correlations are positive in the quarterly data and close to zero in the annual data. Even though these pricing series are better in measuring pricing volatility the lack of strong evidence is probably due to the turnover measure. If the variation in turnover due to its positive correlation with price changes is a big part of the overall variation in trading then the correlation between absolute price changes and volume is not a good measure of the volume-volatility relation. For example, consider a case where the return volatility is high and the next-period return turns out to be low. Then even if there is a positive relation between volume and volatility the resulting volume would be low due to the low return. Therefore, it would be better to look at the conditional mean of turnover instead but for the reason that the turnover series is detrended it cannot be observed. Following the same reasoning as before an indirect evidence can be obtained by looking at the correlation between the absolute turnover with the pricing measures of volatility. In the quarterly data the correlations between $|\Delta pd|$ and $|r^m - \mu(r^m)|$ with $|T|$ are 0.40 and 0.37 respectively. In the annual data these correlations are 0.17 and 0.14 respectively. This statistical evidence is in line with the stylized fact of the positive relation between return volatility and turnover.

We have argued earlier that a positive volume-volatility relation requires a number of conditions in a model with realistic prices one of which is time-variation in disagreement as we will show later. We see here that the model with constant disagreement is unable to produce this effect as seen by the correlation of the market turnover with either the absolute market price-dividend ratio changes or the absolute demeaned market security return. The model correlations of the absolute market security return and the absolute excess return with turnover are highly positive but as discussed earlier these are not good measures of the volume-volatility relation in the light of the positive correlation between returns and volume. The inability of the model with constant disagreement to generate this relationship can also be seen by looking at Figures 4 and 5. Clearly, the most important factor for the return volatility is the habit-endowment ratio and to a much lesser extent wealth dispersion. The conditional mean of the one-period turnover however does not behave uniformly in relation to $\omega$ since it is decreasing when $h$ is high in which case trading is mostly speculative whereas it is increasing when $h$ is low in which case trading is mostly for hedging purposes.

With the qualitative analysis we showed that in the model with constant disagreement the only way of generating a positive volume-volatility relation is if heterogeneity of beliefs as given by $h\Delta$ becomes an important factor for return volatility, return volatility becomes increasing in the belief heterogeneity and speculative turnover becomes much greater than hedging turnover. All these conditions could be met if the level of belief heterogeneity increased by increasing parameter $\rho$ but with a number of other adverse effects. Figures 7 to 9 show how certain price characteristics and the price-volume relations are affected by varying parameter $\rho$ from 0.05 to 0.45 and keeping the rest of the parameters the same. The plots were generated by solving each instant of the model and running 200 simulations with 500 periods each. The continuous lines show the simulation averages.
of the corresponding statistics and the dotted lines show the one standard error bounds.

Figure 9 verifies the conclusions from the qualitative analysis in that a strong volume-volatility relation can indeed be produced with a high value for \( \rho \). However this has an adverse effect on the return-volatility relation, as shown in Figure 8, and on key pricing statistics. The stock-price and the risk-free rate volatilities increase significantly whereas the equity premium decreases drastically from close to 6% down to about 3%. When the heterogeneity of belief increases the risk-free rate becomes more volatile which in turn causes a higher volatility in the stock price. At the same time the correlation of the market security return with the aggregate consumption growth decreases, since the variation of the stock price due to changes in \( h \) and \( \Delta \) is generally unrelated to \( y' - y \), which decreases the equity premium and the return-volume correlation. The latter effect can be remedied by substituting contingent claims on \( y' - y \) with stock derivatives but the former effect cannot.

Lastly, the model independently of whether disagreement is time-varying is able to fit the positive autocorrelation of trading volume as measured by the first lag autocorrelation of the absolute turnover series. The logic behind looking at the absolute series of turnover is once again the fact that the series is detrended and the possibility that the conditional volatility of turnover is positively correlated with its conditional mean. The model generates this positive autocorrelation through the persistence in the level of belief heterogeneity, mainly due to wealth dispersion.

7 Stochastic Disagreement Intensity

The disagreement intensity \(|\rho|\) determines how differently the beliefs of the two agents are affected when the two signals revealed are conflicting. In particular, it affects the conditional volatility of the difference in beliefs as given by \( \Delta \) where the random shock is the difference in the two signals:

\[
\Delta(z, s) = \phi \Delta(z) + (1 - \phi)\rho (s^1 - s^2).
\]

If one accepts that agents disagree occasionally when interpreting common information it is natural to also expect that the amount of disagreement will be time-varying. This could be because the economic uncertainty varies, or because the amount or rate of information release is different across time, or even due to psychological factors that are related to the state of the economy. All these factors could be captured in this model by a time-varying disagreement intensity.

