Problem Sheet 9 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. Solve the optimization problem

$$\max f(x,y) = x^2 + y^2 + y - 1 \quad \text{subject to} \quad x^2 + y^2 \leq 1$$

What is the maximum value?

2. Solve $\max(1 - x^2 - y^2)$ subject to $x \geq 2$ and $y \geq 3$ by a direct argument and then see what the Kuhn-Tucker conditions have to say about the problem.

3. Solve the following problem (assuming it has a solution):

$$\min\ 4\ln(x^2+2) + y^2 \quad \text{subject to} \quad \begin{cases} x^2 + y \geq 2 \\ x \geq 1 \end{cases}$$

4. Mock Final Exam in GRA6035 12/2010, 4
We consider the following optimization problem: Maximize $f(x,y,z) = xy + yz - xz$ subject to the constraint $x^2 + y^2 + z^2 \leq 1$.

a) Write down the first order conditions for this problem, and solve the first order conditions for x, y, z using matrix methods.
b) Solve the optimization problem. Make sure that you check the non-degenerate constraint qualification, and also make sure that you show that the problem has a solution.

5. Final Exam in GRA6035 10/12/2010, 4
We consider the function $f(x,y,z) = xyz$.

a) The function g is defined on the set $D = \{(x,y,z) : x > 0, y > 0, z > 0\}$, and it is given by
$$g(x,y,z) = \frac{1}{f(x,y,z)} = \frac{1}{xyz}$$
Is g a convex or concave function on D?
b) Maximize $f(x,y,z)$ subject to $x^2 + y^2 + z^2 \leq 1$.

6. Final Exam in GRA6035 30/05/2011, Problem 4
We consider the function f, given by $f(x,y) = xy\, e^{x+y}$, with domain of definition $D_f = \{(x,y) : (x+1)^2 + (y+1)^2 \leq 1\}$.

a) Compute the Hessian of f. Is f a convex function? Is f a concave function?
b) Find the maximum and minimum values of f.

Solutions

1 The Lagrangian is $\mathscr{L}(x,y,z,\lambda) = x^2+y^2+y-1-\lambda(x^2+y^2)$, and we write down and solve the Kuhn-Tucker conditions: The first order conditions

$$\mathscr{L}'_x = 2x - \lambda \cdot 2x = 0$$
$$\mathscr{L}'_y = 2y + 1 - \lambda \cdot 2y = 0$$

together with the constraint $x^2+y^2 \le 1$ and the complementary slackness conditions

$$\lambda \ge 0 \quad \text{and} \quad \lambda = 0 \text{ if } x^2+y^2 < 1$$

From the first of the first order conditions, we see that $x = 0$ or $\lambda = 1$. Since $\lambda = 1$ contradicts the second of the first order conditions, we see that $x = 0$. From the second of the first order conditions we have

$$2y(\lambda-1) = 1 \quad \Rightarrow \quad y = \frac{1}{2(\lambda-1)}$$

This implies that

$$x^2+y^2 = \frac{1}{4(\lambda-1)^2} \le 1$$

We first consider the case $x^2+y^2 = 1$. In this case, we have $1 = 4(\lambda-1)^2$, or that $\lambda = \pm 1/2+1$. Both solutions satisfy $\lambda \ge 0$, and this gives two solution

$$(x,y;\lambda) = (0,1;\frac{3}{2}),\ (0,-1;\frac{1}{2})$$

of the Kuhn-Tucker conditions in case $x^2+y^2 = 1$. Secondly, we consider the case when $x^2+y^2 < 1$, which gives $\lambda = 0$ by the complementary slackness condition. Since $x^2+y^2 = 1/4 < 1$, this gives the solution

$$(x,y;\lambda) = (0,-\frac{1}{2};0)$$

of the Kuhn-Tucker conditions in case $x^2+y^2 < 1$. The solutions of the Kuhn-Tucker conditions give $f(0,1) = 1$, $f(0,-1) = -1$ and $f(0,-1/2) = -5/4$, so the solution $(x^*,y^*;\lambda^*) = (0,1;3/2)$ is the best candidate for maximum. The Lagrangian $\mathscr{L}(x,y;\lambda^*) = -x^2/2 - y^2/2 + y - 1$ is clearly concave, so $(x^*,y^*) = (0,1)$ is the maximum point, with maximal value $f^* = f(x^*,y^*) = 1$.

