Problem Sheet 8 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. Solve the Lagrange problem

$$\max f(x,y,z) = e^x + y + z \text{ subject to } \begin{cases} x^2+y^2+z^2 = 1 \\ x+y+z = 1 \end{cases}$$

What is the maximum value?

2. Consider the optimization problem $\max f(x,y;r)$, where r is a parameter and f is the function given by

$$f(x,y;r) = -x^2 - xy - 2y^2 + 2rx + 2ry$$

Find the functions $x^*(r)$ and $y^*(r)$ such that $\mathbf{x}^*(r) = (x^*(r), y^*(r))$ solves the optimization problem, and verify the Envelope Theorem.

3. Consider the optimization problem $\max f(x,y;r,s)$, where r,s are parameters and f is the function given by

$$f(x,y;r,s) = r^2x + 3s^2y - x^2 - 8y^2$$

Find the functions $x^*(r,s)$ and $y^*(r,s)$ such that $\mathbf{x}^*(r,s) = (x^*(r,s), y^*(r,s))$ solves the optimization problem, and verify the Envelope Theorem.

4. Consider the constrained optimization problem with parameter $m \geq 4$:

$$\max U(x,y) = \frac{1}{2}\ln(1+x) + \frac{1}{4}\ln(1+y) \quad \text{subject to} \quad 2x+3y = m$$

a) Solve the optimization problem
b) Show that $U^*(m)$, the optimal value function of the optimization problem, satisfies $\partial U^*(m)/\partial m = \lambda$, where λ is the Lagrange multiplier.

5. Consider the constrained optimization problem

$$\max x^2y^2z^2 \quad \text{subject to} \quad x^2+y^2+z^2 = 1$$

Find all solutions to the Lagrange conditions, and use the Bordered Hessian to determine which of the solutions are local maxima. What is the solution to the contrained optimization problem?

6. Find all solutions to the Lagrange conditions in the Lagrange problem

$$\max xyz \quad \text{subject to} \quad \begin{cases} x^2+y^2 = 1 \\ x+z = 1 \end{cases}$$

Show that the set of admissible points (the set of points satisfying the constraints) is bounded, and use this to solve the optimization problem.

Solutions

1 The Lagrangian is $\mathscr{L}(x,y,z,\lambda) = e^x + y + z - \lambda_1(x^2+y^2+z^2) - \lambda_2(x+y+z)$, and we solve the first order conditions

$$\begin{aligned}
\mathscr{L}'_x &= e^x - \lambda_1 \cdot 2x - \lambda_2 = 0\\
\mathscr{L}'_y &= 1 - \lambda_1 \cdot 2y - \lambda_2 = 0\\
\mathscr{L}'_z &= 1 - \lambda_1 \cdot 2z - \lambda_2 = 0
\end{aligned}$$

together with the contraints $x^2+y^2+z^2 = 1$ and $x+y+z = 1$. From the last two first order conditions, we get

$$\lambda_2 = 1 - 2y\lambda_1 = 1 - 2z\lambda_1$$

This means that either $\lambda_1 = 0$ or $y = z$. We first consider the case with $\lambda_1 = 0$, which implies that $\lambda_2 = 1$ and that $x = 0$ from the first of the first order conditions. The constraints give

$$y+z = 1, \quad y^2+z^2 = 1$$

since $x = 0$, and inserting $y = 1-z$ in the second equation gives $(1-z)^2+z^2 = 1$ or $2z^2-2z = 0$. This gives $z = 0$ or $z = 1$. We therefore find two solutions with $\lambda_1 = 0$:

$$(x,y,z;\lambda_1,\lambda_2) = (0,1,0;0,1),\ (0,0,1;0,1)$$

Both points have $f(x,y,z) = e^0+1 = 2$. Secondly, we consider the case with $\lambda_1 \neq 0$, so that $y = z$. Then the constraints are given by

$$x+2y = 1, \quad x^2+2y^2 = 1$$

Inserting $x = 1-2y$ in the second equation gives $(1-2y)^2+2y^2 = 1$ or $6y^2-4y = 0$. This gives $y = 0$ or $y = 2/3$. For $y = z = 0$, we get $x = 1$, $\lambda_2 = 1$ and $e - 2\lambda_1 = 1$, which gives the solution

$$(x,y,z;\lambda_1,\lambda_2) = (1,0,0;\frac{e-1}{2},1)$$

with $f(x,y,z) = e^1 = e \simeq 2.72$. For $y = z = 2/3$, we get $x = -1/3$, $1-4\lambda_1/3 = \lambda_2$ and $e^{-1/3}+2\lambda_1/3 = \lambda_2$. We solve the last two equations for λ_1,λ_2 and find the solution