7.1 Additional Model Assumptions

We assume that the disagreement intensity varies over time according to the following autoregressive process:

\[
\rho_{t+1} = \phi_{\rho} \rho_t + \frac{1 - \phi_{\rho}}{2} \epsilon_{t+1},
\]
The shock \( \epsilon^\rho \) realizes every whole period, that is at the same time as \( \epsilon^y \), and agents hold homogeneous beliefs as to its conditional distribution. The specific formulation of the shock \( \epsilon^\rho \) is made so that \( \eta \) controls the correlation between endowment growth and changes to the disagreement intensity, \( \rho_t \). Therefore an increase in the disagreement intensity implies a positive value for \( \epsilon^\rho \) when \( \rho_t \) is positive and negative when \( \rho_t \) is negative. In the case where \( \eta = 0.5 \) the correlation is zero and therefore the term \( 2 \cdot 1(\rho_t \geq 0) - 1 \) is redundant.\(^{37}\) The introduction of the parameter \( \eta \) is important for generating the positive correlation between volume and volatility through affecting their joint dynamics. Its importance will be shown with the analysis of the calibrated model.

The state of the economy is now described by \( z = (\alpha, \pi^1, \pi^2, \omega, \rho) \) and the shocks in the second sub-period are \( \epsilon = (\epsilon^y, \epsilon^\rho) \). Even though \( \epsilon^\rho \) introduces a non-fundamental source of uncertainty agents still require two additional independent assets for the second sub-period so that the financial markets are dynamically complete. This is because the disagreement intensity will have an effect on both prices and wealth even though it does not have an effect on consumption. The consumption surplus allocation remains the same which is shown in Lemma 4. The reason is that the consumption surplus processes depend on beliefs and the source of uncertainty for which agents hold heterogeneous beliefs and therefore unaffected by the introduction of \( \epsilon^\rho \).

The two additional assets in the second subperiod will be two contingent claims, one that has positive payoff in state \( \epsilon = (+1, -1) \) and the other with positive payoff in state \( \epsilon = (-1, -1) \). The particular asset structure is not important for the results obtained as long as \( \{\epsilon^y = -1\} \in \epsilon^\pi \) as explained in Section 5.

### 7.2 Price and Volume Implications

A high level for the disagreement intensity implies that any possible disagreement when new information arrives will have have great impact on the difference in beliefs. We already saw that interest rates are affected by the heterogeneity in beliefs depending on the curvature parameter \( \gamma \).

The approximate expression of the additional element that we derived to show the impact of belief heterogeneity on the one-period risk-free rate is the following:

\[
-2h \frac{\sigma^2}{\sigma(z)^2} \frac{1 - \gamma}{\gamma} \left[ (\phi_n \Delta(z))^2 + 2(1 - \phi_n)^2 \rho^2 \right].
\]

The impact on the interest rate is very clear. In the relevant case where \( \gamma \) is above one higher disagreement intensity implies higher expected heterogeneity in beliefs and hence higher risk-free

\(^{37}\)A simpler process could be specified that would make the term \( 2 \cdot 1(\rho_t \geq 0) - 1 \) redundant if \( \rho \) was made to be always positive instead of also switching signs.
rate and the opposite when \( \gamma \) is less than one.

Since interest rates are affected it means that the market security price will also be affected by the level of the disagreement intensity. Continuing with the relevant case where \( \gamma \) is greater than one, the market price-dividend ratio will be decreasing in the disagreement intensity and the magnitude of the effect will depend on the persistence \( \phi_\rho \). Higher persistence implies that for many periods ahead the disagreement is expected to be high increasing in this way the discounting of future cash-flows. Of course, on the other hand, higher \( \phi_\rho \) implies smaller changes to the disagreement intensity and therefore lower volatility due to changes in \(|\rho|\). Another most important effect of \( \phi_\rho \) is on the unconditional average of the disagreement intensity. The unconditional volatility of \( \rho \) is given by \( \frac{1}{2}\sqrt{\frac{1-\phi_\rho}{1+\phi_\rho}} \) which is decreasing in \( \phi_\rho \). The unconditional average of the disagreement intensity is related to the unconditional volatility of \( \rho \) and as such it is also decreasing in \( \phi_\rho \). As a result the average interest rate decreases whereas the equity premium and the correlation between the market security return and consumption growth increase with \( \phi_\rho \).

The return volatility is also affected by the variation in the disagreement intensity. The approximate expression (33) shows that the risk-free rate is proportional to the squared disagreement intensity and as a result the effect is increasing in \(|\rho|\). Roughly, the reason is that when agents have different conditional means for the aggregate endowment growth it means that the same difference in beliefs applies equally to the high as well to the low state. The result is that since the interest rate is increasingly affected when \(|\rho|\) increases it means that the market security price becomes increasingly sensitive to changes in the disagreement intensity and hence its conditional volatility increases. As for the trading volume the effect is straightforward. In the approximate expressions derived in Section 5 the one-period turnover was shown to be proportional to the difference in beliefs \( \Delta(z, s) \) which in turn is increasing in the disagreement intensity. Simply put, higher disagreement intensity implies higher expected disagreement and therefore higher expected speculative turnover. The variation in the disagreement intensity, therefore, creates a positive relation between volume and volatility but as we argued earlier and as we will show below it is still not enough to create a strong unconditional relation when prices follow realistic processes.

