EXTRA LECTURE I:

GRA 6035

MATHEMATICS

EIVIND ERIKSEN, JAN 25 2012

1. INTRODUCTION
2. STRATEGY
3. BASICS OF OPTIMIZATION
   (We did a - c below; d - e will be done Jan 26)

~70% of exam questions

1 - 2 FOCUS ON BASIC PROBLEMS

ONE TYPE OF PROBLEM FOR EACH LECTURE.

ADVANCED PROBLEMS IN THE LAST LECTURES IF THERE IS TIME.

remaining ~30% of exam questions

3 BASICS OF OPTIMALIZATION

a) derivation, computation of the Hessian
b) find stationary points
c) classify types of stationary points

25/01 Extra Lecture 1

d) determine if a function is convex / concave
e) find pts that satisfy first order conditions + constraints in constrained optim. problems.

26/01 Extra Lecture 2

ⓐ Derivation, Hessian matrix.

Find f'_x, f'_y, f'_z and the Hessian matrix in these cases:

i) $f = xy + xz - yz$

ii) $f = x^2 + y^2 + z^2 + z^3 + 2yz - 2x + 12y$

iii) $f = x^2 + 4xy + 4y^2 + e^y - y$

iv) $f = x^2 + y^2 + y^4 + yz - 1$

v) $f = \ln(x+1) + \ln(y+1) - \ln(z-1)$

vi) $f = z \cdot \sqrt{x^2+y^2}$

vii) $f = e^{xyz}$

Solución:

i) $f'_x = y + z$ $\qquad H(f) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

$f'_y = x - z$

$f'_z = x - y$

ii) $f'_x = 2x - 2$

$f'_y = 2y + 2z + 12$ $\qquad H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2+6z \end{pmatrix}$

$f'_z = 2z + 3z^2 + 2y$

iii) $f'_x = 2x + 4y$

$f'_y = 4x + 8y + e^y - 1$ $\qquad H(f) = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 8+e^y & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$f'_z = 0$

iv) $f'_x = 2x$

$f'_y = 2y + 4y^3 + z$

$f'_z = y$

$$H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+12y^2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

v) $f'_x = \dfrac{1}{x+1} = (x+1)^{-1}$

$f'_y = \dfrac{1}{y+1} = (y+1)^{-1}$

$f'_z = -\dfrac{1}{z-1} = -(z-1)^{-1}$

$$H(f) = \begin{pmatrix} -\dfrac{1}{(x+1)^2} & 0 & 0 \\ 0 & -\dfrac{1}{(y+1)^2} & 0 \\ 0 & 0 & +\dfrac{1}{(z-1)^2} \end{pmatrix}$$

vi) $f'_x = z \cdot \dfrac{1 \cdot 2x}{2\sqrt{x^2+y^2}} = \dfrac{xz}{\sqrt{x^2+y^2}}$

$f'_y = z \cdot \dfrac{1 \cdot 2y}{2\sqrt{x^2+y^2}} = \dfrac{yz}{\sqrt{x^2+y^2}}$

$f'_z = \sqrt{x^2+y^2} = \sqrt{x^2+y^2}$

$$H(f) = \begin{pmatrix} \dfrac{y^2 z}{(x^2+y^2)^{3/2}} & \dfrac{-xyz}{(x^2+y^2)^{3/2}} & \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{-xyz}{(x^2+y^2)^{3/2}} & \dfrac{x^2 z}{(x^2+y^2)^{3/2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} & 0 \end{pmatrix}$$

not basic problem

vii) $f'_x = e^{xyz} \cdot yz$

$f'_y = e^{xyz} \cdot xz$

$f'_z = e^{xyz} \cdot xy$

$$H(f) = \begin{pmatrix} y^2z^2 & z(xyz+1) & y(xyz+1) \\ z(xyz+1) & x^2z^2 & x(xyz+1) \\ y(xyz+1) & x(xyz+1) & x^2y^2 \end{pmatrix} \cdot e^{xyz}$$

not <u>basic</u> problem

v) In detail:

$$f(x,y,z) = \ln(x+1) + \ln(y+1) - \ln(z-1)$$

$$\left(\ln(x+1)\right)'_x = \left(\ln(u)\right)'_x = \frac{1}{u} \cdot u'_x = \frac{1}{x+1} \cdot 1$$

