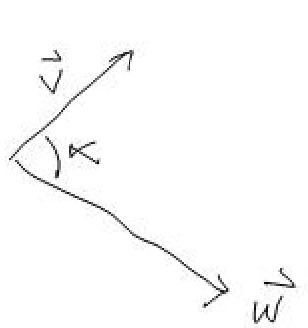
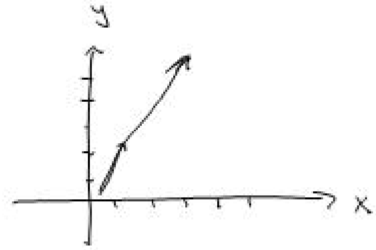
Vektorer i planet (Kap. 13)

7 Vinkelen mellom to vektorer



$$\cos\alpha = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\cdot|\vec{w}|}$$



Parallelle vektorer:

Eks:

$$\left.\begin{array}{l}\vec{v} = (1,2)\\ \vec{w} = (3,5)\end{array}\right\}$$ Er $\vec{v}$ og $\vec{w}$ parallelle?

Defn: To vektorer $\vec{v}$ og $\vec{w}$ er parallelle hvis

$$\left.\begin{array}{l}\text{vinkelen mellom } \vec{v}\\ \text{og } \vec{w} \text{ er } 0^\circ \text{ eller}\\ 180^\circ\end{array}\right\} \Longleftrightarrow \begin{array}{l}\vec{v} = t\cdot\vec{w}\\ \text{for et tall } t\end{array}$$

Eks: (forts).

$$(1,2) = t\cdot(3,5)$$

$$(1,2) = (3t, 5t)$$

$1 = 3t$ og $2 = 5t$

$\underline{t = 1/3}$ $\qquad 2 = 5/3$

umulig.

$\Rightarrow$ ingen løsn. $\Rightarrow$ $\vec{v}$ og $\vec{w}$ er ikke parallelle

$\vec{v} \parallel \vec{w}$ $\quad$ ($\vec{v}$ er parallell med $\vec{w}$)

$$\left\{ \begin{array}{l} \text{vinkelen er} \\ 0^\circ \text{ eller } 180^\circ \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \vec{v} = t \cdot \vec{w} \\ \text{for et tall } t \end{array} \right\} \longleftrightarrow \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = 0$$

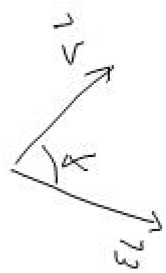
<u>Eks:</u> (fts). $\quad \vec{v} = (1,2)$

$\vec{w} = (3,5)$

$$\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 1 \cdot 5 - 3 \cdot 2 = 5 - 6 = -1 \neq 0$$

$\Rightarrow \vec{v}$ og $\vec{w}$ ikke parallelle.

<u>Normale vektorer</u>



$\vec{v}$ og $\vec{w}$ er normale $\quad (\vec{v} \perp \vec{w})$

$$\left\{ \begin{array}{l} \text{vinkelen} \\ \text{er } 90^\circ \end{array} \right\} \longleftrightarrow \left\{ \vec{v} \cdot \vec{w} = 0 \right\}$$

<u>Eks:</u> $\quad \vec{v} = (1,3) \qquad \vec{w} = (-2,1)$

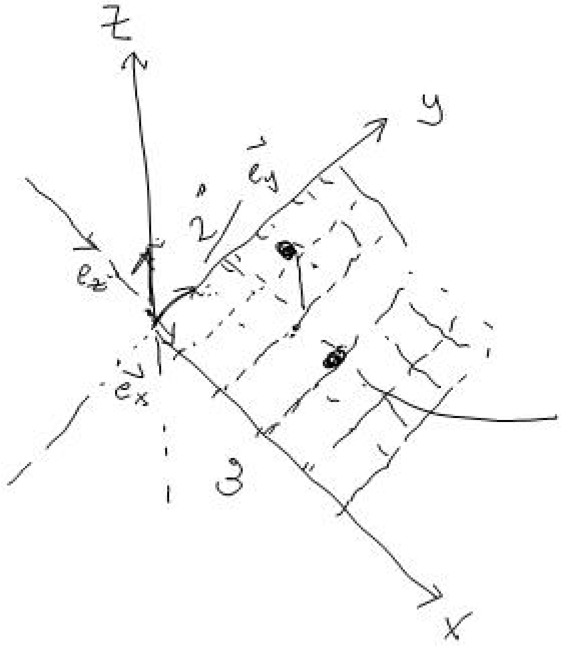
<u>Normale?</u> $\quad \vec{v} \cdot \vec{w} = (1,3) \cdot (-2,1) = 1 \cdot (-2) + 3 \cdot 1 = 1 \neq 0$

$\Rightarrow$ <u>ikke</u> normale

<u>Parallelle?</u> $\quad \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 1 \cdot 1 - 3 \cdot (-2) = 7 \neq 0$

$\Rightarrow$ <u>ikke</u> parallelle

Vektorer i rommet (kap. 14)



Kartesisk koordinatsystem

- alle aksene står normalt på hverandre
- høyrehåndsregel.