### 7.3 Calibration

The parameters to be chosen for this calibration are the autocorrelation parameter \( \phi_\rho \) and \( \eta \) that controls the correlation of changes to the disagreement intensity with aggregate endowment growth and hence the correlation between disagreement intensity and the habit-endowment ratio. There is more than one combination of values for \( \eta \) and \( \phi_\rho \) that can generate the two price-volume relations. For example \( \eta \) could be set to 0.5, that is no correlation between disagreement intensity and \( \omega \), and \( \phi_\rho \) could be set to 0.5 that would create high belief heterogeneity and high volatility in \(|\rho|\). Both can create a positive relation between volume and return volatility but such a configuration would (i) induce the return-volume correlation to be smaller since the correlation of the market security return with endowment growth would be smaller, and (ii) decrease the equity premium and increase
both the risk-free and the price volatility. The model configuration that offers the best fit to the data is one with positive correlation between disagreement intensity and $\omega$, and a high value for $\phi_\rho$, that is a lower level of belief heterogeneity.

The calibrated parameters are shown in the last row of Table 1 and the model statistics are shown in the last two columns of Table 2. The subjective discount factor $\beta$ was also adjusted to fit the level of prices. The reason is that the calibrated model with the stochastic disagreement intensity implies on average a lower level of disagreement than the calibrated model with constant $\rho$. Consequently, the risk-free rate with the same $\beta$ parameter value would imply a lower average risk-free rate and a higher average price-dividend ratio.

The newly calibrated model fits very well the data and generates a strong relation between volume and volatility as indicated by the model statistics. A positive correlation between $|\rho|$ and $\omega$ means that when the economy enters into bad times and the volatility goes up the disagreement intensity also goes up on average which increases speculative trading despite the increased risk-aversion. Despite the fact that the disagreement intensity has a significant effect on return volatility and as such it creates a positive relation between volume and volatility, as shown by Figures 13 and 14, a zero correlation between $|\rho|$ and $\omega$ would give at best a weak volume-volatility relation. This is because $\omega$ creates a negative relation between volume and speculative turnover and cancels out the positive relation created by the disagreement intensity. Figure 18 shows how the volatility-volume relation changes when the parameter $\eta$ varies from 0.5 to 0.1. It is clear that a model in which there is no correlation between $\omega$ and $|\rho|$, that is for $\eta = 0.5$, the volume-volatility relation is almost non-existent unconditionally whereas their correlation increases significantly as $\eta$ decreases.

The parameter $\eta$ also affects the return-volume relation as seen in Figure 17 but not as significantly as the other stylized fact. To understand the effect on the return-volume relation we first need to consider that the time variability of $\rho$ decreases the correlation of the market security return with the endowment growth and hence volume. If $\eta$ on the other hand is small it means that changes to the disagreement intensity are negatively correlated with endowment growth and hence changes in the market security price due to changes in $|\rho|$ are positively correlated with $y' - y$ and hence volume. Consequently, while the uncertainty in $\rho$ decreases the return-endowment growth correlation decreasing $\eta$ on the other hand increases this correlation. Finally, the positive correlation assumed between $\omega$ and $|\rho|$ also affects the volatilities of the market security price and the risk-free rate. As seen in Figures 10 and 11 both $\omega$ and $|\rho|$ affect significantly both the price-dividend ratio and the risk-free rate and in both cases they work in the same direction. Hence, the positive correlation between $\omega$ and $|\rho|$ increases the overall volatilities. The effect, however, as observed by the unconditional model statistics is only marginal.
Figure 15 shows how the parameter $\phi$ affects the price and the risk-free rate volatilities as well as the market risk-premium. Decreasing $\phi$ and hence increasing the overall belief heterogeneity increases volatilities significantly and decreases the equity premium. The increased belief heterogeneity increases the impact of disagreement on prices and decreases the correlation of the market security return with endowment growth. The decreased return-endowment growth correlation also has an impact on the return-volume correlation as shown in Figure 16. However, this is not so important as this effect can be remedied if we assumed call options on the stock instead of endowment growth contingent claims.

The explanation proposed by this theoretical model is that the volume-volatility correlation is positive because speculative turnover is the most important in magnitude and because risk-aversion that drives return volatility and disagreement intensity that drives speculative turnover are both counter-cyclical. In the model the economic cycle is determined by $\omega$, risk-aversion is endogenously counter-cyclical and disagreement is exogenously positively correlated with $\omega$. In order to arrive at this conjecture we first showed that in the absence of time-variation in disagreement intensity speculative turnover is pro-cyclical, while hedging turnover is counter-cyclical, because investors become more risk-averse during bad times and therefore speculate less and hedge more when the volatility goes up. The overall correlation between volume and volatility for the calibrated model was close to zero because the size of the two types of trading was about the same. Then when the disagreement intensity is allowed to be time-varying it was shown that it creates a positive relation between speculative turnover, which in this case is the most important, and volatility. However this positive correlation was shown to be counterbalanced by the negative correlation generated by the time-varying risk-aversion. Consequently, the disagreement intensity if assumed to be positively correlated with risk-aversion so much that even with increased risk-aversion speculative turnover increases, the volatility is strongly correlated with volume.