$$(x,y,z;\lambda_1,\lambda_2) = (-\frac{1}{3},\frac{2}{3},\frac{2}{3};\frac{1-e^{-1/3}}{2},\frac{1+2e^{-1/3}}{3})$$

with $f(x,y,z) = e^{-1/3}+4/3 \simeq 2.05$. The point $(x^*,y^*,z^*) = (1,0,0)$ is the best candidate for max. Since the Lagrangian corresponding to this point is not concave, we argue by elimination. First, the problem must have a solution by the Extreme Value Theorem, since the set of admissible points (the constrained set) is bounded. In fact, since one of the constraints is $x^2+y^2+z^2=1$, we have that $-1 \le x,y,z \le 1$. So the maximum must be one of the points satisfying the Lagrange conditions (that is, one of the points we found above), or an admissible point that does not satisfy NDCQ. In this case, the NDCQ condition is given by

$$\operatorname{rk}\begin{pmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \end{pmatrix} = 2$$

For a point (x,y,z) not to satisfy NDCQ, the rank must be less than two, and this means that all minors of order two must be zero:

$$2x-2y=0, \quad 2x-2z=0, \quad 2y-2z=0$$

The only solution is that $x=y=z$. Since the point must be admissible (that is, satisfy the constraints), we must have $x=y=z=1/3$ since $x+y+z=1$, and then $x^2+y^2+z^2=3(1/3)^2=3/9\neq 1$. So there are no admissible points that do not satisfy NDCQ. We conclude, by elimination, that the point $(x^*,y^*,z^*)=(1,0,0)$ is the maximum point, and the maximum value is $f^*=f(x^*,y^*,z^*)=e$.

2 The stationary points are given by $f'_x=-2x-y+2r=0$, $f'_y=-x-4y+2r=0$. The equations are linear, and there is a unique solution:

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2r \\ 2r \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2r \\ 2r \end{pmatrix} = \begin{pmatrix} 6r/7 \\ 2r/7 \end{pmatrix}$$

The function f is concave, since the Hessian of f is given by

$$f'' = \begin{pmatrix} -2 & -1 \\ -1 & -4 \end{pmatrix}$$

with $D_1=-2$ and $D_2=7$. The stationary point $x^*(r)=6r/7$ and $y^*(r)=2r/7$ is therefore the unique maximum of f. The optimal value function is given by

$$f^*(r) = -(6r/7)^2-(6r/7)(2r/7)-2(2r/7)^2+2r(6r/7)+2r(2r/7) = \frac{8}{7}r^2$$

The Envelope Theorem states that

$$\frac{d}{dr}\,f^*(r) = \frac{\partial f}{\partial r}\,(x=x^*(r),y=y^*(r)) = 2x^*(r)+2y^*(r) = \frac{16}{7}r$$

and we see that this fits with the optimal value function computed above.

3 The stationary points are given by $f'_x=r^2-2x=0$, $f'_y=3s^2-16y=0$. The unique stationary point is therefore given by $x=r^2/2$ and $y=3s^2/16$. The function f is clearly concave, so $x^*(r,s)=r^2/2$ and $y^*(r,s)=3s^2/16$ is the unique maximum of f. The optimal value function is given by

$$f^*(r,s) = r^2(r^2/2) + 3s^2(3s^2/16) - (r^2/2)^2 - 8(3s^2/16)^2 = \frac{1}{4}r^4 + \frac{9}{32}s^4$$

The Envelope Theorem states that

$$\frac{\partial}{\partial r} f^*(r,s) = \frac{\partial f}{\partial r}(x = x^*(r,s), y = y^*(r,s)) = 2rx^*(r,s) = r^3$$

and that

$$\frac{\partial}{\partial s} f^*(r,s) = \frac{\partial f}{\partial s}(x = x^*(r,s), y = y^*(r,s)) = 6sy^*(r,s) = \frac{9}{8}s^3$$

We see that both results fit with the optimal value function computed above.