$u = x+1$

vi) In detail: $f = z \cdot \sqrt{x^2+y^2} = z \cdot \sqrt{u}$, $u = x^2+y^2$

$$f'_x = z \underset{u^{1/2}}{\underbrace{\left(\sqrt{u}\right)'_x}} = z \cdot \frac{1}{2\sqrt{u}} \cdot u'_x = z \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x$$

$$= \frac{xz}{\sqrt{x^2+y^2}}$$

$$f'_y = \frac{yz}{\sqrt{x^2+y^2}}$$ (in the same way as above)

$$f'_z = \sqrt{x^2+y^2} \cdot 1$$

The Hessian in vi) : $f = z \cdot \sqrt{x^2+y^2}$

$$f'_x = \frac{xz}{\sqrt{x^2+y^2}} = \frac{u}{v} \qquad f'_y = \frac{yz}{\sqrt{x^2+y^2}} \qquad f'_z = \sqrt{x^2+y^2}$$

$$f''_{xx} = \frac{u'_x v - u v'_x}{v^2} = \frac{\left(z \cdot \sqrt{x^2+y^2} - xz \cdot \frac{1 \cdot \cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}\right) \cdot \sqrt{x^2+y^2}}{(x^2+y^2) \cdot \sqrt{x^2+y^2}}$$

$$= \frac{z \cdot (\cancel{x^2}+y^2) - \cancel{x^2 z}}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{y^2 z}{(x^2+y^2)^{3/2}}$$

$$f''_{zx} = \frac{1 \cdot 2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$f''_{zy} = \frac{y}{\sqrt{x^2+y^2}}$$

$$f''_{zz} = 0$$

can be computed in a similar way

$$H(f) = \begin{pmatrix} \frac{y^2 z}{(x^2+y^2)^{3/2}} & * & \frac{x}{\sqrt{x^2+y^2}} \\ * & * & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{pmatrix}$$

vii) in detail: $f = e^{xyz} = e^u$, $u = xyz$

$$f'_x = (e^u)'_x = e^u \cdot u'_x = e^{xyz} \cdot yz$$
$$f'_y = (e^u)'_y = e^u \cdot u'_y = e^{xyz} \cdot xz$$
$$f'_z = (e^u)'_z = e^u \cdot u'_z = e^{xyz} \cdot xy$$

$$f''_{xx} = yz \cdot (e^{xyz})'_x = yz \cdot e^{xyz} \cdot yz = \underline{y^2 z^2 e^{xyz}}$$

$$f''_{xy} = (\underbrace{e^{xyz}}_{u} \cdot \underbrace{yz}_{v})'_y = \underbrace{(e^{xyz})'_y}_{u'} \cdot \underbrace{yz}_{v} + \underbrace{e^{xyz}}_{u} \cdot \underbrace{(yz)'_y}_{v'}$$

$$= e^{xyz} \cdot xz \cdot yz + e^{xyz} \cdot z = e^{xyz}(xyz^2 + z)$$
$$= \underline{e^{xyz} \cdot z \cdot (xyz+1)}$$

rest of the second order derivatives can be computed in a similar way

$$H(f) = \begin{pmatrix} y^2z^2 & z(xyz+1) & y(xyz+1) \\ z(xyz+1) & x^2z^2 & x(xyz+1) \\ y(xyz+1) & x(xyz+1) & x^2y^2 \end{pmatrix} e^{xyz}$$

(b) Stationary pts

Solve: $f'_x = 0 \qquad f'_y = 0 \qquad f'_z = 0$

Tip: Solve easy equations first!

Exercise: Find st. pts in i) - viii) above.

