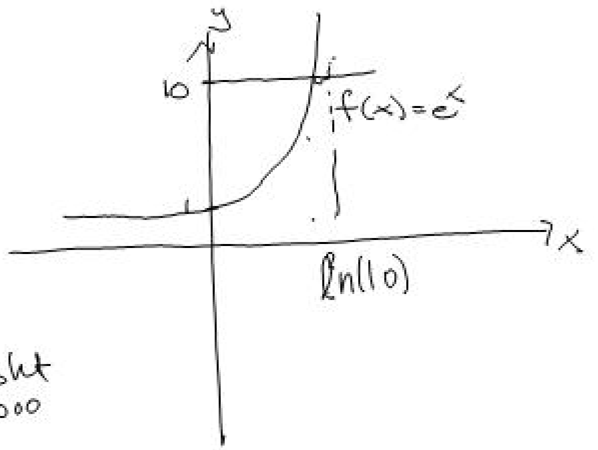
13/03/09: Logaritmer

Oppsummering:

① **Euler-tallet**:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \simeq 2.71828$$

② **Eksponensialfunksjon med grunntall e**

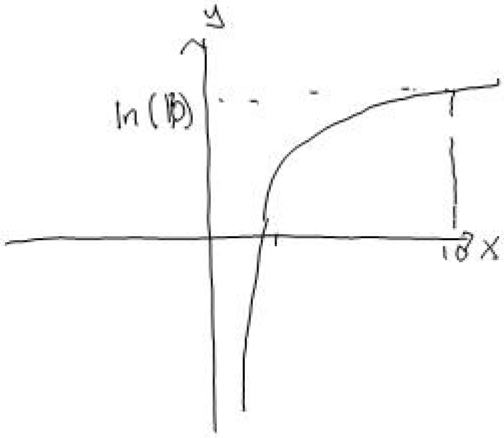


$$f(x) = e^x$$
$$= \exp(x)$$

$$f'(x) = e^x$$

* e^x vokser svært raskt
* $x = 10.000$: $e^{10.000} \simeq 10^{5000}$

③ **Naturlig logaritme**



$$f(x) = \ln x \quad , \quad x > 0$$

omvendt / invers } funksjon til e^x

$$f'(x) = \frac{1}{x}$$

* $\ln(x)$ vokser svært sakte
* $x = 10.000$: $\ln(10000) \simeq 9.2$

Hvorfor blir $(\ln x)' = \frac{1}{x}$?

Husk: $\begin{cases} (e^x)' = e^x \\ (a^x)' = a^x \cdot \left(\lim_{h \to 0} \frac{a^h - 1}{h} \right) \end{cases}$ ← et tall
1 hvis $a = e$

$$\begin{cases} e^{\ln x} = x \\ \ln(e^x) = x \end{cases}$$

Deriver $e^{\ln x} = x$ vha kjerneregelen:

$u = \ln x$
$u' = ?$

$$(e^{\ln x})' = (x)'$$

$$(e^u)' = 1$$

$$\frac{\cancel{e^u} \cdot u'}{\cancel{e^u}} = \frac{1}{e^u}$$

$$u' = \frac{1}{e^u}$$

$$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Derivasjonsformler: $a > 0$

$(e^x)' = e^x$ $\qquad (\ln x)' = \frac{1}{x}$

$(a^x)' = a^x \cdot \ln(a)$ $\qquad (\log_a x)' = \frac{1}{\ln(a)} \cdot \frac{1}{x}$

Likninger med eksponensial- og logaritmefunksjoner

(1) $\ln(a \cdot b) = \ln(a) + \ln(b)$

(2) $\ln(a/b) = \ln(a) - \ln(b)$

(3) $\ln(a^b) = b \cdot \ln(a)$

(4) $\log_a(b) = \ln(b)/\ln(a)$

gjelder også for $\log_a$ $(a > 0)$

(1) $a^m \cdot a^n = a^{m+n}$

(2) $a^m / a^n = a^{m-n}$

(3) $(a^m)^n = a^{m \cdot n}$

Eks:

$$2^x = 3 \implies \begin{aligned} \log_2(2^x) &= \log_2(3) \\ x \cdot \log_2(2) &= \log_2(3) \\ x &= \log_2(3) = \frac{\ln(3)}{\ln(2)} \simeq \underline{\underline{1.58}} \end{aligned}$$

$$\ln(2^x) = \ln(3)$$

$$\frac{x \cdot \cancel{\ln(2)}}{\cancel{\ln(2)}} = \frac{\ln(3)}{\ln(2)} \qquad x = \frac{\ln(3)}{\ln(2)} \simeq \underline{\underline{1.58}}$$

