Problem Sheet 1 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. From linear system to augmented matrix
Write down the coefficient matrix and the augmented matrix of the following linear systems:

$$a) \quad \begin{array}{l} 2x + 5y = 6 \\ 3x - 7y = 4 \end{array} \qquad b) \quad \begin{array}{l} x + y - z = 0 \\ x - y + z = 2 \\ x - 2y + 4z = 3 \end{array}$$

2. From augmented matrix to linear system
Write down the linear system in the variables x, y, z with augmented matrix

$$\begin{pmatrix} 1 & 2 & 0 & | & 4 \\ 2 & -3 & 1 & | & 0 \\ 7 & 4 & 1 & | & 3 \end{pmatrix}$$

3. Solution by substitution
Use substitution to solve the linear system

$$\begin{array}{l} x + y + z = 1 \\ x - y + z = 4 \\ x + 2y + 4z = 7 \end{array}$$

4. Variation of parameters by substitution
For what values of h does the following linear system have solutions?

$$\begin{array}{l} x + y + z = 1 \\ x - y + z = 4 \\ x + 2y + z = h \end{array}$$

5. Gaussian elimination I
Solve the following linear systems by Gaussian elimination:

$$a) \quad \begin{array}{l} x + y + z = 1 \\ x - y + z = 4 \\ x + 2y + 4z = 7 \end{array} \qquad b) \quad \begin{array}{l} 2x + 2y - z = 2 \\ x + y + z = -2 \\ 2x + 4y - 3z = 0 \end{array}$$

6. Gauss-Jordan elimination
Solve the following linear system by Gauss-Jordan elimination:

$$\begin{array}{l} x + y + z = 1 \\ x - y + z = 4 \\ x + 2y + 4z = 7 \end{array}$$

7. Gaussian elimination II

Solve the following linear systems by Gaussian elimination:

$$a) \quad \begin{array}{r} -4x + 6y + 4z = 4 \\ 2x - y + z = 1 \end{array} \qquad b) \quad \begin{array}{r} 6x + y = 7 \\ 3x + y = 4 \\ -6x - 2y = 1 \end{array}$$

8. Variation of parameters

Discuss the number of solutions of the linear system

$$\begin{array}{r} x + 2y + 3z = 1 \\ -x + ay - 21z = 2 \\ 3x + 7y + az = b \end{array}$$

for all values of the parameters a and b.

9. Pivot positions

Find the pivot positions of the following matrix:

$$\begin{pmatrix} 1 & 3 & 4 & 1 & 7 \\ 3 & 2 & 1 & 0 & 7 \\ -1 & 3 & 2 & 4 & 9 \end{pmatrix}$$

10. Basic and free variables

Show that the following linear system has infinitely many solutions, and determine the number of degrees of freedom:

$$\begin{array}{r} x + 6y - 7z + 3w = 1 \\ x + 9y - 6z + 4w = 2 \\ x + 3y - 8z + 4w = 5 \end{array}$$

Find free variables and express the basic variables in terms of the free ones.

11. Linear Systems

Solve the following linear systems by substitution and by Gaussian elimination:

$$a) \quad \begin{array}{r} x - 3y + 6z = -1 \\ 2x - 5y + 10z = 0 \\ 3x - 8y + 17z = 1 \end{array} \qquad b) \quad \begin{array}{r} x + y + z = 0 \\ 12x + 2y - 3z = 5 \\ 3x + 4y + z = -4 \end{array}$$

12. Rank of a matrix I

Find the rank of the following matrices:

$$a) \quad \begin{pmatrix} 1 & 2 \\ 8 & 16 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix} \qquad c) \quad \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{pmatrix}$$

13. Rank of a matrix II
Find the rank of the following matrices:

(a) $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{pmatrix}$ $\qquad b)$ $\begin{pmatrix} 2 & 1 & 3 & 7 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 11 \end{pmatrix}$ $\qquad c)$ $\begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ -2 & -5 & -2 & 0 \end{pmatrix}$

14. Homogeneous linear systems
Prove that any 4×6 homogeneous linear system has non-trivial solutions.

15. Linear system vith parameters
Discuss the ranks of the coefficient matrix A and the augmented matrix $\hat{A}$ of the linear system
$$
\begin{aligned}
x_1 + x_2 + x_3 &= 2q \\
2x_1 - 3x_2 + 2x_3 &= 4q \\
3x_1 - 2x_2 + px_3 &= q
\end{aligned}
$$

for all values of p and q. Use this to determine the number of solutions of the linear system for all values of p and q.