i)
$$f'_x = y+z = 0$$
$$\left.\begin{array}{l} f'_y = x - z = 0 \Rightarrow x = z \\ f'_z = x - y = 0 \Rightarrow x = y \end{array}\right\} \Rightarrow x = y = z$$

$$\Rightarrow y + z = y + y = 0$$
$$2y = 0$$
$$y = 0$$
$$\Downarrow$$
$$\underline{\underline{x = y = z = 0}}$$

ii)
$$f'_x = 2x - 2 = 0 \qquad \Rightarrow x = 1$$
$$f'_y = 2y + 2z + 12 = 0 \qquad \Rightarrow 2y + 2z = -12$$
$$y + z = -6 \qquad y = -6 - z$$
$$f'_z = 2z + 3z^2 + 2y = 0$$
$$\Rightarrow 2z + 3z^2 + 2(-6-z) = 0$$
$$3z^2 - 12 = 0$$
$$z^2 = 4 \Rightarrow z = \pm 2$$
$$y = -8, -4$$
$$\Downarrow$$
$$(x,y,z) = \cancel{(1,2,-8) \text{ or } (1,-2,+4)} \quad \underline{\underline{(1,-8,2) \text{ or } (1,-4,-2)}}$$

iii) $f'_x = 2x+4y=0 \Rightarrow x=-2y \Rightarrow x=0$

$f'_y = 4x+8y+e^y-1=0 \Rightarrow 4(-2y)+8y+e^y-1=0 \Rightarrow e^y=1 \Rightarrow y=0$

$(y=\ln 1)$

$f'_z=0 \Rightarrow z$ free variable

$\Downarrow$

$(x,y,z) = \underline{\underline{(0,0,z)}}$ (z free variable)

iv) $f'_x = 2x = 0 \Rightarrow x=0$

$f'_y = 2y+4y^3+z=0 \Rightarrow z=0$

$f'_z = y = 0 \Rightarrow y=0$

$\Rightarrow (x,y,z) = \underline{\underline{(0,0,0)}}$

v) $f'_x = \frac{1}{x+1}=0$ impossible $\Rightarrow$ no $\underline{\text{solutions}}$

$f'_y = \frac{1}{y+1}=0$

$\Downarrow$

$\underline{\text{no stationary pts}}$

$f'_z = -\frac{1}{z-1}=0$

vi) $f'_x = \frac{xz}{\sqrt{x^2+y^2}}=0$

$f'_y = \frac{yz}{\sqrt{x^2+y^2}}=0$

$f'_z = \sqrt{x^2+y^2}=0 \Rightarrow x=0, y=0 \Rightarrow f'_x, f'_y$ not defined

$\Rightarrow \underline{\text{no stationary pts.}}$

vii) $f'_x = e^{xyz} \cdot yz = 0 \Rightarrow yz = 0$

$f'_y = e^{xyz} \cdot xz = 0 \Rightarrow xz = 0$

$f'_z = e^{xyz} \cdot xy = 0 \Rightarrow xy = 0$

$yz = 0 \Rightarrow \boxed{y = 0 \quad \text{or} \quad z = 0}$

$\underline{y=0}$: $yz = 0$, $xz = 0$, $xy = 0$ $\Rightarrow$ $y = 0, x = 0$

ok $\quad$ | $\quad$ ok $\quad$ or

$x = 0$ or $z = 0$ $\quad$ $y = 0, z = 0$

$\underline{y \neq 0}$: $yz = 0$, $xz = 0$, $xy = 0$ $\Rightarrow$ $x = 0, z = 0$

$\Downarrow$ $\quad$ ok $\quad$ $\Downarrow$

$z = 0$ $\quad$ $x = 0$

$\underline{\text{Stat. pts}}$: $\begin{cases} (0,0,z) & z \text{ free} \\ (0,y,0) & y \text{ free} \\ (x,0,0) & x \text{ free} \end{cases}$

(c) Classification of stationary points

Look at the Hessian matrix at the stationary point;

$$f''(x_0, y_0, z_0)$$

Repeat for each stationary point

where (x_0, y_0, z_0) is the stationary point.

positive definite	$\Rightarrow$ local min.
negative definite	$\Rightarrow$ local max.
indefinite	$\Rightarrow$ saddle point

Classify all stationary pts in i) – vii).

i) Stationary pts: $(0,0,0)$

Hessian at $(0,0,0)$:

$$f''(0,0,0) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$D_1 = 0$

$D_2 = -1 \Rightarrow$ indefinite

$\Downarrow$

$(0,0,0)$ is saddle pt.

ii) Stationary pts: $(1,-8,2)$, $(1,-4,-2)$

$$f''(1,8,+2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 14 \end{pmatrix}$$

$D_1 = 2$

$D_2 = 4$

$D_3 = 48$

positive defn.