As an assumption the positive correlation between risk-aversion and disagreement intensity is plausible. The disagreement intensity essentially determines how volatile the difference in beliefs can be within a given amount of time. Disagreement could be time-varying for many reasons, like the amount of information released or the impact of new information or even due to psychological factors that may affect the way agents interpret new information. It is possible that disagreement increases during bad times because investors become more attentive and more sensitive to information in the fear of making mistakes whereas in good times herding behavior may prevail. If this is so and if such behavior is strong enough to create a positive relation between volume and return volatility is an empirical question.

8 Conclusion

The theoretical model presented in this paper makes a step towards understanding in a unified framework the joint dynamic behavior of assets prices and the volume of trade that are empirically
strongly related with each other. Both prices and trading are central to understanding the workings
and the efficacy of financial markets and therefore uncovering the driving factors behind them is
of first order significance. The driving force behind trading in this model is disagreement about
the interpretation of common information. This assumption goes against the economic notion of
perfect rationality but on the other hand it is hard to accept that economic agents are at all times
perfectly and uniformly aware of the underlying structure of the economy. The question therefore is
not whether there is disagreement between investors but whether disagreement is indeed responsible
for several features of the financial markets including the cause for time-varying hedging needs.

The theory we developed in this article regarding the positive correlation between returns and
volume attributes this phenomenon to a combination of occasional disagreement due to new infor-
mation and the type of assets with which agents choose to speculate. A central empirical prediction
generated by this argument is that the return-volume correlation is related to the difference between
the trading volume on assets or derivatives that pay well in high market return states and the trading
volume on assets or derivatives that pay well when market returns are low. As the volume-volatility
empirical correlation is concerned the results of the paper suggest that disagreement and how it
varies over the economic cycle is central to this phenomenon. Consequently, empirically studying
disagreement, whether and how it arises and how it varies over time might be of central importance
to understanding the financial markets.
References


Appendix A  Proofs

Proof of Remark 1. Use assumption (5) to compute the conditional probability of a signal given $\epsilon^\pi$ using
\[
\mathbb{P}^i(s^j|\epsilon^\pi) = \sum_{s^k} \mathbb{P}^i(s^j, s^k|\epsilon^\pi), \quad k \neq j
\]
and in computing the absolute difference between the beliefs of the two agents for a given signal note that $|1 - 2 \cdot \mathbb{I} \{A\}| = 1$ for any event $A$. \hfill \Box

Proof of Lemma 1. The probabilities $\mathbb{P}^i(\epsilon^\pi|s)$ are derived from the general formula:
\[
\mathbb{P}^i(\epsilon^\pi|s) = \frac{\mathbb{P}^i(s|\epsilon^\pi)\mathbb{P}^i(\epsilon^\pi)}{\mathbb{P}^i(s)}
\]
and note that $\mathbb{P}^i(\epsilon^\pi) = 1/2$ and $\mathbb{P}^i(s) = \sum_{\epsilon^\pi} \mathbb{P}^i(s|\epsilon^\pi) = 1/4$ for $i = 1, 2$. \hfill \Box

Proof of Lemma 2. Let $\chi$ be an $\mathcal{F}_{t+1}$-measurable variable and note that
\[
\mathbb{E}^i(\chi|\mathcal{F}_t) = \mathbb{E} \left[ \mathbb{E}^i(\chi|\mathcal{F}_{t+0.5}) \middle| \mathcal{F}_t \right]
\]
where $\mathbb{P}(s_{t+0.5}|\mathcal{F}_t) = 1/4$ for all $s_{t+0.5}$. Also note that
\[
\mathbb{E}^i(\chi|\mathcal{F}_{t+0.5}) = \mathbb{E}^i \left[ \mathbb{E}(\chi|\pi_t) \middle| \mathcal{F}_{t+0.5} \right] = \mathbb{E}^i \left[ \sum_{\epsilon_{t+1}} \chi(\epsilon_{t+1}, s_{t+0.5})\mathbb{P}(\epsilon_{t+1}|\pi_t) \middle| \mathcal{F}_{t+0.5} \right]
\]
\[
= \sum_{\epsilon_{t+1}} \chi(\epsilon_{t+1}, s_{t+0.5})\mathbb{E}^i \left[ \mathbb{P}(\epsilon_{t+1}|\pi_t) \middle| \mathcal{F}_{t+0.5} \right]
\]
where
\[
\mathbb{E}^i \left[ \mathbb{P}(\epsilon_{t+1}|\pi_t) \middle| \mathcal{F}_{t+0.5} \right] = \mathbb{P}^i(\epsilon_{t+1}|\mathcal{F}_{t+0.5})\mathbb{P}(\epsilon_{t+1}\epsilon_{t+1}|s_{t+0.5})
\]
and $\mathbb{P}^i(\epsilon_{t+1}|\mathcal{F}_{t+0.5})$ is as given in the lemma. \hfill \Box