4 The Lagrangian is $\mathscr{L}(x,y,z,\lambda) = \frac{1}{2}\ln(1+x) + \frac{1}{4}\ln(1+y) - \lambda(2x+3y)$, and we solve the first order conditions

$$\mathscr{L}'_x = \frac{1}{2(x+1)} - \lambda \cdot 2 = 0$$
$$\mathscr{L}'_y = \frac{1}{4(y+1)} - \lambda \cdot 3 = 0$$

together with the contraints $2x+3y = m$. We solve the first order conditions for x and y, and find

$$x = \frac{1}{4\lambda} - 1, \quad y = \frac{1}{12\lambda} - 1$$

The constraint $2x+3y = m$ gives the equation

$$2(\frac{1}{4\lambda} - 1) + 3(\frac{1}{12\lambda} - 1) = m \quad \Rightarrow \quad \frac{3}{4\lambda} = (m+5)$$

This gives the solution

$$\lambda^*(m) = \frac{3}{4(m+5)}, \quad x^*(m) = \frac{m+5}{3} - 1 = \frac{m+2}{3}, \quad y^*(m) = \frac{m+5}{9} - 1 = \frac{m-4}{9}$$

This is the maximum, since the Hessian of $\mathscr{L}(x,y;\lambda^*(m))$ is

$$\begin{pmatrix} -\frac{1}{2(x+1)^2} & 0 \\ 0 & -\frac{1}{4(y+1)^2} \end{pmatrix}$$

and therefore negative semidefinite for all (x,y) in the domain of definition of U (U is defined for all points (x,y) such that $x > -1$ and $y > -1$). The optimal value function $U^*(m)$ is given by

$$U^*(m) = \frac{1}{2}\ln(x^*(m)+1) + \frac{1}{4}\ln(y^*(m)+1) = \frac{1}{2}\ln(\frac{m+5}{3}) + \frac{1}{4}\ln(\frac{m+3}{9})$$

We see that this can be simplified to

$$U^*(m) = \ln\left(\left(\frac{m+5}{3}\right)^{1/2} \cdot \left(\frac{m+5}{9}\right)^{1/4}\right) = \ln\left(\frac{(m+5)^{3/4}}{3}\right)$$

The derivative of the optimal value function is

$$\frac{d}{dm}\, U^*(m) = \frac{3}{(m+5)^{3/4}} \cdot \frac{3}{4}\,\frac{(m+5)^{-1/4}}{3} = \frac{3}{4(m+5)} = \lambda^*(m)$$

5 The Lagrangian is $\mathscr{L}(x,y,z,\lambda) = x^2y^2z^2 - \lambda(x^2+y^2+z^2)$, and we solve the first order conditions

$$\begin{aligned}
\mathscr{L}'_x &= 2xy^2z^2 - \lambda \cdot 2x = 0\\
\mathscr{L}'_y &= 2x^2yz^2 - \lambda \cdot 2y = 0\\
\mathscr{L}'_z &= 2x^2y^2z - \lambda \cdot 2z = 0
\end{aligned}$$

together with the contraint $x^2+y^2+z^2 = 1$. The first order conditions can be reduced to

$$\begin{aligned}
x = 0 \quad &\text{or} \quad y^2z^2 = \lambda\\
y = 0 \quad &\text{or} \quad x^2z^2 = \lambda\\
z = 0 \quad &\text{or} \quad x^2y^2 = \lambda
\end{aligned}$$

If $x = 0$ or $y = 0$ or $z = 0$, then $\lambda = 0$, and we obtain the solutions

$$(x,y,0) \text{ with } x^2+y^2 = 1, \quad (x,0,z) \text{ with } x^2+z^2 = 1, \quad (0,y,z) \text{ with } y^2+z^2 = 1$$

which all satisfy $f = 0$. These points are clearly local minima, since $x^2y^2z^2 \geq 0$. If $x \neq 0, y \neq 0, z \neq 0$, then we have

$$x^2y^2 = x^2z^2 = y^2z^2 = \lambda$$

and this implies that $x^2 = y^2 = z^2 = 1/3$. The solutions are therefore the eight points

$$(x,y,z) = (\pm\sqrt{3}/3, \pm\sqrt{3}/3, \pm\sqrt{3}/3)$$

with $f = 1/27$ and $\lambda = 1/9$. The Bordered Hessian matrix at one of the solutions $(x^*,y^*,z^*;\lambda^*) = (\sqrt{3}/3, \pm\sqrt{3}/3, \pm\sqrt{3}/3; 1/3)$ is given by

$$B = \begin{pmatrix} 0 & 2x & 2y & 2z\\ 2x & 2y^2z^2-2\lambda & 4xyz^2 & 4xy^2z\\ 2y & 4xyz^2 & 2x^2z^2-2\lambda & 4x^2yz\\ 2z & 4xy^2z & 4x^2yz & 2x^2y^2-2\lambda \end{pmatrix} = \begin{pmatrix} 0 & 2x^* & 2y^* & 2z^*\\ 2x^* & 0 & 4/3x^*y^* & 4/3x^*z^*\\ 2y^* & 4/3x^*y^* & 0 & 4/3y^*z^*\\ 2z^* & 4/3x^*z^* & 4/3y^*z^* & 0 \end{pmatrix}$$

We need to compute the $n-m=3-1=2$ last leading principal minors, that is D_3 and D_4. We have

$$D_3 = \frac{32}{27}, \quad D_4 = -\frac{64}{81}$$

Since the sign is alternating and the last sign is negative, and therefore equal to the sign of $(-1)^n = (-1)^3 = -1$, it follows that all eight points are local maxima. Since the set given by $x^2+y^2+z^2=1$ is bounded and NDCQ is satisfied for all admissible points, it follows that these eight points are maxima, and therefore solutions to the Lagrange problem.