$\Downarrow$

$(1,-8,2)$ local min

$$f''(1,-4,-2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & -10 \end{pmatrix}$$

$\left.\begin{array}{l} D_1 = 2 \\ D_2 = 4 \\ D_3 = -48 \end{array}\right\}$ indefinite

$\Downarrow$

$(1,-4,-2)$ saddle point

We use (leading) principal minors:

$D_1 > 0,\ D_2 > 0,\ D_3 > 0 \iff$ positive definite

$D_1 < 0,\ D_2 > 0,\ D_3 < 0 \iff$ negative definite

Doesn't fit pattern $\left.\begin{array}{c} D_1 \geq 0,\ D_2 \geq 0,\ D_3 \geq 0 \\ \text{or} \\ D_1 \leq 0,\ D_2 \geq 0,\ D_3 \leq 0 \end{array}\right\} \iff$ indefinite

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Problems (Contined)

iii) Stat. pts: $(0,0,z)$ (z free)

Hessian:

$$f''(0,0,z) = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$D_1 = 2$ $\quad \Delta_1 = 2, 9, 0$

$D_2 = 18 - 16 = 2$ $\quad \Delta_2 = 2, 0, 0$

$D_3 = 0$

positive semidefinite

$\Downarrow$

second derivative test is inconclusive

We must use another method:

not basic problem

* $f(x,y) = x^2 + 4xy + 4y^2 + e^y - y$ is positive definite as a function of two vars

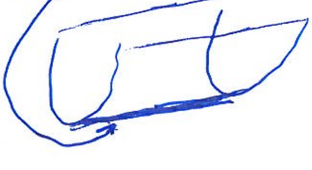
  stat pts: $(0,0)$

  Hessian: $\begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix}$ $\quad D_1 = 2$, $D_2 = 2$

  picture $\cup$

Each $(0,0,z)$ is a local min

* when we extend to three vars, but z is not part of the function expression, we get this picture:



iv) $(x,y,z) = (0,0,0)$ st. pts.

Hessian:

$$f''(0,0,0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$D_1 = 2$
$D_2 = 0$
$D_3 = -2$

indefinite $\Downarrow$ $(0,0,0)$ is saddle pt

v), vi) No stationary pts

vii) Stat. pts: $(x,0,0)$ x free
$(0,y,0)$ y "
$(0,0,z)$ z "

$$f''(x,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix}$$

$D_1 = 0$ $\quad \Delta_1 = 0,0,0$
$D_2 = 0$ $\quad \Delta_2 = 0,0,-x^2$
$D_3 = 0$ $\quad \Delta_3 = 0$

$\Rightarrow$ $(x,0,0)$ saddle pt if $x \neq 0$
test inconclusive if $x = 0$

$$f''(0,y,0) = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix}$$

$D_1 = 0$ $\quad \Delta_1 = 0,0,0$
$D_2 = 0$ $\quad \Delta_2 = 0,-y^2,0$
$D_3 = 0$ $\quad \Delta_3 = 0$

$\Rightarrow$ $(0,y,0)$ saddle pt if $y \neq 0$
test inconclusive if $y = 0$.

$$f''(0,0,z) = \begin{pmatrix} 0 & z & 0 \\ z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$D_1 = 0$ $\quad \Delta_1 = 0,0,0$
$D_2 = -z^2$ $\quad \Delta_2 = -z^2,0,0$
$D_3 = 0$ $\quad \Delta_3 = 0$

$\Rightarrow$ $(0,0,z)$ saddle pt if $z \neq 0$
test inconclusive if $z = 0$

$(x,y,z) = (0,0,0)$: Also saddle point, we must use a different method to show this:

not basic problem $\rightarrow$

$$f(0,0,0) = e^0 = 1$$

$$f(a,a,a) = e^{a^3} > e^0 = 1 \quad \text{if } a > 0$$

$$e^{a^3} < e^0 = 1 \quad \text{if } a < 0$$

$$\Downarrow$$

$(0,0,0)$ saddle point

Tomorrow 26/01 : EXTRA LECTURE 2

- continue with
  - (d) convex / concave functions
  - (e) find pts that satisfy F.O.C. + Constraints.
- Start with matrix problems

EXTRA LECTURE 2:

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MATHEMATICS

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Plan:

1. BASICS OF OPTIMIZATION: (continued)
   - (d) Determine if a function is convex/concave
   - (e) Find points that satisfy first order conditions + constraints in constrained optimalization problems.