16. Midterm Exam in GRA6035 24/09/2010, Problem 3
Compute the rank of the matrix

$$
A = \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & -2 & 4 \end{pmatrix}
$$

17. Mock Midterm Exam in GRA6035 09/2010, Problem 3
Compute the rank of the matrix

$$
A = \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{pmatrix}
$$

18. Midterm Exam in GRA6035 24/05/2011, Problem 3
Compute the rank of the matrix

$$
A = \begin{pmatrix} 2 & 10 & 6 & 8 \\ 1 & 5 & 4 & 11 \\ 3 & 15 & 7 & -2 \end{pmatrix}
$$

Solutions

General remark: In some of the problems, we compute an echelon form. Since the echelon form is not unique, it is possible to get to another echelon form than the one indicated in the solutions below. However, the pivot positions should be the same.

1 From linear system to augmented matrix
The coefficient matrix and the augmented matrix of the system is given by

$$a) \quad \begin{pmatrix} 2 & 5 \\ 3 & -7 \end{pmatrix}, \quad \begin{pmatrix} 2 & 5 & | & 6 \\ 3 & -7 & | & 4 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 & | & 0 \\ 1 & -1 & 1 & | & 2 \\ 1 & -2 & 4 & | & 3 \end{pmatrix}$$

2 From augmented matrix to linear system
The linear system is given by

$$\begin{aligned} x + 2y \phantom{{}+ z} &= 4 \\ 2x - 3y + z &= 0 \\ 7x + 4y + z &= 3 \end{aligned}$$

3 Solution by substitution
We solve the linear system
$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + 4z &= 7 \end{aligned}$$

by substitution. First, we solve the first equation for z and get $z = 1 - x - y$. Then we substitute this expression for z in the last two equations. We get

$$\begin{aligned} -2y &= 3 \\ -3x - 2y &= 3 \end{aligned}$$

We solve the first equation for y, and get $y = -1.5$. Then we substitute this value for y in the second equation, and get $x = 0$. Finally, we substitute both these values in $z = 1 - x - y$ and get $z = 2.5$. The solution is therefore $x = 0$, $y = -1.5$, $z = 2.5$.

4 Variation of parameters by substitution
We solve the linear system
$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + z &= h \end{aligned}$$

by substitution. First, we solve the first equation for z and get $z = 1 - x - y$. Then we substitute this expression for z in the last two equations. We get

$$\begin{aligned} -2y &= 3 \\ y &= h - 1 \end{aligned}$$

We solve the first equation for y, and get $y = -1.5$. Then we substitute this value for y in the second equation, and get $-1.5 = h - 1$. If $h = -0.5$, this holds and the system have solutions (x is a free variable, $y = -1.5$ and $z = 1 - x - y = 2.5 - x$). If $h \neq -0.5$, then this leads to a contradiction and the system have no solutions. Therefore, the linear system have solutions if and only if $h = -0.5$.

5 Gaussian elimination I

The linear systems have the following augmented matrices:

$$a) \quad \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 1 & -1 & 1 & | & 4 \\ 1 & 2 & 4 & | & 7 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 1 & 1 & 1 & | & -2 \\ 2 & 4 & -3 & | & 0 \end{pmatrix}$$

a) To solve the system, we reduce the system to an echelon form using elementary row operations. The row operations are indicated.

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 1 & -1 & 1 & | & 4 \\ 1 & 2 & 4 & | & 7 \end{pmatrix} \qquad \begin{matrix} R_2 \leftarrow R_2 + (-1)R_1 \\ R_3 \leftarrow R_3 + (-1)R_1 \end{matrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -2 & 0 & | & 3 \\ 0 & 1 & 3 & | & 6 \end{pmatrix} \qquad R_3 \leftarrow R_3 + (0.5)R_2$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -2 & 0 & | & 3 \\ 0 & 0 & 3 & | & 7.5 \end{pmatrix}$$

From the last equation we get $z = 2.5$, substitution in the second equation gives $y = -1.5$, and substitution in the first equation gives $x = 0$. Therefore, the solution of a) is $x = 0$, $y = -1.5$, $z = 2.5$.

b) To solve the system, we reduce the system to an echelon form using elementary row operations. The row operations are indicated.

