Revision Problems
GRA 6035 Mathematics

BI Norwegian Business School

Revision Problems

1. Final Exam in GRA6035 06/02/2012, Problem 2
We consider the matrix A and the vector $\mathbf{v}$ given by

$$A = \begin{pmatrix} 1 & 3s+1 & -2 \\ 3 & 7s-2 & 0 \\ 2 & 7s & -4 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -8 \\ 2 \\ 3 \end{pmatrix}$$

a) Compute the determinant and the rank of A.
b) Is $\mathbf{v}$ an eigenvector for A for any value of s? If so, what is the corresponding eigenvalue?
c) Find all eigenvalues of A when $s = 2$.

2. Final Exam in GRA6035 06/02/2012, Problem 1
We consider the function f given by $f(x,y,z) = e^{x^2-y} + y + z^2$.

a) Find all stationary points of f.
b) Is f convex? Is it concave?

3. Lagrange problem with two constraints
Consider the optimization problem

$$\max f(x,y,z) = 2z \text{ subject to } x^2 + y^2 = 2,\ x+y+z = 1$$

a) Write down the Lagrangian $\mathscr{L}$ and the first order conditions for this problem.
b) Solve the optimization problem. What is the maximum value?
c) Write down the NDCQ for this problem. It NDCQ satisfied for all admissible points (x,y,z)? It is necessary to check NDCQ to solve this optimization problem?
d) Change the last constraint to $x+y+z = b$. Show that the problem has a solution, a maximal value, for each value of b. How does this maximum value change if you increase b?

4. Kuhn-Tucker problem with two constraints
Consider the following Kuhn-Tucker optimization problem with two constraints:

$$\max f(x,y,z) = 2z \text{ subject to } x^2 + y^2 \le 2,\ x+y+z \le 1$$

a) Write down the Lagrangian $\mathscr{L}$ and the first order conditions for this problem. Also, write down the complementary slackness conditions.
b) Solve the optimization problem. What is the maximum value?
c) Write down the NDCQ for this problem. It NDCQ satisfied for all admissible points (x,y,z)? It is necessary to check NDCQ to solve this optimization problem?

Solutions

1

a) To compute the determinant of A, we develop it along the third column:

$$\det(A) = \begin{vmatrix} 1 & 3s+1 & -2 \\ 3 & 7s-2 & 0 \\ 2 & 7s & -4 \end{vmatrix} = -2(21s-2(7s-2)) - 4(1(7s-2)-3(3s+1))$$

This gives

$$\det(A) = -2(7s+4) - 4(-2s-5) = \mathbf{-6s+12} = \mathbf{-6(s-2)}$$

This means that A is has rank 3 if $s \neq 2$, since $\det(A) \neq 0$. For $s = 2$, we see that A has rank 2 since $\det(A) = 0$ and there is a minor of order two that is non-zero:

$$\begin{vmatrix} 3 & 0 \\ 2 & -4 \end{vmatrix} = -12 \neq 0$$

Therefore it follows that

$$\mathrm{rk}(A) = \begin{cases} 2 & s = 2 \\ 3 & s \neq 2 \end{cases}$$

b) To check if $\mathbf{v}$ is an eigenvector of A, we compute

$$A\mathbf{v} = \begin{pmatrix} 1 & 3s+1 & -2 \\ 3 & 7s-2 & 0 \\ 2 & 7s & -4 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6s-12 \\ 14s-28 \\ 14s-28 \end{pmatrix}$$

We know that $\mathbf{v}$ is an eigenvector with eigenvalue λ if and only if

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Leftrightarrow \quad \begin{pmatrix} 6s-12 \\ 14s-28 \\ 14s-28 \end{pmatrix} = \lambda \cdot \begin{pmatrix} -8 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -8\lambda \\ 2\lambda \\ 3\lambda \end{pmatrix}$$

From the last two equations, we see that $2\lambda = 3\lambda$, which means that $\lambda = 0$. When we substitute $\lambda = 0$ in all three equations, we see that $s = 2$ is a solution. This means that $\mathbf{v}$ is an eigenvector if and only if $\mathbf{s} = \mathbf{2}$, and the corresponding eigenvalues is $\lambda = \mathbf{0}$.

c) We substitute $s = 2$ in A, and find that

$$A = \begin{pmatrix} 1 & 7 & -2 \\ 3 & 12 & 0 \\ 2 & 14 & -4 \end{pmatrix}$$

The we write down the characteristic equation $A - \lambda I = 0$, which gives

$$\begin{vmatrix} 1-\lambda & 7 & -2 \\ 3 & 12-\lambda & 0 \\ 2 & 14 & -4-\lambda \end{vmatrix} = -3(7(-4-\lambda)+28)+(12-\lambda)((1-\lambda)(-4-\lambda)+4) = 0$$

After we simplify this equation, we get

$$-3(-7\lambda)+(12-\lambda)(\lambda^2+3\lambda) = \lambda(-\lambda^2+9\lambda+57) = 0$$

The eigenvalues of A for $s=2$ are therefore $\lambda = \mathbf{0}$ and $\lambda = \frac{\mathbf{-9\pm\sqrt{309}}}{\mathbf{-2}}$.

2

a) We compute the partial derivatives $f_x' = 2xe^u$, $f_y' = -e^u+1$ and $f_z' = 2z$, where we write $u = x^2-y$. The stationary points are given by the equations

$$2xe^u = 0, \quad 1-e^u = 0, \quad 2z = 0$$

The first equation gives $x=0$ and the third gives $z=0$. From the second equation, we get that $e^u = 1$, or that $u = x^2-y = 0$, and this gives $y=0$ (since $x=0$). The stationary points are therefore given by $(\mathbf{x},\mathbf{y},\mathbf{z}) = (\mathbf{0},\mathbf{0},\mathbf{0})$.

b) We compute the second order partial derivatives of f and form the Hessian matrix

$$f'' = \begin{pmatrix} (2+4x^2)e^u & -2xe^u & 0 \\ -2xe^u & e^u & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

We see that the matrix has leading principal minors $D_1 = (2+4x^2)e^u > 0$, $D_2 = 2e^{2u} > 0$ and $D_3 = 4e^{2u} > 0$. Since all leading principal minors are positive, f is **convex but not concave**.

3 There are two solution of the Lagrange conditions, $(x,y,z) = (-1,-1,3)$ with $\lambda_1 = 1$ and $\lambda_2 = 2$ gives $f(-1,-1,3) = 6$, and $(x,y,z) = (1,1,-1)$ with $\lambda_1 = -1$ and $\lambda_2 = 2$ gives $f(1,1,-1) = -2$. The first point is a candidate for maximum, and it is a maximum since $\mathscr{L}(x,y,z;1,2)$ is concave. The set of admissible points is bounded and all admissible points satisfy NDCQ, and this gives another argument for the fact that this point is the maximum.

4 There is one solution of the Kuhn-Tucker conditions, $(x,y,z) = (-1,-1,3)$ with $\lambda_1 = 1$ and $\lambda_2 = 2$, and this point has value $f(-1,-1,3) = 6$. Since $\mathscr{L}(x,y,z;1,2)$ is concave, this point is the maximum. The set of admissible points is bounded and all admissible points satisfy NDCQ, and this is another argument for the fact that this point is the maximum.