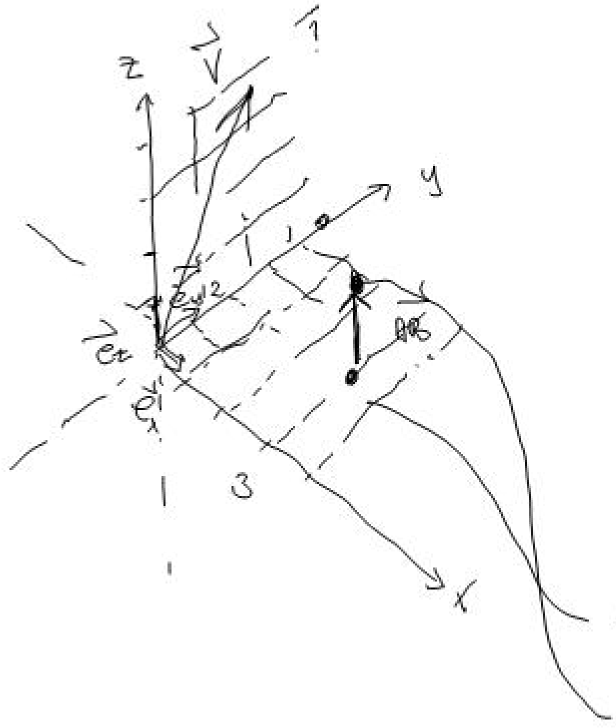
Vektorer i rommet (kap. 14)



Kartesisk koordinatsystem

- alle aksene skal stå normalt på hverandre
- x-aksen, y-aksen, z-aksen danner et høyrehåndssystem.

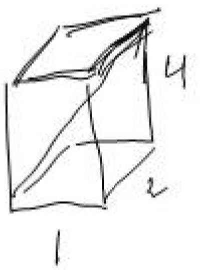
$A = (3, 2, 0)$

$B = (3, 2, 2)$

Enhetsvektorer:

$\vec{e}_x, \vec{e}_y, \vec{e}_z$: lengde 1, langs positive koordinatakser

$\vec{v} = (1, 2, 4) = 1 \cdot \vec{e}_x + 2 \cdot \vec{e}_y + 4 \cdot \vec{e}_z$



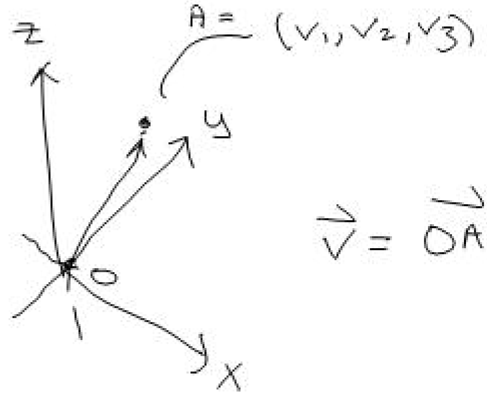
* $\vec{AB} = (0, 0, 2)$

* Hvis vi parallellforskyver $\vec{v} = (v_1, v_2, v_3)$

slik at startpkt blir origo $= (0, 0, 0)$,

så blir sluttpkt. (v_1, v_2, v_3).

$\vec{v} = (v_1, v_2, v_3)$



$\vec{v} = \vec{OA}$.

Regneoperasjoner:

① Vektoraddisjon / subtraksjon

$\vec{v} = (v_1, v_2, v_3)$

$\vec{w} = (w_1, w_2, w_3)$

$$\boxed{\begin{aligned} \vec{v} + \vec{w} &= (v_1 + w_1, v_2 + w_1, v_3 + w_3) \\ \vec{v} - \vec{w} &= (v_1 - w_1, v_2 - w_2, v_3 - w_3) \end{aligned}}$$

