Problem Sheet 11 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. Find the general solution of the differential equation $\dot{y}+\frac{1}{2}y=\frac{1}{4}$. Determine the equilibrium state of the equation. Is it stable? Draw some typical solutions.

2. Find the general solution of the following differential equations:

a) $\dot{y}+y=10$
b) $\dot{y}-3y=27$
c) $4\dot{y}+5y=100$

3. Find the general solution of the following differential equations, and in each case, find the particular solution satisfying $y(0)=1$:

a) $\dot{y}-3y=5$
b) $3\dot{y}+2y+16=0$
c) $\dot{y}+2y=t^2$

4. Find the general solution of the following differential equations:

a) $t\dot{y}+2y+t=0 \quad (t\neq 0)$
b) $\dot{y}-y/t=t \quad (t>0)$
c) $\dot{y}-\dfrac{t}{t^2-1}y=t \quad (t>1)$
d) $\dot{y}-\dfrac{2}{t}y+\dfrac{2a^2}{t^2}=0 \quad (t>0)$

5. Determine which of the following differential equations are exact, and find the general solution in those cases where it is exact:

a) $(2y+t)\dot{y}+2+y=0$
b) $y^2\dot{y}+2t+y=0$
c) $(t^5+6y^2)\dot{y}+5yt^4+2=0$

6. Solve the differential equation $2t+3y^2\dot{y}=0$, first as a separable differential equation and then as an exact differential equation.

7. Find the general solution of the following differential equations:

a) $\ddot{y}=t$
b) $\ddot{y}=e^t+t^2$

8. Solve the initial value problem $\ddot{y}=t^2-t, \quad y(0)=1,\ \dot{y}(0)=2.$

9. Solve the initial value problem $\ddot{y}=\dot{y}+t, \quad y(0)=1,\ y(1)=2.$

10. Find the general solution of the following differential equations:

a) $\ddot{y}-3\dot{y}=0$
b) $\ddot{y}+4\dot{y}+8y=0$
c) $3\ddot{y}+8\dot{y}=0$

d) $4\ddot{y}+4\dot{y}+y=0$
e) $\ddot{y}+\dot{y}-6y=0$
f) $\ddot{y}+3\dot{y}+2y=0$

11. Consider the equation $\ddot{y}+a\dot{y}+by=0$ when $a^2/4-b=0$, so that that the characteristic equation has a double root $r=-a/2$. Let $y(t)=u(t)e^{rt}$ and prove that this function is a solution if and only if $\ddot{u}=0$. Conclude that the general solution is $y=(A+Bt)e^{rt}$.

12. Final Exam in GRA6035 30/05/2011, 3c
Solve the initial value problem $(2t+y)-(4y-t)y'=0,\ y(0)=0$.

13. Final Exam in GRA6035 10/12/2010, 3c
Solve the initial value problem

$$\frac{t}{y^2}y'=\frac{1}{y}-3t^2, \quad y(1)=\frac{1}{3}$$

Solutions

1 Since the differential equation is linear first order with constant coeffients $a = 1/2$ and $b = 1/4$, it has general solution

$$y = \frac{b}{a} + Ce^{-at} = \frac{1}{2} + Ce^{-t/2}$$

Since $y \to 1/2$ when $t \to \infty$ for all values of C, the equation is stable (and also globally asymptotically stable), with equilibrium state $y = 1/2$.

2 Since the differential equations are linear first order, they have general solutions

$$y = \frac{b}{a} + Ce^{-at}$$

This gives

a) $y = 10 + Ce^{-t}$
b) $y = -9 + Ce^{3t}$
c) $y = 20 + Ce^{-5t/4}$

3 Since the first two differential equation are linear first order, they have general solutions

$$y = \frac{b}{a} + Ce^{-at}$$

This gives

a) $y = -5/3 + Ce^{3t}$, and $y(0) = 1$ gives $C = 8/3$
b) $y = -8 + Ce^{-2t/3}$, and $y(0) = 1$ gives $C = 9$

The last equation has integration factor $u = e^{2t}$, and this gives $ye^{2t} = \int t^2 e^{2t}\, dt$. We solve the integral by using integration by parts twice, and get

$$ye^{2t} = (t^2/2 - t/2 + 1/4)e^{2t} + \mathscr{C} \quad \Rightarrow \quad y = t^2/2 - t/2 + 1/4 + \mathscr{C} e^{-2t}$$

The condition $y(0) = 1$ gives $\mathscr{C} = 3/4$.

