LØSNINGSFORSLAG TIL EKSAMEN I MATEMATIKK 200 (FO210A) 10.12.08

Oppgave 1 :

a) $2B = 2\begin{bmatrix} 1 & -3 & 1 \\ -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -6 & 2 \\ -2 & 4 & 8 \end{bmatrix}$

A^2B er ikke definert.

$$BA = \begin{bmatrix} 1 & -3 & 1 \\ -1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 3 & 4 & -2 \\ 1 & 1 & -2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 6 & -2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 3 & 4 & -2 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 3 & 4 & -2 & 1 & 0 & 0 \end{array}\right] \quad -1 \quad -3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & -1 \\ 0 & -2 & -2 & 1 & 0 & -3 \end{array}\right] \quad -1$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & -2 & -2 & 1 & 0 & -3 \end{array}\right] \quad 2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 2 & 1 & -2 & -1 \end{array}\right] \quad \tfrac{1}{2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & \tfrac{1}{2} & -1 & -\tfrac{1}{2} \end{array}\right] \quad -2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & -\frac{1}{2} \end{array}\right] \begin{array}{c} \leftarrow \\ -2 \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & -3 \\ 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -4 & -6 \\ -2 & 2 & 4 \\ 1 & -2 & -1 \end{bmatrix}$$

b) $$\begin{vmatrix} a+1 & a+2 & -2 \\ 1 & a-1 & -a \\ 1 & 2 & a-2 \end{vmatrix} = \begin{vmatrix} a & a & -a \\ 1 & a-1 & -a \\ 1 & 2 & a-2 \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 1 & a-2 & 1-a \\ 1 & 1 & a-1 \end{vmatrix}$$

$$= a \begin{vmatrix} a-2 & 1-a \\ 1 & a-1 \end{vmatrix} = a(a-1) \begin{vmatrix} a-2 & -1 \\ 1 & 1 \end{vmatrix} = a(a-1)^2$$

Eksakt en løsning når $a \neq 0$ og $a \neq 1$.

$\underline{a = 0}$:

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & -1 & 0 & -2 \\ 1 & 2 & -2 & 1 \end{bmatrix} \begin{array}{c} -1 \quad -1 \\ \leftarrow \\ \leftarrow \end{array}$$

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -3 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \\ -\frac{1}{3} \\ \end{array}$$

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & -\frac{2}{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \leftarrow \\ -2 \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & -\frac{2}{3} & -1 \\ 0 & 1 & -\frac{2}{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Uendelig mange løsninger når $a = 0$.

$$\begin{array}{ccccccc} x & & - & \frac{2}{3}z & = & -1 \\ & y & - & \frac{2}{3}z & = & 1 \end{array}$$

$x = -1 + \frac{2}{3}z$

$y = 1 + \frac{2}{3}z$

$x = -1 + 2t$

$y = 1 + 2t$

$z = 3t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$\underline{a = 1}$:

$$\begin{bmatrix} 2 & 3 & -2 & 0 \\ 1 & 0 & -1 & -2 \\ 1 & 2 & -1 & 1 \end{bmatrix} \begin{array}{l} \leftarrow \\ \leftarrow \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 2 & 3 & -2 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix} \begin{array}{l} -2 \quad -1 \\ \leftarrow \\ \leftarrow \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 3 & 0 & 4 \\ 0 & 2 & 0 & 3 \end{bmatrix} \begin{array}{l} \\ \frac{1}{3} \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 2 & 0 & 3 \end{bmatrix} \begin{matrix} \\ -2 \\ \leftarrow \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Ingen løsning når $a = 1$.

c) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & -4 & -6 \\ -2 & 2 & 4 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$x = -1$, $y = 1$, $z = 1$.

Oppgave 2 :

a) $M1$: Rotasjon 270° om origo.

$M2$: Translasjon med vektoren $(-1, 1)$.

$M3$: Skjærtransformasjon med faktor 1 i y-retning.

Den første og siste transformasjonen er lineær.

b) Vi utfører først en translasjon som flytter punktet A til origo, deretter en speiling om linjen som nå går gjennom origo og danner 45° med den positive x-aksen, og til slutt en translasjon som flytter punktet A tilbake der det var.

$$M = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Oppgave 3 :

a) $\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1 \cdot 5 - 2 \cdot 2 = 1 \neq 0$

Det viser at vektorene $(1, 2)$ og $(2, 5)$ er lineært uavhengige og derfor en basis for $\mathbf{R}^2$.

$(1, 4) = a\,(1, 2) + b\,(2, 5)$

$$\begin{array}{rcrcr} a & + & 2\,b & = & 1 \\ 2\,a & + & 5\,b & = & 4 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \end{bmatrix} \begin{array}{l} -2 \\ \leftarrow \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{array}{l} \leftarrow \\ -2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix}$$

$a = -3$
$b = 2$

$(1, 4) = -3\,(1, 2) + 2\,(2, 5)$

$(1, 4)$ har koordinatene -3 og 2 med hensyn på basisen $\{(1, 2), (2, 5)\}$.

b) $T(1, 4) = -3\,T(1, 2) + 2\,T(2, 5) = -3\,(2, -1) + 2\,(4, 1) = (2, 5)$

Oppgave 4 :

a) $p(\lambda) = \begin{vmatrix} \lambda - 5 & -1 \\ -2 & \lambda - 6 \end{vmatrix} = (\lambda - 5)(\lambda - 6) - 2 = \lambda^2 - 11\lambda + 28 = 0$

$\lambda = \frac{11 \pm \sqrt{121 - 112}}{2} = \frac{11 \pm 3}{2}$

A har egenverdiene $\lambda = 4$ og $\lambda = 7$.

$\underline{\lambda = 4}$:

$$\begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x + y = 0 \ , \ y = -x$

$x = t$
$y = -t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ , \ t \neq 0$$

$\underline{\lambda = 7}$:

$$\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x - y = 0 \ , \ y = 2x$

$x = t$
$y = 2t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ , \ t \neq 0$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \ , \ D = \begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}$$

b) $\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$

La $\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} u \\ v \end{bmatrix}$.

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = P^{-1}AP \begin{bmatrix} u \\ v \end{bmatrix} = D \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 4u \\ 7v \end{bmatrix}$$

$$u = c_1\, e^{4t}$$
$$v = c_2\, e^{7t}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} u \\ v \end{bmatrix} = c_1\, e^{4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\, e^{7t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x = c_1\, e^{4t} + c_2\, e^{7t}$$
$$y = -\, c_1\, e^{4t} + 2\, c_2\, e^{7t}$$