Problem Sheet 4 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. Check if the vector $\mathbf{v}$ is an eigenvector of the matrix A when

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If $\mathbf{v}$ is an eigenvector, what is the corresponding eigenvalue?

2. Find the eigenvalues and eigenvectors of the following matrices:

$$a) \quad \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 2 & 4 \\ -2 & 6 \end{pmatrix} \qquad c) \quad \begin{pmatrix} 1 & 4 \\ 6 & -1 \end{pmatrix}$$

3. Find the eigenvalues and eigenvectors of the following matrices:

$$a) \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

4. Let A be a square matrix and let λ be an eigenvalue of A. Suppose that A is an invertible matrix, and prove that $\lambda \neq 0$ and that $1/\lambda$ is an eigenvalue of A^{-1}.

5. Consider the square matrix A and the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ given by

$$A = \begin{pmatrix} 1 & 18 & 30 \\ -2 & -11 & -10 \\ 2 & 6 & 5 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

Show that $\mathbf{v}_i$ is an eigenvector for A for $i = 1,2,3$ and find the corresponding eigenvalues. Use this to find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

6. Find an invertible matrix P such that $D = P^{-1}AP$ is diagonal when

$$A = \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}$$

7. Show that the following matrix is not diagonalizable:

$$A = \begin{pmatrix} 3 & 5 \\ 0 & 3 \end{pmatrix}$$

8. Initially, two firms A and B (numbered 1 and 2) share the market for a certain commodity. Firm A has 20% of the marked and B has 80%. In course of the next year, the following changes occur:

A keeps 85% of its customers, while losing 15% to B
B keeps 55% of its customers, while losing 45% to A

We can represent market shares of the two firms by means of a *market share vector*, defined as a column vector $\mathbf{s}$ whose components are all nonnegative and sum to 1. Define the matrix $\mathbf{T}$ and the initial share vector $\mathbf{s}$ by

$$T = \begin{pmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$$

The matrix T is called the *transition matrix*.

a) Compute the vector $T\mathbf{s}$, and show that it is also a market share vector.
b) Find the eigenvalues and eigenvectors of T.
c) Find a matrix P such that $D = P^{-1}TP$ is diagonal, and show that $T^n = PD^nP^{-1}$.
d) Compute the limit of D^n as $n \to \infty$ and use this to find the limit of $T^n\mathbf{s}$ as $n \to \infty$. Explain that the we will approach an *equilibrium*, a situation where the market shares of A and B are constant. What are the equilibrium marked shares?

9. Determine if the following matrix is diagonalizable:

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 5 \end{pmatrix}$$

If this is the case, find an invertible matrix P such that $P^{-1}AP$ is diagonal, and use this to compute A^{17}.

10. Final Exam in GRA6035 10/12/2010, Problem 2
We consider the matrix A and the vector $\mathbf{v}$ given by

$$A = \begin{pmatrix} 1 & 7 & -2 \\ 0 & s & 0 \\ 1 & 1 & 4 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) Compute the determinant and the rank of A.
b) Find all eigenvalues of A. Is $\mathbf{v}$ an eigenvector for A?
c) Determine the values of s such that A is diagonalizable.

11. Mock Final Exam in GRA6035 12/2010, Problem 1
We consider the matrix A given by

$$A = \begin{pmatrix} 1 & 1 & -4 \\ 0 & t+2 & t-8 \\ 0 & -5 & 5 \end{pmatrix}$$

a) Compute the determinant and the rank of A.
b) Find all eigenvalues of A.
c) Determine the values of t such that A is diagonalizable.

12. Final Exam in GRA6035 30/05/2011, Problem 2
We consider the matrix A and the vector $\mathbf{v}$ given by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & s & s^2 \\ 1 & -1 & 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

a) Compute the determinant and the rank of A.
b) Find all values of s such that $\mathbf{v}$ is an eigenvector for A.
c) Compute all eigenvalues of A when $s = -1$. Is A diagonalizable when $s = -1$?

Challenging Matrix Problems for Advanced Students

These matrix problems are quite challenging and are meant for advanced students. It is recommended that you work through the ordinary problems and exam problems from Problem Sheet 1-4 and make sure that you master them before you attempt Problem 13-15 (which are optional).