4 See [FMEA] Problem 5.4.6.

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a) We consider $(2 + y) + (2y + t)\dot{y} = 0$ with $f = 2 + y$ and $g = 2y + t$. Since $f'_y = 1 = g'_t$, the equation is exact. We find $h = y^2 + yt + 2t$ satisfy $h'_t = f$ and $h'_y = g$, so the general solution is

$$y^2 + yt + 2t = C \quad \Rightarrow \quad y = \frac{-t \pm \sqrt{t^2 - 8t + 4C}}{2}$$

b) $(2t + y) + y^2\dot{y} = 0$ with $f = 2t + y$ and $g = y^2$. Since $f'_y = 1$ and $g'_t = 0$, the equation is not exact.

c) $(5yt^4+2)+(t^5+6y^2)\dot{y}=0$ with $f=5yt^4+2$ and $g=t^5+6y^2$. Since $f'_y=5t^4=g'_t$, the equation is exact. We find $h=t^5y+2y^3+2t$ satisfy $h'_t=f$ and $h'_y=g$, so the general solution (in implicit form) is

$$t^5y+2y^3+2t=C$$

It is difficult to find the solution in explicit form in this problem (it is a third degree equation in y).

6 The differential equation $2t+3y^2\dot{y}=0$ can be written as $3y^2\dot{y}=-2t$, and is therefore separable with solution $y^3=-t^2+C$, which gives $y=\sqrt[3]{C-t^2}$. We write $f=2t$ and $g=3y^2$; then $h=t^2+y^3$ has the property that $h'_t=f$ and $h'_y=g$, so the equation is exact with solution $t^2+y^3=C$, which again gives $y=\sqrt[3]{C-t^2}$.

7 We solve the differential equation by direct integration:

a) $\ddot{y}=t \implies \dot{y}=\frac{1}{2}t^2+C_1 \implies y=\frac{1}{6}t^3+C_1t+C_2$
b) $\ddot{y}=e^t+t^2 \implies \dot{y}=e^t+\frac{1}{3}t^3+C_1 \implies y=e^t+\frac{1}{12}t^4+C_1t+C_2$

8 We have $\ddot{y}=t^2-t \implies \dot{y}=\frac{1}{3}t^3-\frac{1}{2}t^2+C_1 \implies y=\frac{1}{12}t^4-\frac{1}{6}t^3+C_1t+C_2$. The initial condition $y(0)=1$ gives $\frac{1}{12}0^4-\frac{1}{6}0^3+C_10+C_2=C_2=1$, and $\dot{y}(0)=2$ gives $\frac{1}{3}0^3-\frac{1}{2}0+C_1=C_1=2$. The particular solution is therefore

$$y(t)=\frac{1}{12}t^4-\frac{1}{6}t^3+2t+1$$

9 Substitute $u=\dot{y}$. Then $\ddot{y}=\dot{y}+t \Leftrightarrow \dot{u}=u+t \Leftrightarrow \dot{u}-u=t$. The integrating factor is e^{-t}, and we get

$$ue^{-t}=\int te^{-t}dt=-e^{-t}-te^{-t}+C_1$$

From this we obtain $u=(-e^{-t}-te^{-t}+C_1)e^t=C_1e^t-t-1$ and we integrate to find y from $u=\dot{y}$, and get $y=\int(C_1e^t-t-1)dt=C_1e^t-t-\frac{1}{2}t^2+C_2$. The initial condition $y(0)=1$ gives $C_1+C_2=1 \implies C_2=1-C_1$, and the condition $y(1)=2$ gives $C_1e-1-\frac{1}{2}+C_2=C_1e-3/2+1-C_1=2$. This gives $C_1(e-1)=5/2$, or

$$C_1=\frac{5}{2(e-1)}, \quad C_2=1-\frac{5}{2(e-1)}=\frac{2e-7}{2(e-1)}$$

The particular solution is therefore

$$y(t)=\frac{5}{2(e-1)}\cdot e^t-t-\frac{1}{2}t^2+\frac{2e-7}{2(e-1)}$$

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a) The characteristic equation is $r^2-3r=0 \implies r=0,3 \implies y(t)=C_1+C_2e^{3t}$.