2 We see that $f(x,y) = 1-x^2-y^2$ will decrease as x increases and as y increases, since x and y are positive. Therefore the maximum will occur at $x = 2, y = 3$, which are the smallest possible values of x and y, and the maximum value is

$$f(2,3) = 1-4-9 = -12$$

To write down the Kuhn-Tucker conditions, we write the problem in standard form

$$\max 1 - x^2 - y^2 \quad \text{subject to} \quad \begin{cases} -x \leq -2 \\ -y \leq -3 \end{cases}$$

The Lagrangian is $\mathscr{L}(x,y,z,\lambda) = 1 - x^2 - y^2 + \lambda_1 x + \lambda_2 y$, and we write down and solve the Kuhn-Tucker conditions: The first order conditions

$$\mathscr{L}'_x = -2x + \lambda_1 = 0$$
$$\mathscr{L}'_y = -2y + \lambda_2 = 0$$

together with the constraints $x \geq 2$ and $y \geq 3$ and the complementary slackness conditions

$$\lambda_1, \lambda_2 \geq 0 \quad \text{and} \quad \lambda_1 = 0 \text{ if } x > 2, \lambda_2 = 0 \text{ if } y > 3$$

If $x > 2$, then $\lambda_1 = 0$, and this gives $x = 0$, a contradiction. If $y > 3$, then $\lambda_2 = 0$, and this gives $y = 0$, a contradiction. The only solution to the Kuhn-Tucker conditions is therefore $x = 2$, $y = 3$, $\lambda_1 = 4$ and $\lambda_2 = 6$. This point is the maximum since $\mathscr{L}(x,y;4,6) = 1 - x^2 - y^2 + 4x + 6y$ is concave.

3 We first write the problem in standard form as

$$\max\ -4\ln(x^2+2) - y^2 \quad \text{subject to} \quad \begin{cases} -x^2 - y \leq -2 \\ -x \ \leq -1 \end{cases}$$

The Lagrangian is $\mathscr{L}(x,y,z,\lambda) = -4\ln(x^2+2) - y^2 + \lambda_1(x^2+y) + \lambda_2 x$, and we write down and solve the Kuhn-Tucker conditions: The first order conditions

$$\mathscr{L}'_x = \frac{-8x}{x^2+2} + \lambda_1 \cdot 2x + \lambda_2 = 0$$
$$\mathscr{L}'_y = -2y + \lambda_1 = 0$$

together with the constraints $x^2 + y \geq 2$ and $x \geq 1$ and the complementary slackness conditions

$$\lambda_1, \lambda_2 \geq 0 \quad \text{and} \quad \lambda_1 = 0 \text{ if } x^2 + y > 2, \lambda_2 = 0 \text{ if } x > 1$$

Since there are two constrains, there are four cases to consider. Case 1 is given by $x^2 + y = 2$, $x = 1$. This gives $x = 1$ and $y = 1$, and by the first order conditions that $\lambda_1 = 2$ and $\lambda_2 = 8/3 - 4 < 0$. This contradicts the complementary slackness condition, and there are no solutions in this case. Case 2 is the case $x^2 + y = 2$ and $x > 1$. In this case, $\lambda_2 = 0$ and $y = 2 - x^2$. The first order conditons give $\lambda_1 = 2y$ and $-8x/(x^2+2) + 4xy = 0$. Inserting $y = 2 - x^2$ into the last equation we get

$$x\left(4(2-x^2) - \frac{8}{x^2+2}\right) = x \cdot \frac{4(2-x^2)(2+x^2) - 8}{x^2+2} = 0$$

Since $x > 1$, this implies that $4(4-x^4) = 8$ or $x^4 = 2$, and this gives $x = \sqrt[4]{2}$ and $y = 2-\sqrt{2}$, with $\lambda_1 = 2(2-\sqrt{2})$ and $\lambda_2 = 0$. Since $x > 1$ and $\lambda_1 \geq 0$, this gives the solution

$$(x,y;\lambda_1,\lambda_2) = (\sqrt[4]{2}, 2\sqrt{2}; 2(2-\sqrt{2}), 0)$$

of the Kuhn-Tucker conditions in Case 2. Case 3 is the case $x^2+y > 2$ and $x = 1$. This gives $\lambda_1 = 0$ by the complementary slackness conditons, and then $y = 0$ by the first order conditions. But then $x^2+y = 1$, and this contradicts $x^2+y > 2$. Case 4 is the case $x^2+y > 2$ and $x > 1$. This gives $\lambda_1 = \lambda_2 = 0$ by the complementary slackness conditions, this gives $x = y = 0$ by the first order conditions. Again, this contradicts the constraint $x^2+y > 2$. We conclude that there is only one solution to the Kuhn-Tucker conditions. We have assumed that the problem has a solution. It follows that this solution must occur at the point $(x,y) = (\sqrt[4]{2}, 2-\sqrt{2})$ or at an admissible point where the NDCQ is not satisfied. In case 1, we see that NDCQ is given by

$$\text{rk}\begin{pmatrix} -2x & -1 \\ -1 & 0 \end{pmatrix} = 2$$

Since the determinant is -1, the rank is two at all points, and NDCQ holds. Similarly, we see that in case 2 and case 3 that NDCQ holds, since each row contains an entry that cannot be zero. In case 4, there is no NDCQ. We can therefore conclude that $(x^*, y^*) = (\sqrt[4]{2}, 2-\sqrt{2})$ is the minimum.

4 Mock Final Exam in GRA6035 12/2010, Problem 4

See handwritten solution on the course page of GRA 6035 Mathematics 2010/11.