13. Solve the equation

$$\begin{vmatrix} x & 2 & 3 \\ 2 & x & 3 \\ 2 & 3 & x \end{vmatrix} = 0$$

14. Solve the equation

$$\begin{vmatrix} x+1 & 0 & x & 0 & x-1 & 0 \\ 0 & x & 0 & x-1 & 0 & x+1 \\ x & 0 & x-1 & 0 & x+1 & 0 \\ 0 & x-1 & 0 & x+1 & 0 & x \\ x-1 & 0 & x+1 & 0 & x & 0 \\ 0 & x+1 & 0 & x & 0 & x-1 \end{vmatrix} = 9$$

15. Solve the linear system

$$\begin{array}{rcl} x_2 + x_3 + \ldots + x_{n-1} + x_n &=& 2 \\ x_1 + x_3 + \ldots + x_{n-1} + x_n &=& 4 \\ x_1 + x_2 + \ldots + x_{n-1} + x_n &=& 6 \\ \vdots && \vdots \\ x_1 + x_2 + x_3 + \ldots + x_{n-1} &=& 2n \end{array}$$

Solutions

1 We compute that

$$A\mathbf{v} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\mathbf{v}$$

This means that $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda = 3$.

2 a) We solve the characteristic equation to find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 5 = 0 \quad \Rightarrow \lambda = -1, -5$$

For each eigenvalue, we compute the eigenvectors using an echelon form of the coefficient matrix, and express the eigenvectors in terms of the free variables. For $\lambda = -1$, we get eigenvectors

$$\begin{pmatrix} 3 & -7 \\ 3 & -7 \end{pmatrix} \dashrightarrow \begin{pmatrix} 3 & -7 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad 3x - 7y = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{7}{3}y \\ y \end{pmatrix} = y \begin{pmatrix} 7/3 \\ 1 \end{pmatrix}$$

For $\lambda = -5$, we get eigenvectors

$$\begin{pmatrix} 7 & -7 \\ 3 & -3 \end{pmatrix} \dashrightarrow \begin{pmatrix} 7 & -7 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad 7x - 7y = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b) We solve the characteristic equation to find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & 4 \\ -2 & 6-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 20 = 0 \quad \Rightarrow \quad \text{no solutions}$$

Since there are no solutions, there are no eigenvalues and no eigenvectors.
c) We solve the characteristic equation to find the eigenvalues:

$$\begin{vmatrix} 1-\lambda & 4 \\ 6 & -1-\lambda \end{vmatrix} = \lambda^2 - 25 = 0 \quad \Rightarrow \lambda = 5, -5$$

For each eigenvalue, we compute the eigenvectors using an echelon form of the coefficient matrix, and express the eigenvectors in terms of the free variables. For $\lambda = 5$, we get eigenvectors

$$\begin{pmatrix} -4 & 4 \\ 6 & -6 \end{pmatrix} \dashrightarrow \begin{pmatrix} -4 & 4 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad -4x + 4y = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda = -5$, we get eigenvectors

$$\begin{pmatrix} 6 & 4 \\ 6 & 4 \end{pmatrix} \dashrightarrow \begin{pmatrix} 6 & 4 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad 6x + 4y = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}y \\ y \end{pmatrix} = y \begin{pmatrix} -2/3 \\ 1 \end{pmatrix}$$

3 a) We solve the characteristic equation to find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)(4-\lambda) = 0 \quad \Rightarrow \lambda = 2,3,4$$

For each eigenvalue, we compute the eigenvectors using an echelon form of the coefficient matrix, and express the eigenvectors in terms of the free variables. For $\lambda = 2$, we get eigenvectors

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \quad y = 0,\ 2z = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 3$, we get eigenvectors

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \quad x = z = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = 4$, we get eigenvectors

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \quad x = y = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

b) We solve the characteristic equation to find the eigenvalues. Since the equation (of degree three) is a bit difficult to solve, we first change the determinant by adding the second row to the first row:

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2-\lambda & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix}$$

Then we transpose the matrix, and subtract the first row from the second row:

$$= \begin{vmatrix} 2-\lambda & 0 & 2 \\ 2-\lambda & 1-\lambda & 0 \\ 0 & 1 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2+\lambda) = 0$$