2. MATRICES / LIN. ALGEBRA:
   - (a) Compute determinants and ranks
   - (b) Finding eigenvalues / eigenvectors
   - (c) Determine if a matrix is diagonalizable

   } ~~Next~~ EXTRA LECTURE 3 27/01

(d) Convex / Concave

We use the Hessian matrix of $f(x,y,z)$,

$$H(f) = f'' = \begin{pmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{xy} & f''_{yy} & f''_{yz} \\ f''_{xz} & f''_{yz} & f''_{zz} \end{pmatrix}$$

Criteria:

$f''(x,y,z)$ pos. semidefinite for all x,y,z $\iff$ f convex

$f''(x,y,z)$ neg. semidefinite for all x,y,z $\iff$ f concave

Check if f is convex or concave (or both) in cases i) – vii).

Solution:

i)
$$f'' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \begin{array}{l} D_1 = 0 \\ D_2 = -1 \Rightarrow \text{indefinite} \Rightarrow \end{array} \left\{ \begin{array}{l} \text{not convex} \\ \text{not concave} \end{array} \right.$$

ii)
$$f'' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2+6z \end{pmatrix} \qquad \begin{array}{l} D_1 = 2 \\ D_2 = 4 \\ D_3 = 24z \end{array}$$

– can be both pos. and neg. $\Rightarrow$ $\left\{ \begin{array}{l} \text{not convex} \\ \text{not concave} \end{array} \right.$

iii)
$$f'' = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 8+e^y & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} D_1 = 2 \\ D_2 = 2e^y > 0 \\ D_3 = 0 \end{array} \qquad \begin{array}{l} \Delta_1 = 2,\ 8+e^y > 0,\ 0 \\ \Delta_2 = 2e^y > 0,\ 0,\ 0 \\ \Delta_3 = 0 \end{array}$$

$\Downarrow$

pos. semidefinite

$\Downarrow$

convex,
not concave

iv)
$$f'' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+12y^2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \left. \begin{array}{l} D_1 = 2 \\ D_2 = 4+24y^2 > 0 \\ D_3 = -2 < 0 \end{array} \right\} \Rightarrow \begin{array}{l} \text{not convex} \\ \text{not concave} \end{array}$$

v) $$f'' = \begin{pmatrix} -\frac{1}{(x+1)^2} & 0 & 0 \\ 0 & -\frac{1}{(y+1)^2} & 0 \\ 0 & 0 & \frac{1}{(z-1)^2} \end{pmatrix}$$

$$D_1 = -\frac{1}{(x+1)^2} < 0$$

$$D_2 = +\frac{1}{(x+1)^2(y+1)^2} > 0$$

$$D_3 = +\frac{1}{(x+1)^2(y+1)^2(z-1)^2} > 0$$

$\Downarrow$

not concave
not convex

vi) $$f'' = \begin{pmatrix} \frac{y^2 z}{(x^2+y^2)^{3/2}} & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$D_1 = \frac{y^2 z}{(x^2+y^2)^{3/2}}$$

can be pos. and neg.

$\Downarrow$

not concave
not convex

vii) $$f'' = \begin{pmatrix} y^2z^2 & z(xyz+1) & y(xyz+1) \\ z(xyz+1) & x^2z^2 & x(xyz+1) \\ y(xyz+1) & x(xyz+1) & x^2y^2 \end{pmatrix} e^{xyz}$$

$$D_1 = y^2z^2e^{xyz} \geq 0 \qquad \Delta_1 = x^2z^2e^{xyz},\ x^2y^2e^{xyz} \geq 0$$

$$D_2 = x^2y^2z^4 - z^2(xyz+1)^2 = -2xyz^3 - z^2 = -z^2(1+2xyz)$$

can be both pos. and neg.

$\Downarrow$

not convex
not concave

(e) Find all points that satisfy first order conditions (FOC), constraints (C) and, in case of inequality constraints, the complementary slackness conditions (CSC).