Proof of Lemma 3. Let
\[
M_{t,t+1}^i = \delta \left( \frac{2C_{t+1} - \beta X_{t+1}}{2C_t - \beta X_t} \right)^{-\gamma}
\]
which from (19), the definition of $\alpha^i$ in (20) and the definition of $\omega$ in (21) can be written as
\[
M_{t,t+1}^i = \left( \frac{Y_{t+1}}{Y_t} \right)^{-\gamma} \left( \frac{1 - 2\beta \exp(\omega_{t+1})}{1 - 2\beta \exp(\omega_t)} \right)^{-\gamma} \left( \frac{\alpha_{t+1}^i}{\alpha_t^i} \right)^{-\gamma}.
\]
Now, note from the first order conditions (14) and (15) and that fact that $R_{t+0.5}^1 = P_{t+0.5}$ that $M_{t,t+1}^i$ is agent $i$’s one period stochastic discount factor for the stock. Finally, note from the result (18) that $M_{t,t+1}^i$ is also the half-period stochastic discount factor but given the new information set $\mathcal{F}_{t+0.5}$. The exponent of the aggregate endowment growth $1 - \gamma$ is obtained by dividing both sides with $Y_t$ to obtain the equilibrium condition for the price-dividend ratio. \hfill \Box

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Proof of Lemma 4. Financial markets are dynamically complete and therefore all state prices are the same across agents which implies that

$$\mathbb{P}^1(s_{t+0.5}, \epsilon_{t+1} | \mathcal{F}_t)M_{t,t+1}^1 = \mathbb{P}^2(s_{t+0.5}, \epsilon_{t+1} | \mathcal{F}_t)M_{t,t+1}^2$$

where $$\mathbb{P}^i(s_{t+0.5}, \epsilon_{t+1} | \mathcal{F}_t)$$ is as given in Lemma 2 and $$M_{t,t+1}^i$$ as given at the proof of Lemma 3. The result is obtained by rearranging the above equation.

Proof of Lemma 5. The budget constraints, as given in the optimization problem ($$\mathcal{O}$$), are binding in equilibrium and note that $$C_{t+0.5}^i = b_i^t$$ because it has been shown to be $$\mathcal{F}_t$$-measurable from the fact that the marginal rate of substitution from $$t$$ to $$t+0.5$$ is constant. Therefore it must be that

$$\bar{\theta}_{t+0.5}^i \mathbb{P}_{t+0.5} = \bar{\theta}_{t+0.5}^i R_{t+0.5} = \mathbb{E}_{t+0.5}^s \left[ M_{t,t+1}^s \bar{\theta}_{t+0.5}^i R_{t+1} \right]$$

where

$$M_{t,t+1}^s = \xi(\epsilon^y | z, s)e^{-\gamma(\epsilon' - y)} \left( \frac{1 - 2\beta e^\omega}{1 - 2\beta e^\omega} \right)^{-\gamma}$$

as shown in equation (24). We also have that $$\bar{\theta}_{t}^i \mathbb{P}_t = \mathbb{E}_t \left[ \bar{\theta}_{t}^i R_{t+0.5} \right]$$ from (18). Let $$W_t = \bar{\theta}_{t-0.5}^i R_t$$ and $$W_{t+0.5} = \bar{\theta}_{t}^i R_{t+0.5}$$, apply conditional expectations to the budget constraints and divide both sides with $$Y_t$$ to obtain the result.

Proof of Remark 2. The first property is an outcome of Hölder’s inequality and it’s reverse. The other two properties are obtained with partial differentiation and using the fact that $$\pi^1 > \pi^2$$.

Appendix B  Approximations

For a given state $$z = (\alpha, \pi^1, \pi^2, \omega)$$ let $$\Delta(z) := \pi^1 - \pi^2$$ and $$h(z) := \alpha(1 - \alpha)$$. The law of motion of the state vector is denoted by $$z' = L(z, s, \epsilon)$$. $$\Delta(z, s)$$ represents the corresponding variable in the middle of a period when signal $$s$$ is observed. The log-consumption growth is $$\gamma' - y$$ and let $$\mu(z)$$ and $$\sigma(z)$$ denote the conditional mean and volatility of log-consumption growth.

The stochastic discount factor is given by

$$M(z, z') = \delta \left( \frac{e^{\gamma' - y} - 2\beta e^{\phi_y \omega}}{1 - 2\beta e^\omega} \right)^{-\gamma} \xi(\epsilon^y | z, s).$$

The first component using Taylor expansion wrt. $$\gamma' - y$$ around $$-(1 - \phi_x)\omega$$ can be expressed as:

$$\left( \frac{e^{\gamma' - y} - 2\beta e^{\phi_y \omega}}{1 - 2\beta e^\omega} \right)^{-\gamma} = e^{\gamma(1 - \phi_x)\omega} \left[ 1 - \gamma(\omega)(\gamma' - y + (1 - \phi_x)\omega) \right.$$

$$+ \frac{\gamma(\omega)^2}{2}(\gamma' - y + (1 - \phi_x)\omega)^2 \left( 1 + \frac{1}{\gamma} - \frac{1}{\gamma(\omega)} \right) + O((\gamma' - y)^3) \right].$$
where $\gamma(\omega) := \gamma/(1 - 2\beta e^\omega)$ indicates the level of risk-aversion in state $z$. The second element is given by