$(3,2,0)$ betyr $\begin{cases} x=3 \\ y=2 \\ z=0 \end{cases}$

$(2,2,1)$ betyr $\begin{cases} x=2 \\ y=2 \\ z=1 \end{cases}$

Vektor i rommet:

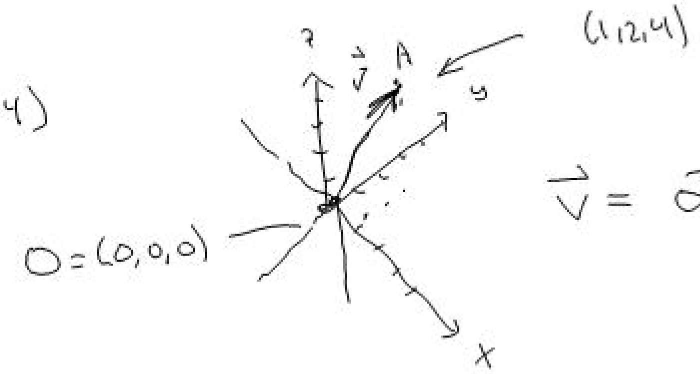
Enhetsvektorene: $\left\{ \vec{e_x}, \vec{e_y}, \vec{e_z} \right.$

$\vec{v} = (1,2,4)$ $\qquad \vec{v} = 1\cdot\vec{e_x} + 2\cdot\vec{e_y} + 4\cdot\vec{e_z}$

$\vec{w} = (-1,2,-3)$ $\qquad \vec{w} = -1\cdot\vec{e_x} + 2\cdot\vec{e_y} - 3\vec{e_z}$

Hvis vi parallellforskyver $\vec{v} = (1,2,4)$ slik at startpunkt blir origo $O = (0,0,0)$, da blir endepkt. til $\vec{v}$ punktet $(1,2,4)$.

Eks: $\vec{v} = (1,2,4)$



$\vec{v} = \overrightarrow{OA}$

Regneoperasjoner i rommet

① Vektor addisjon / subtraksjon:

$\vec{v} = (v_1, v_2, v_3)$

$\vec{w} = (w_1, w_2, w_3)$

$$\boxed{\begin{aligned} \vec{v} + \vec{w} &= (v_1 + w_1,\ v_2 + w_2,\ v_3 + w_3) \\ \vec{v} - \vec{w} &= (v_1 - w_1,\ v_2 - w_2,\ v_3 - w_3) \end{aligned}}$$

② Skalar multiplikasjon

$\vec{v} = (v_1, v_2, v_3)$ $\quad$ c : et tall (skalar)

$$\boxed{c \cdot \vec{v} = c \cdot (v_1, v_2, v_3) = (cv_1,\ cv_2,\ cv_3)}$$

Eks: $(1,2,4) + (3,-1,2) = (4,1,6)$

$2 \cdot (0,1,3) = (0,2,6)$

(3) Skalar produkt / prikk produkt

$\vec{v} = (v_1, v_2, v_3)$

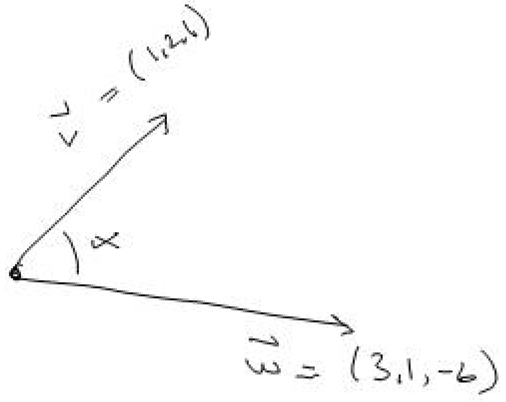
$\vec{w} = (w_1, w_2, w_3)$

$$\boxed{\begin{aligned} \vec{v} \cdot \vec{w} &= v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3 \\ \vec{v} \cdot \vec{w} &= |\vec{v}| \cdot |\vec{w}| \cdot \cos(\alpha) \end{aligned}}$$