② Skalarmultiplikasjon

c (tall) , $\vec{v} = (v_1, v_2, v_3)$

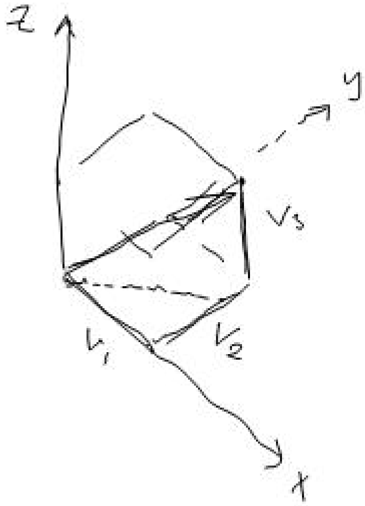
$$\boxed{c \cdot \vec{v} = (cv_1, cv_2, cv_3)}$$

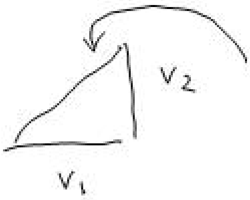
Eks: $(1,2,4) + (0,-1,3) = (1,1,7)$

$3 \cdot (0,-1,3) = (0,-3,9)$

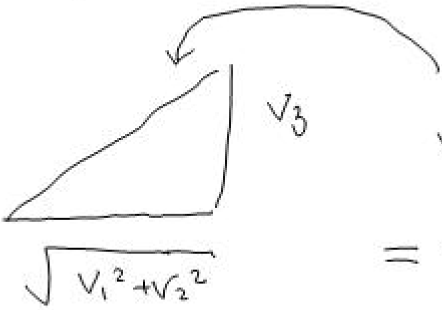
③ Lengden til en vektor

$\vec{v} = (v_1, v_2, v_3)$





$\sqrt{v_1^2 + v_2^2}$



$$\sqrt{\left(\sqrt{v_1^2+v_2^2}\right)^2 + v_3^2}$$

$$= \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$\boxed{|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}}$$

Eks: $\vec{v} = (1, 2, 4)$

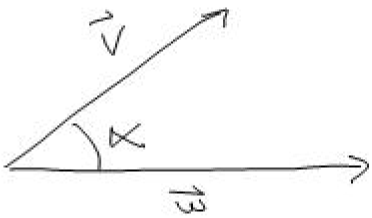
$$|\vec{v}| = \sqrt{1^2 + 2^2 + 4^2} = \underline{\underline{\sqrt{21}}}$$

(4) Skalarprodukt (prikkprodukt)

$\vec{v} = (v_1, v_2, v_3)$

$\vec{w} = (w_1, w_2, w_3)$

$$\boxed{\begin{aligned} \vec{v} \cdot \vec{w} &= v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3 \\ \vec{v} \cdot \vec{w} &= |\vec{v}| \cdot |\vec{w}| \cdot \cos\alpha \end{aligned}}$$

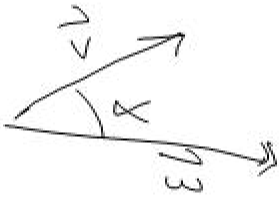


α : vinkelen mellom $\vec{v}$ og $\vec{w}$, mellom 0° og 180°.

(5) Vinkelen mellom to vektorer

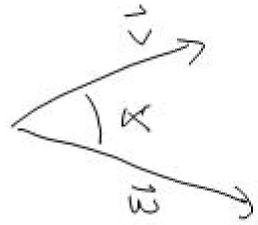
$\vec{v} = (v_1, v_2, v_3)$

$\vec{w} = (w_1, w_2, w_3)$

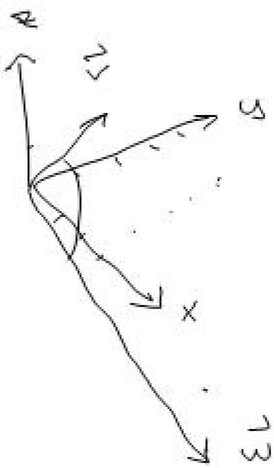


$$\boxed{\cos\alpha = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| \cdot |\vec{w}|}} = \frac{v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3}{|\vec{v}| \cdot |\vec{w}|}$$