$$\xi(\epsilon^y|z, s) = \frac{q(\epsilon^y|z, s)}{\mathbb{P}^*(\epsilon^y|z, s)}$$

where $q$ can be expressed using Taylor expansion for each $(\mathbb{P}^i)^{1/\gamma}$ around $\mathbb{P}^* = \alpha \mathbb{P}^1 + (1 - \alpha) \mathbb{P}^2$ to get

$$\xi(\epsilon^y|z, s) = \left[1 + h(z) \frac{1 - \gamma}{2\gamma} \left(\frac{\Delta(z, s)}{\mathbb{P}^*(\epsilon^y|z, s)}\right)^2 + \mathcal{O}\left(\frac{\Delta(z, s)}{\mathbb{P}^*(\epsilon^y|z, s)}\right)^3\right]^{\gamma},$$

and then using again Taylor expansion wrt. $\Delta(z, s)$ around zero to get

$$\xi(\epsilon^y|z, s) = 1 + h(z) \frac{1 - \gamma}{2\gamma} \left(\frac{\Delta(z, s)}{\mathbb{P}^*(\epsilon^y|z, s)}\right)^2 + \mathcal{O}(\Delta(z, s)^3).$$

Applying conditional expectations to both sides:

$$\mathbb{E}\left[\xi(\epsilon^y|z, s)|z\right] = 1 + 2h(z)\sigma^2 \frac{1 - \gamma}{\gamma} \mathbb{E}\left[\frac{\Delta(z, s)^2}{\sigma(z, s)^2}\bigg|z\right] + \mathcal{O}(\Delta(z, s)^3).$$

From the expressions derived for the stochastic discount factor the continuously compounded risk-free rate can be approximated by:

$$r^f(z) \approx -\log(\delta) + \gamma(\omega)\mu(z) + [\gamma(\omega) - \gamma](1 - \phi_x)\omega - 2h(z)\sigma^2 \frac{1 - \gamma}{\gamma} \mathbb{E}\left[\frac{\Delta(z, s)^2}{\sigma(z, s)^2}\bigg|z\right]$$

$$- \frac{\gamma(\omega)^2}{2} \left[\sigma(z)^2 + (\mu(z) + (1 - \phi_x)\omega)^2\right] \left(1 + \frac{1}{\gamma} - \frac{1}{\gamma(\omega)}\right). \quad (B1)$$

If the variation in the conditional volatility of endowment growth is neglected then the belief heterogeneity component in the approximate risk-free rate expression can be substituted with the following:

$$-2h(z) \frac{\sigma^2}{\sigma(z)^2} \frac{1 - \gamma}{\gamma} \left[(\phi_\pi \Delta(z))^2 + 2(1 - \phi_\pi)^2 \rho^2\right],$$

since $\Delta(z, s) = \phi_\pi \Delta(z) + (1 - \phi_\pi)\rho(s^1 - s^2)$. 

The wealth share of the first agent in state $z$ is denoted with $\vartheta(z) := \hat{W}(z)/(\hat{P}(z) + 1)$ and $\theta$ denotes the equilibrium holding of the market security by the first agent which for state $z$ is given by

$$\theta(z) = \frac{\hat{W}(z, s^*)}{\hat{P}(z, s^*)} = \vartheta(z) + \frac{\lambda(z, s^*)}{\hat{P}(z, s^*)} \quad (B2)$$

where $s^*$ is the state for which there does not exist a contingent claim with positive payoff and

$$\lambda(z, s) = \mathbb{E}\left[\vartheta(z') - \vartheta(z) \bigg|M(z, z') e^{y - y} \left[\hat{P}(z') + 1\right]\bigg|z, s\right]. \quad (B3)$$
Equivalently $\lambda(z)$ is defined as the above expectation but given state $z$ and thus,

$$\hat{\vartheta}(z) = c(z) + \lambda(z), \tag{B4}$$

where $c(z)$ is the consumption share of the first agent in state $z$ and is equal to $\alpha(1 - 2\beta e^\omega) + \beta e^\omega$. The quantity $\lambda$ is non-zero to the extent that wealth share changes are correlated with discounted prices. In the case where $\beta = 0$ and $\gamma = 1$ the correlation is identically zero whereas for $\gamma$’s other than one it becomes non-zero and in certain middle of the period states it becomes significant. When $\beta$ is positive then this correlation increases because wealth share changes are correlated with the habit-endowment surplus ratio.

In state $(z, s)$ the optimal stock holding for the first agent is given by $\vartheta(z, s) = \vartheta(L(z, s, e^*))$ where $e^*$ is the state for which there does not exist a contingent claim with positive payoff. Hence the equilibrium volume of trade in the market security over a period is given by

$$T(z, s) = \left| \vartheta(L(z, s, e^*)) - \vartheta(z) - \frac{\lambda(z, s^*)}{P(z, s*)} \right|$$

and

$$T(z, s, \epsilon) = \left| \vartheta(L(z, s, \epsilon)) + \frac{\lambda(L(z, s, \epsilon), s^*)}{P(L(z, s, \epsilon), s^*)} - \vartheta(L(z, s, e^*)) \right|. \tag{B5}$$

The most important component of the volume of the first round of trade is the change in the wealth share of which the most important part is the change in the consumption share. Similarly, the volume of the second round of trade in the case where $\epsilon$ is different than $e^*$ is mostly given by the difference in the consumption share between the two states.