5 Final Exam in GRA6035 10/12/2010, Problem 4

The Hessian of f is indefinite for all $(x,y,z) \neq (0,0,0)$ since it is given by

$$f'' = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}$$

and has principal minors $-z^2, -y^2, -x^2$ of order two. Hence f is neither convex nor concave. We compute the Hessian of g, and find

$$g'' = \frac{1}{xyz}\begin{pmatrix} \frac{2}{x^2} & \frac{1}{xy} & \frac{1}{xz} \\ \frac{1}{xy} & \frac{2}{y^2} & \frac{1}{yz} \\ \frac{1}{xz} & \frac{1}{yz} & \frac{2}{z^2} \end{pmatrix}$$

Hence the leading principal minors are

$$D_1 = \frac{1}{xyz}\frac{2}{x^2} > 0, \quad D_2 = \frac{1}{(xyz)^2}\frac{3}{(xy)^2} > 0, \quad D_3 = \frac{1}{(xyz)^3}\frac{4}{(xyz)^2} > 0$$

This means that g is convex.

The set $\{(x,y,z) : x^2+y^2+z^2 \leq 1\}$ is closed and bounded, so the problem has solutions by the extreme value theorem. The NDCQ is satisfied, since the rank of

$\left(2x\ 2y\ 2z\right) = 1$ when $x^2+y^2+z^2 = 1$. We form the Lagrangian

$$\mathscr{L} = xyz - \lambda(x^2+y^2+z^2-1)$$

and solve the Kuhn-Tucker conditions, consisting of the first order conditions

$$\begin{aligned}\mathscr{L}'_x &= yz - \lambda \cdot 2x = 0\\ \mathscr{L}'_y &= xz - \lambda \cdot 2y = 0\\ \mathscr{L}'_z &= xy - \lambda \cdot 2z = 0\end{aligned}$$

together with one of the following conditions: i) $x^2+y^2+z^2 = 1$ and $\lambda \geq 0$ or ii) $x^2+y^2+z^2 < 1$ and $\lambda = 0$. We first solve the equations/inequalities in case i): If $x = 0$, then we see that $y = 0$ or $z = 0$ from the first equation, and we get the solutions $(x,y,z;\lambda) = (0,0,\pm 1;0),(0,\pm 1,0;0)$. If $x \neq 0$, we get $2\lambda = yz/x$ and the remaining first order conditions give $(x^2-y^2)z = 0$ and $(x^2-z^2)y = 0$. If $y = 0$, we get the solution $(\pm 1,0,0;0)$. Otherwise, we get $x^2 = y^2 = z^2$, hence

$$(x,y,z;\lambda) = \left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}};\pm\frac{1}{2\sqrt{3}}\right)$$

The condition that $\lambda \geq 0$ give that either all three coordinates (x,y,z) are positive, or that one is positive and two are negative. In total, we obtain four different solutions. We note that $f(x,y,z) = \frac{1}{3\sqrt{3}}$ for each of these four solutions, while $f(x,y,z) = 0$ for either of the first three solutions. Finally, we consider case ii), where $\lambda = 0$. This gives $xy = xz = yz = 0$, and we obtain

$$(x,y,z;\lambda) = (a,0,0;0),(0,a,0;0),(0,0,a;0)$$

The condition that $x^2+y^2+z^2 < 1$ give $a^2 \leq 1$ or $a \in (-1,1)$. For all these solutions, we get $f(x,y,z) = 0$. We can therefore conclude that the solution to the optimization problem is $(x^*,y^*,z^*) = (\pm 1/\sqrt{3},\pm 1/\sqrt{3},\pm 1/\sqrt{3})$ with an even number of variables with negative values. The maximum value is $f = \frac{1}{3\sqrt{3}}$.

6 Final Exam in GRA6035 30/05/2011, Problem 4

a) We compute the Hessian of f, and find

$$f'' = e^{x+y}\begin{pmatrix}(x+2)y & (x+1)(y+1)\\ (x+1)(y+1) & x(y+2)\end{pmatrix}$$

The principal minors are

$$\Delta_1 = e^{x+y}(x+2)y,\ \Delta_1 = e^{x+y}x(y+2), \quad D_2 = (e^{x+y})^2(1-(x+1)^2-(y+1)^2)$$

Since $(x+1)^2+(y+1)^2 \leq 1$, D_f is a ball with center in $(-1,-1)$ and radius $r=1$, and it follows that $x,y<0$ and $x+2, y+2 \geq 0$, and therefore $\Delta_1 \leq 0$ and $D_2 \geq 0$. This means that f is concave, but not convex.

b) Since D_f is closed and bounded, f has maximum and minimum values. We compute the stationary points of f: We have

$$f'_x = (x+1)ye^{x+y} = 0, \quad f'_y = x(y+1)e^{x+y} = 0$$

and $(x,y)=(0,0)$ and $(x,y)=(-1,-1)$ are the solutions. Hence there is only one stationary point $(x,y)=(-1,-1)$ in D_f, and the $f(-1,-1)=\mathbf{e^{-2}}$ is the maximum value of f since f is concave. The minimum value most occur for (x,y) on the boundary of D_f. We see that $f(x,y) \geq 0$ for all $(x,y) \in D_f$ while $f(-1,0)=f(0,-1)=0$. Hence $\mathbf{f(-1,0)=f(0,-1)=0}$ is the minimum value of f.