Eks: $\vec{v} = (1,2,1)$
$\vec{w} = (3,4,-7)$



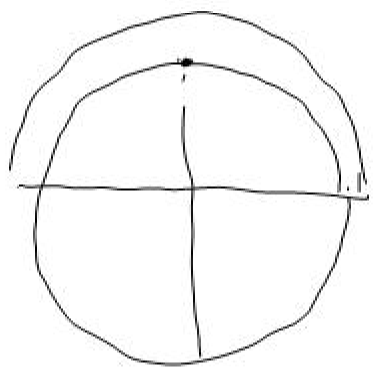
$$\vec{v}\cdot\vec{w} = (1,2,1)\cdot(3,4,-7)$$
$$= 1\cdot 3 + 2\cdot 4 + 1\cdot(-7)$$
$$= \underline{4}$$



$$|\vec{v}| = \sqrt{1^2+2^2+1^2} = \underline{\sqrt{6}}$$
$$|\vec{w}| = \sqrt{3^2+4^2+(-7)^2} = \underline{\sqrt{74}}$$

$$\cos\alpha = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\cdot|\vec{w}|} = \frac{4}{\sqrt{6}\cdot\sqrt{74}}$$

$$\alpha = \cos^{-1}\left(\frac{4}{\sqrt{6}\cdot\sqrt{74}}\right) \simeq 79^\circ$$



$$\begin{cases} \cos\alpha < 0 & : \quad \alpha > 90^\circ \\ \cos\alpha > 0 & : \quad \alpha < 90^\circ \\ \cos\alpha = 0 & : \quad \alpha = 90^\circ \end{cases}$$

Eks: $(1,2t,0)\cdot(t^2,-1,\sqrt{t})$
$$= 1\cdot t^2 + 2t\cdot(-1) + 0\cdot\sqrt{t}$$
$$= \underline{\underline{t^2 - 2t}}$$

$$|(1,2t,0)| = \sqrt{1^2+(2t)^2+0^2} = \underline{\underline{\sqrt{1+4t^2}}}$$

Spesialtilfeller:

$\vec{v}$ og $\vec{w}$ er normale ($\vec{v} \perp \vec{w}$)

$\alpha = 90^\circ \iff \vec{v} \cdot \vec{w} = 0$

$\vec{v}$ og $\vec{w}$ er parallelle ($\vec{v} \parallel \vec{w}$)

$$\left.\begin{array}{c} \alpha = 0^\circ \\ \text{eller} \\ \alpha = 180^\circ \end{array}\right\} \iff \left\{\begin{array}{l} \vec{w} = t \cdot \vec{v} \\ \text{for en } t \end{array}\right.$$

Eks: $\vec{v} = (1,2,1)$
$\vec{w} = (3,4,-7)$

$\vec{w} = t \cdot \vec{v}$

$(3,4,-7) = t \cdot (1,2,1)$

$(3,4,-7) = (t, 2t, t)$

$3 = t \Rightarrow t = 3$
$4 = 2t \Rightarrow 4 = 6$ umulig.
$-7 = t$

$\Rightarrow$ ingen løsning.

$\Rightarrow \vec{v} \not\parallel \vec{w}$.
($\vec{v}$ og $\vec{w}$ ikke parallelle)

⑥ Vektorprodukt (kryssprodukt)

$$\left.\begin{array}{l}\vec{v} = (v_1, v_2, v_3) \\ \vec{w} = (w_1, w_2, w_3)\end{array}\right\} \quad \underline{\vec{v} \times \vec{w} = \text{en vektor}}$$

Eksempel:

$\vec{v} = (1, 2, 0)$

$\vec{w} = (4, 1, 0)$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 1 & 2 & 0 \\ 4 & 1 & 0 \end{vmatrix}$$

$$= \vec{e}_x \cdot \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} - \vec{e}_y \cdot \begin{vmatrix} 1 & 0 \\ 4 & 0 \end{vmatrix} + \vec{e}_z \cdot \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$$

$$= \vec{e}_x \cdot (2 \cdot 0 - 0 \cdot 1) - \vec{e}_y \cdot (1 \cdot 0 - 0 \cdot 4) + \vec{e}_z \cdot (1 \cdot 1 - 2 \cdot 4)$$

$$= 0 \cdot \vec{e}_x + 0 \cdot \vec{e}_y - 7 \cdot \vec{e}_z$$

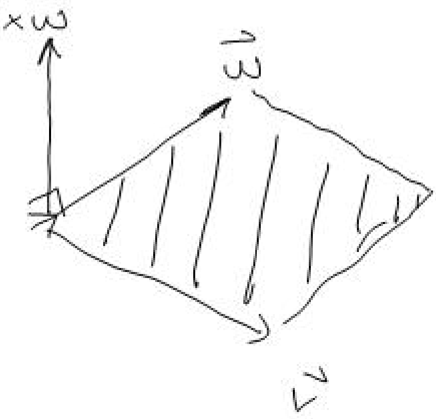
$$= \underline{\underline{(0, 0, -7)}}$$

$$\boxed{\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1)}$$

~~E~~genskaper til kryssproduktet

(a) $|\vec{v} \times \vec{w}| =$

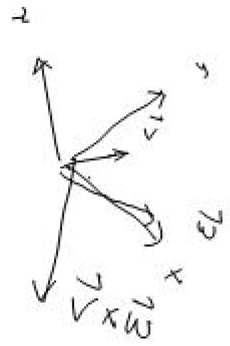
arealet av parallellogrammet utspent av $\vec{v}$ og $\vec{w}$



(b) Retning:

- $\vec{v} \times \vec{w}$ står normalt på $\vec{v}$ og på $\vec{w}$
- $\vec{v}, \vec{w}, \vec{v} \times \vec{w}$ danner et høyrehåndssystem.

Eks: $\vec{v} = (1,2,0)$
$\vec{w} = (4,1,0)$



Konsekvens:

$$\boxed{\vec{v} \times \vec{w} = - \vec{w} \times \vec{v}}$$

Eks:

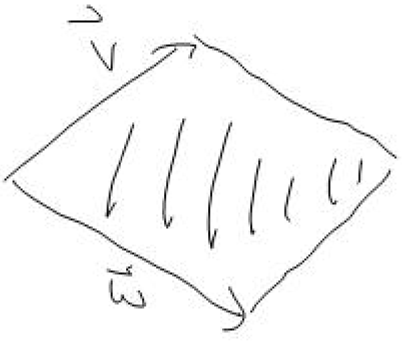
$\vec{v} = (3, 4, -1)$

$\vec{w} = (2, -1, 2)$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 3 & 4 & -1 \\ 2 & -1 & 2 \end{vmatrix} =$$

$$= \Big(4 \cdot 2 - (-1)\cdot(-1),\ -\big(3 \cdot 2 - (-1)\cdot 2\big),\ 3\cdot(-1) - 4 \cdot 2\Big)$$

$$= \underline{\underline{(7, -8, -11)}}$$



Areal:

$$A = |\vec{v} \times \vec{w}| =$$

$$= \sqrt{7^2 + (-8)^2 + (-11)^2}$$

$$= \sqrt{49 + 64 + 121}$$

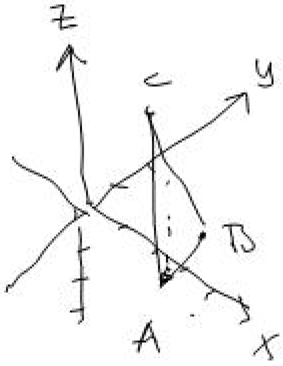
$$= \underline{\underline{\sqrt{234}}}$$

Eks:

$A = (1,2,-3)$

$B = (4,-1,2)$

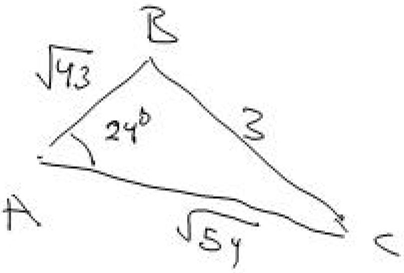
$C = (2,0,4)$



$\vec{AB} = (3,-3,5)$

$\vec{AC} = (1,-2,7)$

$\vec{BC} = (-2,1,2)$



$AB = \sqrt{3^2+(-3)^2+5^2} = \underline{\sqrt{43}}$

$AC = \sqrt{1^2+(-2)^2+7^2} = \underline{\sqrt{54}}$

$BC = \sqrt{(-2)^2+1^2+2^2} = \underline{3}$

$\underline{\angle A}$: (1) Cosinussetningen

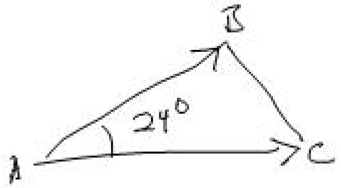
$3^2 = \sqrt{43}^2 + \sqrt{54}^2 - 2\cdot\sqrt{43}\cdot\sqrt{54}\cdot\cos A$

$9 = 43+54 - 2\cdot\sqrt{43}\cdot\sqrt{54}\cdot\cos A$

$\cos A = \dfrac{\cancel{+88}\ 44}{\cancel{+2}\sqrt{43}\cdot\sqrt{54}} = \dfrac{44}{\sqrt{43}\cdot\sqrt{54}}$

$\angle A \simeq \underline{\underline{24^\circ}}$

(2) <u>Vektorregning:</u>



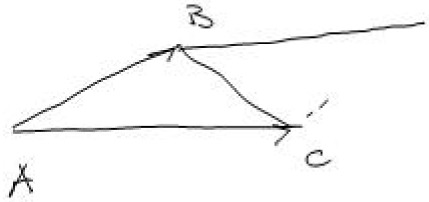
$$\cos A = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|\cdot|\vec{AC}|}$$

$$= \frac{(3,-3,5)\cdot(1,-2,7)}{\sqrt{43}\cdot\sqrt{54}}$$

$$= \frac{3\cdot 1 + (-3)\cdot(-2) + 5\cdot 7}{\sqrt{43}\cdot\sqrt{54}}$$

$$= \frac{44}{\sqrt{43}\cdot\sqrt{54}} \implies \underline{\underline{A \cong 24^\circ}}$$

<u>Areal:</u> $A = \frac{1}{2}\cdot AB\cdot AC\cdot \sin A$



$$A = \frac{1}{2}|\vec{AB}\times\vec{AC}|$$

$$= \frac{1}{2}\cdot\sqrt{121+256+9} = \underline{\underline{\frac{\sqrt{386}}{2}}}$$

$$\underline{\underline{\simeq 9.8}}$$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 3 & -3 & 5 \\ 1 & -2 & 7 \end{vmatrix}$$

$$= \left(-3\cdot 7 - 5\cdot(-2),\ \overset{-(3\cdot 7 - 5\cdot 1)}{5\cdot 1 - 3\cdot 7},\ \overset{3\cdot(-2) - 1\cdot(-3)}{\cancel{(-3)\cdot 7 - 5\cdot(-2)}}\right)$$

$$= \left(-21+10,\ 5-21,\ \overset{-6+3}{-\cancel{21+10}}\right) = \underline{\left(-11,\ -16,\ \overset{-3}{\cancel{-11}}\right)}$$

$$A = \frac{1}{2}|\vec{AB}\times\vec{AC}| = \frac{1}{2}\cdot\sqrt{(-11)^2+(-16)^2+(-11)^2} = \frac{1}{2}\sqrt{121+256+121}$$

$$= \frac{1}{2}\sqrt{498} \simeq \underline{\underline{11.2}}$$