We see that the eigenvalues are $\lambda = 2, 0, -1$. For each eigenvalue, we compute the eigenvectors using an echelon form of the coefficient matrix, and express the eigenvectors in terms of the free variables. For this operation, we must use the matrix **before** the transposition, since the operation of transposing the coefficient matrix will not preserve the solutions of the linear system (but it will preserve the determinant). For $\lambda = 2$, we get eigenvectors

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & -4 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 & -4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad y = z,\, x = 2z \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda = 0$, we get eigenvectors

$$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad 2x = 2z,\, y = -z \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda = -1$, we get eigenvectors

$$\begin{pmatrix} 3 & 3 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad 2x = z,\, 2y = -z \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

4 If A is invertible, then $\det(A) \neq 0$. Hence $\lambda = 0$ is not an eigenvalue. If it were, then $\lambda = 0$ would solve the characteristic equation, and we would have $\det(A - 0 \cdot I_n) = \det(A) = 0$; this is a contradiction. For the last part, notice that if $\mathbf{v}$ is an eigenvector for A with eigenvalue λ, then we have

$$A\mathbf{v} = \lambda v \quad \Rightarrow \quad \mathbf{v} = A^{-1}\lambda\mathbf{v} = \lambda A^{-1}\mathbf{v} \quad \Rightarrow \quad \lambda^{-1}\mathbf{v} = A^{-1}\mathbf{v}$$

This means that $\lambda^{-1} = 1/\lambda$ is an eigenvalue of A^{-1} (with eigenvector $\mathbf{v}$).

5 We form that matrix P with $\mathbf{v}_i$ as the i'th column, $i = 1, 2, 3$, and compute AP:

$$AP = \begin{pmatrix} 1 & 18 & 30 \\ -2 & -11 & -10 \\ 2 & 6 & 5 \end{pmatrix} \begin{pmatrix} -3 & -5 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 25 & 15 \\ -5 & 0 & -5 \\ 0 & -5 & 5 \end{pmatrix}$$

We have $AP = (A\mathbf{v}_1|A\mathbf{v}_2|A\mathbf{v}_3)$, and looking at the columns of AP we see that

$$A\mathbf{v}_1 = -5\mathbf{v}_1,\; A\mathbf{v}_2 = -5\mathbf{v}_2,\; A\mathbf{v}_3 = 5\mathbf{v}_3$$

The vectors are therefore eigenvectors, with eigenvalues $\lambda = -5, -5, 5$. The matrix P is invertible since the three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, and we have $AP = PD \quad \Rightarrow \quad A = PDP^{-1}$ when D is the diagonal matrix

$$D = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

6 We use the eigenvalues and eigenvectors we found in Problem a). Since there are two distinct eigenvalues, the matrix A is diagonalizable, and $D = P^{-1}AP$ when we put

$$P = \begin{pmatrix} 7/3 & 1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

7 We find the eigenvalues of A by solving the characteristic equation:

$$\begin{vmatrix} 3-\lambda & 5 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0 \quad \Rightarrow \quad \lambda = 3$$

Hence there is only one eigenvalue (with multiplicity 2). The corresponding eigenvectors are found by reducing the coefficient matrix to an echelon form. For $\lambda = 3$, we get eigenvectors

$$\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad 5y = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

In particular, there is only one free variable and therefore not more than one linearly independent eigenvector. This means that there are too few linearly independent eigenvectors (only one eigenvector while $n = 2$), hence A is not diagonalizable.

8

a) We compute the matrix product

$$T\mathbf{s} = \begin{pmatrix} 0.53 \\ 0.47 \end{pmatrix}$$

and see that the result is a market share vector.

b) We find the eigenvalues of T by solving the characteristic equation

$$\begin{vmatrix} 0.85-\lambda & 0.45 \\ 0.15 & 0.55-\lambda \end{vmatrix} = \lambda^2 - 1.4\lambda + 0.4 = 0$$