6 The Lagrangian is $\mathscr{L}(x,y,z,\lambda) = xyz - \lambda_1(x^2+y^2) - \lambda_2(x+z)$, and we solve the first order conditions

$$\begin{aligned}
\mathscr{L}'_x &= yz - \lambda_1 \cdot 2x - \lambda_2 = 0 \\
\mathscr{L}'_y &= xz - \lambda_1 \cdot 2y = 0 \\
\mathscr{L}'_z &= xy - \lambda_2 = 0
\end{aligned}$$

together with the contraints $x^2+y^2=1$ and $x+z=1$. We first consider the case when $y=0$. Then $\lambda_2=0$ and $xz=x\lambda_1=0$ by the first order conditions. But $x=\pm 1 \neq 0$ by the first constraint, so $z=\lambda_1=0$. Finally, $x=1$ by the second constraint, and we find the solution

$$(x,y,z;\lambda_1,\lambda_2) = (1,0,0;0,0)$$

with $f=0$. We then consider the case $y \neq 0$. Then we have

$$\lambda_1 = \frac{xz}{2y}, \quad \lambda_2 = xy$$

by the last two first order conditions, and the first of the first order conditions give

$$yz - \frac{xz}{2y} \cdot 2x - xy = 0 \quad \Rightarrow \quad y^2z - x^2z - xy^2 = 0$$

From the constraints, we have $y^2 = 1-x^2$ and $z=1-x$. Inserting this in the equation above, we get

$$(1-x^2)(1-x) - x^2(1-x) - x(1-x^2) = 0$$

We see that $(1-x)$ is a factor in the left hand side, and we can therefore write the equation as

$$(1-x)(1-x^2-x^2-x(1+x)) = (1-x)(-3x^2-x+1) = 0$$

and the solutions are $x=1$ and $x=-1/6 \pm \sqrt{13}/6$. The first solution, $x=1$, gives $y=0$, which contradicts $y \neq 0$. We therefore get two solutions to the Lagrange conditions in the case $y \neq 0$:

$$x = -\frac{1}{6} - \frac{\sqrt{13}}{6} \qquad y = \pm\sqrt{\frac{11}{18} - \frac{\sqrt{13}}{18}} \qquad z = \frac{7}{6} + \frac{\sqrt{13}}{6}$$

$$x = -\frac{1}{6} + \frac{\sqrt{13}}{6} \qquad y = \pm\sqrt{\frac{11}{18} + \frac{\sqrt{13}}{18}} \qquad z = \frac{7}{6} - \frac{\sqrt{13}}{6})$$

Using approximations for (x,y,z), we see that the highest value of f is obtained at the point

$$x = -\frac{1}{6} - \frac{\sqrt{13}}{6} \simeq -0.768, \quad y = -\sqrt{\frac{11}{18} - \frac{\sqrt{13}}{18}} \simeq -0.641, \quad z = \frac{7}{6} + \frac{\sqrt{13}}{6} \simeq 1.768$$

with $f \simeq 0.87$. Since $\mathscr{L}(x,y,z;\lambda_1^*,\lambda_2^*)$ is not concave, we try to argue by elimination: First, the Lagrange problem has a solution since the set of admissible points is bounded. In fact, since $x^2+y^2=1$, we have that $-1 \le x,y \le 1$, and the second constraint $x+z=1$ then means that $0 \le z \le 2$. The NDCQ condition is in this case

$$\text{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 0 & 1 \end{pmatrix} = 2$$

For a point (x,y,z) not to satisfy NDCQ, the rank must be less than two, and this means that all minors of order two must be zero:

$$-2y = 0, \quad 2x = 0, \quad 2y = 0$$

The only solution is that $x = y = 0$. Since the point must be admissible (that is, satisfy the constraints), we must have $x^2+y^2=1$. But this is not the case, so there are no admissible points that do not satisfy NDCQ. We conclude, by elimination, that the point $(x^*,y^*,z^*) \simeq (-0.768,-0.641,1.768)$ is the maximum point, and the maximum value is $f^* = f(x^*,y^*,z^*) \simeq 0.87$.