Ex: max/min $f(x,y) = \ln(x+1) + \ln(y+1)$ subj. to $\begin{cases} y \leq 5 \\ x+y \leq 2 \end{cases}$

ADMISSIBLE PTS: $\boxed{\begin{array}{l} y \leq 5 \\ x+y \leq 2 \end{array}}$

"

PT that satisfy the constraint (C)

FIRST ORDER CONDITIONS: (FOC)

$$\mathcal{L} = \ln(x+1) + \ln(y+1) - \lambda_1 \cdot (y) - \lambda_2 (x+y)$$

$$\mathcal{L}'_x = \frac{1}{x+1} - \lambda_2 \cdot 1 = 0$$

$$\mathcal{L}'_y = \frac{1}{y+1} - \lambda_1 \cdot 1 - \lambda_2 \cdot 1 = 0$$

COMPL. SLACKNESS COND: (CSC)

$\lambda_1 \geq 0$

$\lambda_2 \geq 0$

If $y < 5$ then $\lambda_1 = 0$

If $x+y < 2$ then $\lambda_2 = 0$

Exercises: Find the pts that satisfy FOC + C (+ CSC)

i) max/min $f(x,y,z) = 12x - 9y^2 + 2z^3$ subj. to $\begin{cases} z - x = 0 \\ y - z = 0 \end{cases}$

ii) max/min $f(x,y) = \ln(x+1) + \ln(y+1)$ subj. to $\begin{cases} y \leq 5 \\ x + y \leq 2 \end{cases}$

iii) max/min $f(x,y,z) = 2z$ — " — $\begin{cases} x^2 + y^2 = 2 \\ x + y + z = 1 \end{cases}$

Solution:

i) FOC: $\mathcal{L} = 12x - 9y^2 + 2z^3 - \lambda_1(z - x) - \lambda_2(y - z)$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 12 + \lambda_1 = 0 \\ \mathcal{L}'_y &= -18y - \lambda_2 = 0 \\ \mathcal{L}'_z &= 6z^2 - \lambda_1 + \lambda_2 = 0 \end{aligned}}$$

C: $\boxed{\begin{aligned} z - x &= 0 \\ y - x &= 0 \end{aligned}}$

No CSC's since equality constraints

$\Downarrow$

$x = y = z$

$12 + \lambda_1 = 0 \Rightarrow \lambda_1 = -12$

$-18y - \lambda_2 = 0 \Rightarrow \lambda_2 = -18y$

$6z^2 - \lambda_1 + \lambda_2 = 0 \Rightarrow 6y^2 + 12 - 18y = 0$ abc

$y = 1, 2$

$y = 1$: $x = y = z = 1$, $\lambda_1 = -12$, $\lambda_2 = -18$

$y = 2$: $x = y = z = 2$, $\lambda_1 = -12$, $\lambda_2 = -36$

ii) FOC: $\boxed{\begin{array}{l} \dfrac{1}{x+1} - \lambda_2 = 0 \\ \dfrac{1}{y+1} - \lambda_1 - \lambda_2 = 0 \end{array}}$

See above

C: $\boxed{\begin{array}{l} y \le 5 \\ x+y \le 2 \end{array}}$

CSC: $\boxed{\begin{array}{l} \lambda_1 \ge 0 \\ \lambda_2 \ge 0 \\ y < 5 \Rightarrow \lambda_1 = 0 \\ x+y < 2 \Rightarrow \lambda_2 = 0 \end{array}}$

Cases:

A) $\left.\begin{array}{l} y=5 \\ x+y=2 \end{array}\right\}$ $\begin{array}{l} y = \underline{5} \\ x = 2-y = \underline{-3} \end{array}$ $\lambda_2 = \dfrac{1}{-3+1} = -1/2$ impossible $\Rightarrow$ no soln in A)

B) $\left.\begin{array}{l} y=5 \\ x+y<2 \end{array}\right\} \Rightarrow \lambda_2 = 0 \Rightarrow \dfrac{1}{x+1} = 0$ impossible $\Rightarrow$ no soln in B)

C) $\left.\begin{array}{l} y<5 \\ x+y=2 \end{array}\right\}$ $\Rightarrow \lambda_1 = 0 \Rightarrow \lambda_2 = \dfrac{1}{x+1} = \dfrac{1}{y+1} \Rightarrow x = y$

$\Rightarrow x = y = 1$, $\lambda_2 = 1/2$, $\dfrac{1}{2} = \lambda_1 + \lambda_2 = \lambda_2 \Rightarrow \lambda_2 = 1/2$