Finally, using Taylor expansion of the probabilities $(\mathbb{P}^i)^{1/\gamma}$ around $\mathbb{P}^*$ the law of motion of the consumption surplus share is expressed by

$$\alpha' = \alpha + h(z) \frac{\Delta(z, s) e^\omega}{\gamma \mathbb{P}^*(\epsilon|z, s)} f_1(z, s),$$

where

$$f_1(z, s) = \frac{1 + \frac{1 - \gamma}{2\gamma} \frac{\Delta(z, s) e^\omega}{\mathbb{P}^*(\epsilon|z, s)} (1 - 2\alpha)}{1 + h(z) \frac{1 - \gamma}{2\gamma} \frac{\Delta(z, s)^2}{\mathbb{P}^*(\epsilon|z, s)^2}} + \mathcal{O}((\Delta(z, s)^3), \tag{B5}$$

which is identically one when $\gamma = 1$ or when $\Delta(z, s) = 0$. 45
### Appendix C  Tables and Figures

Table 1: Calibrated model parameters.

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\phi_\pi$</th>
<th>$\phi_x$</th>
<th>$\delta$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\kappa$</th>
<th>$\rho$</th>
<th>$\phi_\rho$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant $\rho$</td>
<td>2.01</td>
<td>3.37</td>
<td>0.056</td>
<td>0.90</td>
<td>0.995</td>
<td>0.425</td>
<td>3.50</td>
<td>0.10</td>
<td>0.15</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Stochastic $\rho$</td>
<td>2.01</td>
<td>3.37</td>
<td>0.056</td>
<td>0.90</td>
<td>0.988</td>
<td>0.425</td>
<td>3.50</td>
<td>0.10</td>
<td>-</td>
<td>0.95</td>
<td>0.20</td>
</tr>
</tbody>
</table>

The two rows refer to the two corresponding model specifications considered. The calibration aimed to fit first the consumption growth, risk-free rate, market price-dividend ratio and equity premium statistics as well as the statistics related to the relation between turnover and prices. These statistics and more are shown in Table 2.
Table 2: Data and calibrated model statistics.

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Constant $\rho$</th>
<th>Stochastic $\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Quarterly</td>
<td>Annual</td>
<td>Avg.</td>
</tr>
<tr>
<td>$\mu(\Delta y)$</td>
<td>1.89</td>
<td>2.01</td>
<td>2.01 (0.15)</td>
</tr>
<tr>
<td>$\sigma(\Delta y)$</td>
<td>1.33</td>
<td>3.37</td>
<td>3.37 (0.01)</td>
</tr>
<tr>
<td>$\rho_1(\Delta y)$</td>
<td>0.27</td>
<td>0.05</td>
<td>0.05 (0.04)</td>
</tr>
<tr>
<td>$corr(\Delta y, r^m)$</td>
<td>0.27</td>
<td>0.59</td>
<td>0.95 (0.01)</td>
</tr>
<tr>
<td>$corr(\Delta y, r^m - rf)$</td>
<td>0.23</td>
<td>0.60</td>
<td>0.98 (0.00)</td>
</tr>
<tr>
<td>$\mu(r^f)$</td>
<td>0.68</td>
<td>1.51</td>
<td>0.45 (0.75)</td>
</tr>
<tr>
<td>$\sigma(r^f)$</td>
<td>2.49</td>
<td>5.32</td>
<td>3.70 (0.56)</td>
</tr>
<tr>
<td>$\rho_1(r^f)$</td>
<td>0.58</td>
<td>0.49</td>
<td>0.87 (0.03)</td>
</tr>
<tr>
<td>$corr(r^f, T)$</td>
<td>-0.03</td>
<td>-0.17</td>
<td>0.03 (0.08)</td>
</tr>
<tr>
<td>$\mu(pd)$</td>
<td>3.35</td>
<td>3.20</td>
<td>3.25 (0.10)</td>
</tr>
<tr>
<td>$\sigma(pd)$</td>
<td>0.44</td>
<td>0.42</td>
<td>0.42 (0.05)</td>
</tr>
<tr>
<td>$\rho_1(pd)$</td>
<td>0.86</td>
<td>0.85</td>
<td>0.91 (0.02)</td>
</tr>
<tr>
<td>$corr(\Delta pd, T)$</td>
<td>0.42</td>
<td>0.40</td>
<td>0.45 (0.09)</td>
</tr>
<tr>
<td>$corr(</td>
<td>\Delta pd</td>
<td>, T)$</td>
<td>0.29</td>
</tr>
<tr>
<td>$corr(pd, [T])$</td>
<td>-0.38</td>
<td>-0.45</td>
<td>-0.02 (0.15)</td>
</tr>
<tr>
<td>$\mu(r^m)$</td>
<td>6.50</td>
<td>6.09</td>
<td>6.19 (0.31)</td>
</tr>
<tr>
<td>$\sigma(r^m)$</td>
<td>21.07</td>
<td>18.14</td>
<td>20.51 (0.76)</td>
</tr>
<tr>
<td>$\rho_1(r^m)$</td>
<td>-0.05</td>
<td>0.06</td>
<td>0.02 (0.05)</td>
</tr>
<tr>
<td>$corr(r^m, T)$</td>
<td>0.42</td>
<td>0.52</td>
<td>0.46 (0.08)</td>
</tr>
<tr>
<td>$corr(</td>
<td>r^m</td>
<td>, T)$</td>
<td>0.30</td>
</tr>
<tr>
<td>$corr(</td>
<td>r^m - \mu(r^m)</td>
<td>, T)$</td>
<td>0.22</td>
</tr>
<tr>
<td>$\mu(r^m - rf)$</td>
<td>5.81</td>
<td>4.58</td>
<td>5.73 (0.51)</td>
</tr>
<tr>
<td>$\sigma(r^m - rf)$</td>
<td>21.25</td>
<td>18.18</td>
<td>19.82 (0.70)</td>
</tr>
<tr>
<td>$\rho_1(r^m - rf)$</td>
<td>-0.05</td>
<td>0.07</td>
<td>-0.01 (0.05)</td>
</tr>
<tr>
<td>$\mu(r^m - rf)/\sigma(r^m - rf)$</td>
<td>0.27</td>
<td>0.25</td>
<td>0.29 (0.03)</td>
</tr>
<tr>
<td>$corr(r^m - rf, T)$</td>
<td>0.42</td>
<td>0.56</td>
<td>0.47 (0.10)</td>
</tr>
<tr>
<td>$corr(</td>
<td>r^m - rf</td>
<td>, T)$</td>
<td>0.30</td>
</tr>
<tr>
<td>$\rho_1([T])$</td>
<td>0.31</td>
<td>0.24</td>
<td>0.32 (0.09)</td>
</tr>
</tbody>
</table>

