Geometry of noncommutative algebras

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Abstract: There has been several attempts to generalize commutative algebraic geometry to the noncommutative situation. Localizations with good properties rarely exist for noncommutative algebras, and this makes a direct generalization difficult. Our point of view, following Laudal, is that the points of the noncommutative geometry should be represented as simple modules, and that noncommutative deformations should be used to obtain a suitable localization in the noncommutative situation.

Let $A$ be an algebra over an algebraically closed field $k$. If $A$ is commutative and finitely generated over $k$, then any simple $A$-module has the form $M = A/m$, the residue field, for a maximal ideal $m \subseteq A$, and the commutative deformation functor $\text{Def}_M$ has formal moduli $\hat{A}_m$. In the general case, we may replace the $A$-module $A/m$ with the simple $A$-module $M$, and use the formal moduli of the commutative deformation functor $\text{Def}_M$ as a replacement for the complete local ring $\hat{A}_m$. We recall the construction of the commutative scheme $\text{simp}(A)$, with points in bijective correspondence with the simple $A$-modules of finite dimension over $k$, and with complete local ring at a point $M$ isomorphic to the formal moduli of the corresponding simple module $M$.

The scheme $\text{simp}(A)$ has good properties, in particular when there are no infinitesimal relations between different points, i.e. when $\text{Ext}^1_A(M, M') = 0$ for all pairs of non-isomorphic simple $A$-modules $M, M'$. It does not, however, characterize $A$. We use noncommutative deformation theory to define localizations, in general.

We consider the quantum plane, given by $A = k\langle x, y \rangle/(xy - qyx)$, as an example. This is an Artin-Schelter algebra of dimension two.