This gives eigenvalues $\lambda = 1, 0.4$. The eigenvectors for $\lambda = 1$ is given by $-0.15x + 0.45y = 0$, or $x = 3y$; and for $\lambda = 0.4$, the eigenvectors are given by $0.45x + 0.45y = 0$, or $x = -y$. Hence eigenvectors are given in terms of the free variables by

$$\mathbf{v}_1 = y \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \; \mathbf{v}_2 = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

c) From the eigenvectors, we see that

$$P = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

This gives $T^n = (PDP^{-1})^n = PD^nP^{-1}$.

d) When $n \to \infty$, we get that

$$D^n = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 0.4^n \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence the limit of T^n as $n \to \infty$ is given by

$$T^n \to \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3/4 & 3/4 \\ 1/4 & 1/4 \end{pmatrix}$$

and

$$T^n\mathbf{s} \to \begin{pmatrix} 3/4 & 3/4 \\ 1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$$

The equilibrium marked shares are 75% for A and 25% for B.

9 We find the eigenvalues of A by solving the characteristic equation:

$$\begin{vmatrix} 4-\lambda & 1 & 2 \\ 0 & 3-\lambda & 0 \\ 1 & 1 & 5-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 9\lambda + 18) = 0 \quad \Rightarrow \quad \lambda = 3, 3, 6$$

Hence there is one eigenvalue $\lambda = 3$ with multiplicity 2, and one eigenvalue $\lambda = 6$ with multiplicity 1. The corresponding eigenvectors are found by reducing the coefficient matrix to an echelon form. For $\lambda = 3$, we get eigenvectors

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad x = -y - 2z$$

Since there is two degrees of freedom, there are two linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ for $\lambda = 3$, given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-2z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = y\mathbf{v}_1 + z\mathbf{v}_2$$

Since $\lambda = 6$ is an eigenvalue of multiplicity one, we get one eigenvector $\mathbf{v}_3$ given by

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -3 & 0 \\ 1 & 1 & -1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad x = z,\ y = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Hence A is diagonalizable, and we have that $P^{-1}AP = D$ is diagonal with

$$P = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

We use this to compute A^{17}, since $A = PDP^{-1}$. We do not show the computation of P^{-1}, which is straight-forward:

$$A^{17} = (PDP^{-1})^{17} = PD^{17}P^{-1} = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3^{17} & 0 & 0 \\ 0 & 3^{17} & 0 \\ 0 & 0 & 6^{17} \end{pmatrix} \frac{1}{3} \begin{pmatrix} 0 & 3 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

This gives

$$A^{17} = 3^{16}\begin{pmatrix} 2 & -1 & -2 \\ 0 & 3 & 0 \\ -1 & -1 & 1 \end{pmatrix} + 6^{16}\begin{pmatrix} 2 & 2 & 4 \\ 0 & 0 & 0 \\ 2 & 2 & 4 \end{pmatrix}$$

10 Final Exam in GRA6035 10/12/2010, Problem 2

a) The determinant of A is given by

$$\det(A) = \begin{vmatrix} 1 & 7 & -2 \\ 0 & s & 0 \\ 1 & 1 & 4 \end{vmatrix} = s(4+2) = 6s$$

It follows that the rank of A is 3 if $s \neq 0$ (since $\det(A) \neq 0$). When $s = 0$, A has rank 2 since $\det(A) = 0$ but the minor

$$\begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} = 6 \neq 0$$

Therefore, we get

$$\mathrm{rk}(A) = \begin{cases} 3, & s \neq 0 \\ 2, & s = 0 \end{cases}$$

b) We compute the characteristic equation of A, and find that

$$\begin{vmatrix} 1-\lambda & 7 & -2 \\ 0 & s-\lambda & 0 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (s-\lambda)(\lambda^2 - 5\lambda + 6) = (s-\lambda)(\lambda-2)(\lambda-3)$$

Therefore, the eigenvalues of A are $\lambda = s, 2, 3$. Furthermore, we have that

$$A\mathbf{v} = \begin{pmatrix} 6 \\ s \\ 6 \end{pmatrix}$$

We see that $\mathbf{v}$ is an eigenvector for A if and only if $s = 6$, in which case $A\mathbf{v} = 6\mathbf{v}$.

c) If $s \neq 2, 3$, then A has three distinct eigenvalues, and therefore A is diagonalizable. If $s = 2$, we check the eigenspace corresponding to the double root $\lambda = 2$: The coefficient matrix of the system has echelon form

$$\begin{pmatrix} -1 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank two, so there is only one free variable. If $s = 3$, we check the eigenspace corresponding to the double root $\lambda = 3$: The coefficient matrix of the system has echelon form

$$\begin{pmatrix} -2 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank two, so there is only one free variable. In both cases, there are too few linearly independent eigenvectors, and A is not diagonalizable. Hence A is diagonalixable if $s \neq 2, 3$.