*The model statistics are obtained from 1000 simulations of 500 annual periods each. Avg. and Std. are the average and the standard deviation respectively of each statistic from the simulations. Standard errors can be computed by dividing the Std. statistic by $\sqrt{1000}$. The first two columns show the annual (1901-2003) and quarterly (1927:1-2007:4) statistics respectively. The third and fourth columns refer to the constant $\rho$ model specification and the last two columns to the stochastic $\rho$ specification. The notation $\mu(x)$, $\sigma(x)$ and $\rho_1(x)$ denote the sample mean, volatility and first-lag autocorrelation of a variable $x$ and $corr(x_1, x_2)$ denotes the sample correlation between two variables $x_1$ and $x_2$. 
Figure 1: Constant $\rho$ model - Log market price-dividend ratio $pd$

Figure 2: Constant $\rho$ model - One-period risk free rate $r^f$

Figure 3: Constant $\rho$ model - Market one-period expected excess return $E(r^m - r^f)$
Figure 4: Constant $\rho$ model - One-period market return volatility $\sigma(r^m)$

Figure 5: Constant $\rho$ model - One-period market security expected turnover $E(T)$

Figure 6: Constant $\rho$ model - First agent’s market security holding $\theta$
Figure 7: Constant $\rho$ model - Impact of $\rho$ on prices

Figure 8: Constant $\rho$ model - Impact of $\rho$ on the return-volume relation

Figure 9: Constant $\rho$ model - Impact of $\rho$ on the volatility-volume relation
Figure 10: Stochastic $\rho$ model - Log market price-dividend ratio $pd$

Figure 11: Stochastic $\rho$ model - One-period risk free rate $r^f$

Figure 12: Stochastic $\rho$ model - Market one-period expected excess return $E(r^m - r^f)$
Figure 13: Stochastic $\rho$ model - One-period market return volatility $\sigma(r^m)$

- $(\omega=-0.2, \rho=0)$
- $(\pi^1=0.5, \pi^2=0.5, \omega=-0.2)$
- $(\alpha=0.5, \pi^1=0.5, \pi^2=0.5)$

Figure 14: Stochastic $\rho$ model - One-period market security expected turnover $E(T)$

- $(\omega=-0.2, \rho=0)$
- $(\pi^1=0.5, \pi^2=0.5, \omega=-0.2)$
- $(\alpha=0.5, \pi^1=0.5, \pi^2=0.5)$

Figure 15: Stochastic $\rho$ model - Impact of $\phi_\rho$ on prices

- $\sigma(pd)$
- $\sigma(r_f)$
- $\mu(r^m-r_f)$
Figure 16: Stochastic $\rho$ model - Impact of $\phi_\rho$ on the return-volume relation

Figure 17: Stochastic $\rho$ model - Impact of $\eta$ on the return-volume relation

Figure 18: Stochastic $\rho$ model - Impact of $\eta$ on the volatility-volume relation