b) Characteristic equation is $r^2+4r+8=0$. This has no real solutions. Thus we put $\alpha=-\frac{1}{2}a=-\frac{1}{2}4=-2, \beta=\sqrt{b-\frac{1}{4}a^2}=\sqrt{8-\frac{1}{4}4^2}=2$. From this the general solution is $y(t)=e^{\alpha t}(A\cos\beta t+B\sin\beta t)=e^{-2t}(A\cos 2t+B\sin 2t)$.
c) $3\ddot{y}+8\dot{y}=0 \iff \ddot{y}+\frac{8}{3}\dot{y}=0$. The characteristic equation is $r^2+\frac{8}{3}r=0 \implies r=0$ or $r=-\frac{8}{3}$. The general solution is $y(t)=C_1e^{0t}+C_2e^{-\frac{8}{3}t}=C_1+C_2e^{-\frac{8}{3}t}$.
d) $4\ddot{y}+4\dot{y}+y=0$ has characteristic equation $4r^2+4r+1=0$. There is one solution $r=-\frac{1}{2}$. The general solution is $y(t)=(C_1+C_2t)e^{-\frac{1}{2}t}$.
e) $\ddot{y}+\dot{y}-6y=8$ has characteristic equation $r^2+r-6=0$. It has the solutions $r=-3$ and $r=2$. The general solution is thus

$$y(t)=C_1e^{-3t}+C_2e^{2t}$$

f) $\ddot{y}+3\dot{y}+2y=0$ has characteristic equation $r^2+3r+2=0$. The solutions are $r=-1$ and $r=-2$. The general solution is thus

$$y(t)=C_1e^{-t}+C_2e^{-2t}$$

11 $y=ue^{rt} \implies \dot{y}=\dot{u}e^{rt}+ure^{rt}=e^{rt}(\dot{u}+ur) \implies \ddot{y}=\ddot{u}e^{rt}+\dot{u}re^{rt}+r(\dot{u}e^{rt}+ure^{rt})=e^{rt}(\ddot{u}+2r\dot{u}+ur^2)$. From this we get

$$\begin{aligned}\ddot{y}+a\dot{y}+by&=e^{rt}[(\ddot{u}+2r\dot{u}+ur^2)+a(\dot{u}+ur)+bu]\\&=e^{rt}[\ddot{u}+(2r+a)\dot{u}+(r^2+ar+b)u]\end{aligned}$$

The characteristic equation is assumed to have one solution $r=\frac{-a}{2}$. Putting $r=\frac{-a}{2}$ into the eypression we get

$$\ddot{y}+a\dot{y}+by=e^{rt}\ddot{u}$$

So $y=ue^{rt}$ is a solution if and only if $e^{rt}\ddot{u}=0 \Leftrightarrow \ddot{u}=0$. The differential equation $\ddot{u}=0$ has the general solution $u=A+Bt$. Thus $y=(A+Bt)e^{rt}$ is the general solution of $\ddot{y}+a\dot{y}+by=0$.

12 The differential equation can be written in the form

$$(2t+y)+(t-4y)y'=0$$

and we see that it is exact. Hence its solution can be written in the form $u(y,t)=C$, where $u(y,t)$ is a function that satisfies

$$\frac{\partial u}{\partial t}=2t+y \quad \text{and} \quad \frac{\partial u}{\partial y}=t-4y$$

One solution is $u(y,t)=t^2+ty-2y^2$, and the initial condition $y(0)=0$ gives $C=0$. Hence

$$t^2+ty-2y^2=0 \quad \Leftrightarrow \quad y=\frac{-t\pm 3t}{-4}$$

The solution to the initial value problem is therefore

$$y = -\frac{1}{2}t \text{ or } y = t$$

13 The differential equation can be written in the form

$$\left(3t^2 - \frac{1}{y}\right) + \frac{t}{y^2}y' = 0$$

and we see that it is exact. Hence it can be written of the form $u(y,t) = C$, where $u(y,t)$ is a function that satisfies

$$\frac{\partial u}{\partial t} = 3t^2 - \frac{1}{y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{t}{y^2}$$

One solution is $u(y,t) = t^3 - t/y$, and this gives

$$t^3 - \frac{t}{y} = C \quad \Leftrightarrow \quad y = \frac{t}{t^3 - C}$$

The initial condition gives $1/(1-C) = 1/3$ or $C = -2$. The solution to the initial value problem is therefore

$$y = \frac{t}{t^3 + 2}$$