One soln: $\underline{x=y=1}$, $\underline{\lambda_1 = 0}$, $\underline{\lambda_2 = 1/2}$

D) $\left.\begin{array}{l} y<5 \\ x+y<2 \end{array}\right\}$ $\begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = 0 \end{array}$ $\Rightarrow \dfrac{1}{x+1} = 0$ impossible $\Rightarrow$ no soln. in D

Concl: Solution $(x,y) = (1,1)$, $\lambda_1 = 0$, $\lambda_2 = 1/2$

(iii) <u>FOC:</u> $\mathcal{L} = 2z - \lambda_1(x^2+y^2) - \lambda_2(x+y+z)$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= -\lambda_1 \cdot 2x - \lambda_2 = 0 \\ \mathcal{L}'_y &= -\lambda_1 \cdot 2y - \lambda_2 = 0 \\ \mathcal{L}'_z &= 2 - \lambda_2 = 0 \end{aligned}}$$

C: $\boxed{\begin{aligned} &x^2+y^2=2 \\ &x+y+z=1 \end{aligned}}$

$\lambda_1 \neq 0$ since $\lambda_1 = 0$ is impossible

$\lambda_2 = 2$

$$\left.\begin{aligned} -2\lambda_1 x - 2 = 0 &\Rightarrow \lambda_1 x = -1 \\ -2\lambda_1 y - 2 = 0 &\Rightarrow \lambda_1 y = -1 \end{aligned}\right\} \Rightarrow x = -\frac{1}{\lambda_1} = y$$

$\Downarrow$

$x = y$

$x^2+y^2 = 2x^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

<u>$x=1$:</u> $y=1$, $z = 1-x-y = -1$, $\lambda_1 = -\frac{1}{x} = -1$, $\lambda_2 = 2$

<u>$x=-1$:</u> $y=-1$, $z = 1-x-y = 3$, $\lambda_1 = -\frac{1}{x} = 1$, $\lambda_2 = 2$

$\Downarrow$

<u>Solutions:</u> $(x, y, z; \lambda_1, \lambda_2) = (1, 1, -1; -1, 2)$

$(-1, -1, 3; 1, 2)$

where $\mathcal{C}(t)$ is a function of t (or a constant considered as a function in x). The second equation is $h'_t = Q(x,t)$, and we use the expression above for h:

$$h'_t = Q(x,t) \quad \Rightarrow \quad xe^{x+t} + txe^{x+t} + \mathcal{C}'(t) = (t+1)xe^{x+t} + \mathcal{C}'(t) = (t+1)xe^{x+t}$$

We see that this condition holds if and only if $\mathcal{C}'(t) = 0$, or if $\mathcal{C} = C_1$ is a constant. In conclusion, we may choose $h = txe^{x+t} + C_1$, and the general solution of the exact differential equation is $h = C_2$, where C_2 is another constant. This gives

$$txe^{x+t} = B$$

where $B = C_2 - C_1$ is a new constant. The initial condition is $x(1) = 1$, and this gives $1 \cdot e^2 = B$, or $B = e^2$. The solution can therefore be written in implicit form as

$$\mathbf{txe^{x+t} = e^2}$$

It is not necessary (or possible) to solve this equation for x. (If we first take absolute values on both sides of the equation, and then the natural logarithm, we obtain the equation from question a).

Question 4.

We consider the optimization problem

$$\min 2x^2 + y^2 + 3z^2 \text{ subject to } \begin{cases} x - y + 2z & = 3 \\ x + y & = 3 \end{cases}$$

(a) The Lagrangian for this problem is given by $\mathcal{L} = 2x^2 + y^2 + 3z^2 - \lambda_1(x - y + 2z) - \lambda_2(x + y)$, and the first order conditions are

$$\begin{aligned} \mathcal{L}'_x &= 4x - \lambda_1 - \lambda_2 = 0 \\ \mathcal{L}'_y &= 2y + \lambda_1 - \lambda_2 = 0 \\ \mathcal{L}'_z &= 6z - 2\lambda_1 = 0 \end{aligned}$$

We solve the first order conditions for x, y, z and get

$$x = \frac{\lambda_1 + \lambda_2}{4}, \quad y = \frac{\lambda_2 - \lambda_1}{2}, \quad z = \frac{\lambda_1}{3}$$