Eks:

$$\frac{\cancel{1000} \cdot 1.07^x}{\cancel{1000}} = \frac{1500}{1000}$$

$$1.07^x = 1.5$$

$$\ln(1.07^x) = \ln(1.5)$$

$$\frac{x \cdot \cancel{\ln 1.07}}{\cancel{\ln 1.07}} = \frac{\ln 1.5}{\ln 1.07}$$

$$x = \frac{\ln 1.5}{\ln 1.07} \simeq \underline{\underline{6}}$$

$$\ln(1000 \cdot 1.07^x) = \ln(1500)$$

$$\ln(1000) + \ln(1.07^x) = \ln(1500)$$

$$x \cdot \ln(1.07) = \ln(1500) - \ln(1000)$$

$$x \cdot \ln(1.07) = \ln\left(\frac{1500}{1000}\right)$$

$$x = \frac{\ln(1.5)}{\ln(1.07)} \simeq \underline{\underline{6}}$$

Eks:

$$3^x = 12 \cdot 2^x$$

$$\ln(3^x) = \ln(12 \cdot 2^x)$$

$$x \cdot \ln(3) = \ln(12) + x \cdot \ln(2)$$

$$x \cdot \ln(3) - x \cdot \ln(2) = \ln(12)$$

$$\frac{x \cdot \cancel{(\ln 3 - \ln 2)}}{\cancel{(\ln 3 - \ln 2)}} = \frac{\ln(12)}{\ln(3) - \ln(2)}$$

$$x = \frac{\ln(12)}{\ln(3) - \ln(2)} = \frac{\ln(12)}{\ln(3/2)} \simeq \underline{\underline{6.13}}$$

$$= \frac{\ln(3 \cdot 2^2)}{\ln(3) - \ln(2)}$$

$$= \frac{\ln(3) + 2 \cdot \ln(2)}{\ln(3) - \ln(2)}$$

Eks:

$$e^x - 3 = e^{-x}$$

$$\ln(e^x - 3) = \ln(e^{-x})$$

kommer ikke videre

$$e^{-x} = \frac{1}{e^x}$$

$$e^x - 3 = e^{-x} \quad | \cdot e^x$$

$$e^x \cdot e^x - 3e^x = 1$$

e^{2x}

$$e^x \cdot e^x - 3e^x - 1 = 0$$

$$u = e^x$$

$$u^2 - 3u - 1 = 0$$

$$u = \frac{3 \pm \sqrt{9 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{3 \pm \sqrt{13}}{2}$$

$$e^x = \frac{3 + \sqrt{13}}{2} \quad \text{eller} \quad e^x = \frac{3 - \sqrt{13}}{2} < 0$$

$$x = \ln\left(\frac{3 + \sqrt{13}}{2}\right) \simeq \underline{\underline{1.19}}$$

ingen løsning.

Eks:

$$\ln(x) = 4 \qquad | \text{ bruker } e^x$$

$$e^{\ln x} = e^4$$

$$x = \underline{\underline{e^4}} \simeq \underline{\underline{54{,}6}}$$

$$\ln(x) = -7$$

$$e^{\ln x} = e^{-7}$$

$$x = \underline{\underline{e^{-7}}} \simeq \underline{\underline{0.0009}}$$

Eks:

$$\ln(x^2) - \ln x = 4$$

$$2 \cdot \ln x - \ln x = 4$$

$$\ln x = 4$$

$$x = \underline{\underline{e^4}} \simeq \underline{\underline{55}}$$

$$\ln(x)^2 - \ln x = 4$$

$$\ln(x)^2 - \ln x - 4 = 0$$

$$\ln x = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-4)}}{2 \cdot 1} = \frac{1 \pm \sqrt{17}}{2}$$

$$\ln x = \frac{1 + \sqrt{17}}{2} \quad \text{eller} \quad \ln x = \frac{1 - \sqrt{17}}{2}$$

$$x = e^{\frac{1+\sqrt{17}}{2}} = \exp\left(\frac{1+\sqrt{17}}{2}\right) \simeq \underline{\underline{13.0}}$$

$$x = e^{\frac{1-\sqrt{17}}{2}} = \exp\left(\frac{1-\sqrt{17}}{2}\right) \simeq \underline{\underline{0.21}}$$

Eks:
$$\log_{13} x = 4 \quad | \text{ bruker } 13^x \text{ på begge sider}$$
$$13^{\log_{13} x} = 13^4$$
$$x = 13^4 \approx \underline{\underline{28561}}$$

$$\log x = 2 \quad | \text{ bruker } 10^x$$
$$10^{\log x} = 10^2$$
$$x = \underline{\underline{100}}$$

Eks: (11.44b i boken)

$$\ln(8x^2) - 2\ln(2x) = \ln(8) + \ln(x^2) - 2(\ln(2) + \ln x)$$
$$= \ln 8 + \cancel{2\ln x} - 2\ln 2 - \cancel{2\ln x}$$
$$= \ln 8 - 2\ln 2 = \ln(2^3) - 2\ln 2$$
$$= 3\ln 2 - 2\ln 2 = \underline{\underline{\ln 2}}$$

$$\ln(8x^2) - 2\ln(2x) = \ln(8x^2) - \ln((2x)^2)$$
$$= \ln \frac{8x^2}{(2x)^2} = \underline{\underline{\ln 2}}$$