Ek:

$$(1,2,1) \cdot (3,1,-6) = 1 \cdot 3 + 2 \cdot 1 + 1 \cdot (-6)$$

$$= \underline{-1}$$



(4) Lengden til en vektor

$\vec{v} = (v_1, v_2, v_3)$

$$\boxed{|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}}$$

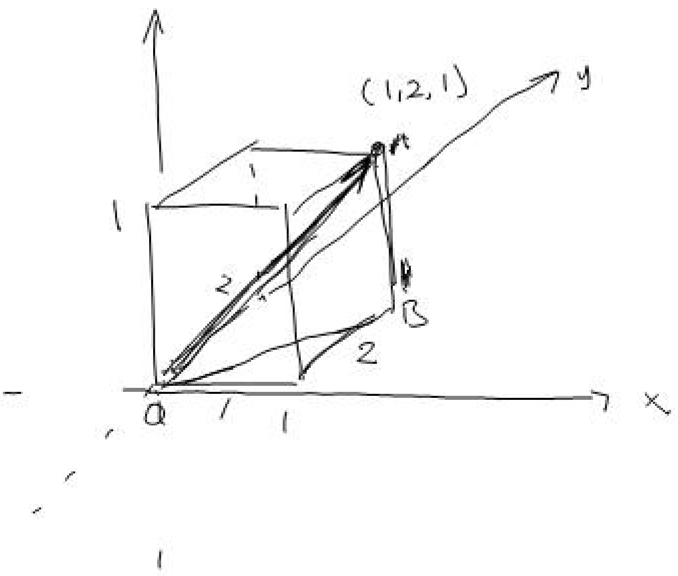
Ek:

$$|(1,2,1)| = \sqrt{1^2 + 2^2 + 1^2}$$

$$= \underline{\underline{\sqrt{6}}}$$

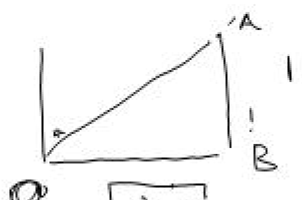
$$|(3,1,-6)| = \sqrt{3^2 + 1^2 + (-6)^2}$$

$$= \underline{\underline{\sqrt{46}}}$$



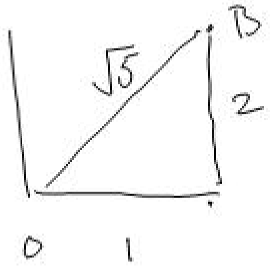
$\vec{v} = (1,2,1)$

Tverrsnitt:



$\sqrt{1^2+2^2}$

$= \sqrt{5}$



$$OA^2 = \left(\sqrt{1^2+2^2}\right)^2 + 1^2$$

$$= 1^2 + 2^2 + 1^2$$

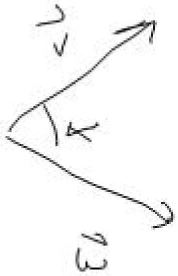
$$OA = \sqrt{1^2+2^2+1^2}$$

$$= \underline{\underline{\sqrt{6}}}$$

⑤ Vinkelen mellom to vektorer

$\vec{v} = (v_1, v_2, v_3)$

$\vec{w} = (w_1, w_2, w_3)$



$$\begin{cases} \vec{v} \cdot \vec{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3 \\ \vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos\alpha \end{cases}$$

$$\boxed{\cos\alpha = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| \cdot |\vec{w}|}} = \frac{v_1 w_1 + v_2 \cdot w_2 + v_3 \cdot w_3}{\sqrt{v_1^2 + v_2^2 + v_3^2} \cdot \sqrt{w_1^2 + w_2^2 + w_3^2}}$$

Eks:

$$\left.\begin{array}{l} \vec{v} = (1,2,1) \\ \vec{w} = (3,1,-6) \end{array}\right\} \quad \cos\alpha = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| \cdot |\vec{w}|}$$

$\vec{v} \cdot \vec{w} = 1 \cdot 3 + 2 \cdot 1 + 1 \cdot (-6) = -1$

$|\vec{v}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$

$|\vec{w}| = \sqrt{3^2 + 1^2 + (-6)^2} = \sqrt{46}$

$$\cos\alpha = \frac{-1}{\sqrt{6} \cdot \sqrt{46}} \cong -0.06$$

$$\alpha \cong \underline{\underline{93^\circ}}$$

Spesialtilfeller:

* **Normale vektorer:** $(\vec{v} \perp \vec{w})$

$$\left.\begin{array}{l}\vec{v} = (v_1, v_2, v_3) \\ \vec{w} = (w_1, w_2, w_3)\end{array}\right\} \quad \vec{v} \perp \vec{w} \iff \boxed{\vec{v} \cdot \vec{w} = 0}$$

* **Parallelle vektorer:** $(\vec{v} \parallel \vec{w})$

$$\left.\begin{array}{l}\vec{v} = (v_1, v_2, v_3) \\ \vec{w} = (w_1, w_2, w_3)\end{array}\right\} \quad \vec{v} \parallel \vec{w} \iff \boxed{\begin{array}{l}\vec{v} = t \cdot \vec{w} \\ \text{for et tall } t\end{array}}$$

Eks:

$$\left.\begin{array}{l}\vec{v} = (1, 2, 1) \\ \vec{w} = (3, 4, -7)\end{array}\right\} \quad \vec{v} \perp \vec{w}\,?$$

$$\vec{v} \cdot \vec{w} = (1,2,1)\,(3,4,-7)$$
$$= 1 \cdot 3 + 2 \cdot 4 + 1 \cdot (-7) =$$
$$= 4 \neq 0 \Rightarrow \vec{v} \not\perp \vec{w}$$

(ikke normal)

$\vec{v} \parallel \vec{w}\,?$

$$\left.\begin{array}{l}(1,2,1) = t \cdot (3,4,-7) \\ (1,2,1) = (3t, 4t, -7t) \\ \\ 1 = 3t \Rightarrow t = 1/3 \\ 2 = 4t \qquad 2 = 4 \cdot 1/3 \text{ umulig.} \\ 1 = -7t \Rightarrow \text{ingen løsn.}\end{array}\right\} \quad \vec{v} \not\parallel \vec{w}$$

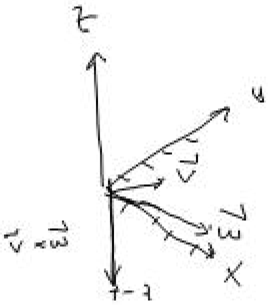
(ikke parallelle)

Regneregler for vektorer i rommet:

akkurat de samme!

⑥ Vektorprodukt (kryssprodukt)

$$\left.\begin{aligned}\vec{v} &= (v_1, v_2, v_3)\\ \vec{w} &= (w_1, w_2, w_3)\end{aligned}\right\} \quad \vec{v}\times\vec{w} \text{ er en vektor}$$



Eksempel:

$\vec{v} = (1, 2, 0)$

$\vec{w} = (4, 1, 0)$

$\vec{v}\times\vec{w}$:

$$\begin{vmatrix} \vec{e_x} & \boxed{\vec{e_y}} & \vec{e_z} \\ 1 & 2 & 0 \\ 4 & 1 & 0 \end{vmatrix} = \vec{e_x}\cdot\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} - \vec{e_y}\cdot\begin{vmatrix} 1 & 0 \\ 4 & 0 \end{vmatrix} + \vec{e_z}\cdot\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$$

$$= \vec{e_x}\cdot(2\cdot 0 - 0\cdot 1) - \vec{e_y}\cdot(1\cdot 0 - 0\cdot 4) + \vec{e_z}\cdot(1\cdot 1 - 2\cdot 4)$$

$$= \vec{e_x}\cdot 0 - \vec{e_y}\cdot 0 + \vec{e_z}\cdot(-7)$$

$$= (0, 0, -7)$$

$$\boxed{\vec{v}\times\vec{w} = (v_2w_3 - v_3w_2,\ v_3w_1 - v_1w_3,\ v_1w_2 - v_2w_1)}$$

Egenskaper for kryssprodukt

$$\left.\begin{array}{l}\vec{v} = (v_1, v_2, v_3)\\ \vec{w} = (w_1, w_2, w_3)\end{array}\right\}\quad \vec{v}\times\vec{w}$$

* $|\vec{v}\times\vec{w}|$ = arealet av parallellogrammet utspent av $\vec{v}$ og $\vec{w}$

$= |\vec{v}|\cdot|\vec{w}|\cdot\sin\alpha$

* <u>Retningen til $\vec{v}\times\vec{w}$:</u>
  - normal på $\vec{v}$ og på $\vec{w}$
  - $\vec{v}, \vec{w}, \vec{v}\times\vec{w}$ danner et høyrehåndssystem