When we substitute these expressions into the two constraints $x - y + 2z = 3$ and $x + y = 3$, we get the equations

$$17\lambda_1 - 3\lambda_2 = 36, \quad -\lambda_1 + 3\lambda_2 = 12$$

Adding the two equations, we get $16\lambda_1 = 48$, or $\lambda_1 = 3$, and the last equation gives $\lambda_2 = 5$. When we substitute this into the expressions for x, y, z we get $(x, y, z) = (2, 1, 1)$. This means that $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; 3, 5)$ **is the unique point that satisfies the first order conditions and the constraints**. Alternatively, one may observe that the first order conditions and the constraints form a 5×5 linear system. If we substitute $(x, y, z) = (2, 1, 1)$ in this system, we find that $\lambda_1 = 3$ and $\lambda_2 = 5$; hence $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; 3, 5)$ is one solution of the system. To show that this is the only solution, we may check that the determinant of the coefficient matrix is non-zero. We first use some elementary row operations that preserve the determinant:

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & 0 & 0 & 17/12 & -1/4 \\ 0 & 0 & 0 & -1/4 & 3/4 \end{vmatrix}$$

Then we see that the determinant is given by $4 \cdot 2 \cdot 6 \cdot (17/4 \cdot 3/4 - 1/4 \cdot 1/4) = 48 \neq 0$.

(b) The bordered Hessian at $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; 3, 5)$ is the matrix

$$B = \begin{pmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 6 \end{pmatrix}$$

Since there are $n = 3$ variables and $m = 2$ constratints, we have to compute the $n - m = 1$ last principal minors; that is, just the determinant $D_5 = |B|$. We first use an elementary row operation to simplify the computation, then develop the determinant along the last column:

$$|B| = \begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 2 & 0 & -3 & 3 & 0 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 4 & 0 \\ -1 & 1 & 0 & 2 \\ 2 & 0 & -3 & 3 \end{vmatrix}$$

Then we develop the last determinant along the first row, and get

$$|B| = 2 \begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 4 & 0 \\ -1 & 1 & 0 & 2 \\ 2 & 0 & -3 & 3 \end{vmatrix} = 2 \left(\begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 0 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 4 \\ -1 & 1 & 0 \\ 2 & 0 & -3 \end{vmatrix} \right) = 2(10 + 14) = 48$$

Since $|B| = 48 > 0$ has the same sign as $(-1)^m = (-1)^2 = 1$, we conclude that **the point $(x, y, z) = (2, 1, 1)$ is a local minimum for $2x^2 + y^2 + 3z^2$** (among the admissible points). The local minimum value is $f(2, 1, 1) = 8 + 1 + 3 = \mathbf{12}$.

(c) We fix $\lambda_1 = 3$ and $\lambda_2 = 5$, and consider the Lagrangian

$$\mathcal{L}(x, y, z) = 2x^2 + y^2 + 3z^2 - 3(x - y + 2z) - 5(x + y)$$

This function is clearly convex, since the Hessian matrix

$$\mathcal{L}'' = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

is positive definite (with eigenvalues $4, 2, 6$). Therefore, the point $(x, y, z) = (2, 1, 1)$ **solves the minimum problem**. The Kuhn-Tucker problem can be reformulated in standard form as

$$\max -(2x^2 + y^2 + 3z^2) \text{ subject to } \begin{cases} -(x - y + 2z) & \leq -3 \\ -(x + y) & \leq -3 \end{cases}$$

Therefore, we see that the Lagrangian of the Kuhn-Tucker problem is

$$-(2x^2 + y^2 + 3z^2) + \lambda_1(x - y + 2z) + \lambda_2(x + y) = -\mathcal{L}$$

and the first order conditions of the Kuhn-Tucker problem are the same as in the original problem. Hence $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; 3, 5)$ is still a solution of the first order conditions and the constraints, and $\lambda_1, \lambda_2 \geq 0$ also solves the complementary slackness conditions. When we fix $\lambda_1 = 3$ and $\lambda_2 = 5$, $-\mathcal{L}$ is concave since $\mathcal{L}$ is convex, and this means that $(x, y, z) = (2, 1, 1)$ **also solves the Kuhn-Tucker problem.**