11 Mock Final Exam in GRA6035 12/2010, Problem 1

a) The determinant of A is given by

$$\det(A) = \begin{vmatrix} 1 & 1 & -4 \\ 0 & t+2 & t-8 \\ 0 & -5 & 5 \end{vmatrix} = 10t - 30 = 10(t-3)$$

It follows that the rank of A is 3 if $t \neq 3$ (since $\det(A) \neq 0$). When $t = 3$, A has rank 2 since $\det(A) = 0$ but the minor

$$\begin{vmatrix} 1 & -4 \\ 0 & 5 \end{vmatrix} = 5 \neq 0$$

Therefore, we get

$$\text{rk}(A) = \begin{cases} 3, & t \neq 3 \\ 2, & t = 3 \end{cases}$$

b) We compute the characteristic equation of A, and find that

$$\begin{vmatrix} 1-\lambda & 1 & -4 \\ 0 & t+2-\lambda & t-8 \\ 0 & -5 & 5-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - (t+7)\lambda + 10(t-3)) = 0$$

Since $\lambda^2 - (t+7)\lambda + 10(t-3) = 0$ has solutions $\lambda = 10$ and $\lambda = t-3$, the eigenvalues of A are $\lambda = 1, 10, t-3$.

c) When A has three distinct eigenvalues, it is diagonalizable. We see that this happens for all values of t except $t = 4$ and $t = 13$. Hence A is diagonalizable for $t \neq 4, 13$. If $t = 4$, we check the eigenspace corresponding to the double root $\lambda = 1$: The coefficient matrix of the system has echelon form

$$\begin{pmatrix} 0 & 1 & -4 \\ 0 & 5 & -4 \\ 0 & -5 & 4 \end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & 1 & -4 \\ 0 & 0 & 16 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank two, so there is only one free variable. If $t = 13$, we check the eigenspace corresponding to the double root $\lambda = 10$: The coefficient matrix of the system has echelon form

$$\begin{pmatrix} -9 & 1 & -4 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{pmatrix} \dashrightarrow \begin{pmatrix} -9 & 1 & -4 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank two, so there is only one free variable. In both cases, there are too few linearly independent eigenvectors, and A is not diagonalizable. Hence A is diagonalizable if $t \neq 4, 13$.

12 Final Exam in GRA6035 30/05/2011, Problem 2

a) The determinant of A is given by

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & s & s^2 \\ 1 & -1 & 1 \end{vmatrix} = 2s^2 - 2 = 2(s-1)(s+1)$$

It follows that the rank of A is 3 if $s \neq \pm 1$ (since $\det(A) \neq 0$). When $s = \pm 1$, A has rank 2 since $\det(A) = 0$ but there is a non-zero minor of order two in each case. Therefore, we get

$$\text{rk}(A) = \begin{cases} 3, & s \neq \pm 1 \\ 2, & s = \pm 1 \end{cases}$$

b) We compute that

$$A\mathbf{v} = \begin{pmatrix} 1 \\ 1 + s - s^2 \\ -1 \end{pmatrix}, \quad \lambda \mathbf{v} = \begin{pmatrix} \lambda \\ \lambda \\ -\lambda \end{pmatrix}$$

and see that $\mathbf{v}$ is an eigenvector for A if and only if $\lambda = 1$ and $1 + s - s^2 = 1$, or $s = s^2$. This gives $s = 0, 1$.

c) We compute the characteristic equation of A when $s = -1$, and find that

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = \lambda(2 + \lambda - \lambda^2) = -\lambda(\lambda - 2)(\lambda + 1)$$

Therefore, the eigenvalues of A are $\lambda = 0, 2, -1$ when $s = -1$. Since A has three distinct eigenvalues when $s = -1$, it follows that A is diagonalizable.