$$\begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 1 & 1 & 1 & | & -2 \\ 2 & 4 & -3 & | & 0 \end{pmatrix} \qquad \begin{matrix} R_2 \leftarrow R_2 + (-0.5)R_1 \\ R_3 \leftarrow R_3 + (-1)R_1 \end{matrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 0 & 0 & 1.5 & | & -3 \\ 0 & 2 & -2 & | & -2 \end{pmatrix} \qquad \begin{matrix} R_2 \leftarrow R_3 \\ R_3 \leftarrow R_2 \end{matrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 0 & 2 & -2 & | & -2 \\ 0 & 0 & 1.5 & | & -3 \end{pmatrix}$$

From the last equation we get $z = -2$, substitution in the second equation gives $y = -3$, and substitution in the first equation gives $x = 3$. Therefore, the solution of b) is $x = 3$, $y = -3$, $z = -2$.

6 Gauss-Jordan elimination

We reduce the system to the reduced echelon form using elementary row operations:

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 1 & -1 & 1 & | & 4 \\ 1 & 2 & 4 & | & 7 \end{pmatrix} \qquad \begin{array}{l} R_2 \leftarrow R_2 + (-1)R_1 \\ R_3 \leftarrow R_3 + (-1)R_1 \end{array}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -2 & 0 & | & 3 \\ 0 & 1 & 3 & | & 6 \end{pmatrix} \qquad R_3 \leftarrow R_3 + (0.5)R_2$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -2 & 0 & | & 3 \\ 0 & 0 & 3 & | & 7.5 \end{pmatrix} \qquad \begin{array}{l} R_2 \leftarrow (-1/2) \cdot R_2 \\ R_3 \leftarrow (1/3) \cdot R_3 \end{array}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 0 & | & -1.5 \\ 0 & 0 & 1 & | & 2.5 \end{pmatrix} \qquad R_1 \leftarrow R_1 + (-1)R_2 + (-1)R_3$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & -1.5 \\ 0 & 0 & 1 & | & 2.5 \end{pmatrix}$$

We read off the solution of the system: $x = 0$, $y = -1.5$, $z = 2.5$.

7 Gaussian elimination II

a) We reduce the linear system to an echelon form:

$$\begin{pmatrix} -4 & 6 & 4 & | & 4 \\ 2 & -1 & 1 & | & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} -4 & 6 & 4 & | & 4 \\ 0 & 2 & 3 & | & 3 \end{pmatrix}$$

We see that the system has infinitely many solutions (z is a free variable and x, y are basic variables). We reduce the system to a reduced echelon form:

$$\begin{pmatrix} -4 & 6 & 4 & | & 4 \\ 0 & 2 & 3 & | & 3 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & -1.5 & -1 & | & -1 \\ 0 & 1 & 1.5 & | & 1.5 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 1.25 & | & 1.25 \\ 0 & 1 & 1.5 & | & 1.5 \end{pmatrix}$$

We see that $x + 1.25z = 1.25$, $y + 1.5z = 1.5$. Therefore the solution is given by $x = 1.25 - 1.25z$, $y = 1.5 - 1.5z$ (z is a free variable).

b) We reduce the linear system to an echelon form:

$$\begin{pmatrix} 6 & 1 & | & 7 \\ 3 & 1 & | & 4 \\ -6 & -2 & | & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 6 & 1 & | & 7 \\ 0 & 0.5 & | & 0.5 \\ 0 & -1 & | & 8 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 6 & 1 & | & 7 \\ 0 & 0.5 & | & 0.5 \\ 0 & 0 & | & 9 \end{pmatrix}$$

We see that the system has no solutions.

8 Variation of parameters

We find the augmented matrix of the linear system and reduce it to an echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & a & -21 & 2 \\ 3 & 7 & a & b \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & a+2 & -18 & 3 \\ 0 & 1 & a-9 & b-3 \end{array}\right)$$

We interchange the last two rows to avoid division with $a+2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a-9 & b-3 \\ 0 & a+2 & -18 & 3 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a-9 & b-3 \\ 0 & 0 & -18-(a-9)(a+2) & 3-(b-3)(a+2) \end{array}\right)$$

We compute $-18-(a-9)(a+2) = 7a-a^2$. So when $a \neq 0$ and $a \neq 7$, the system has a unique solution. When $a = 0$, we compute $3-(b-3)(a+2) = 9-2b$. So when $a = 0$ and $b \neq 9/2$, the system is inconsistent, and when $a = 0$, $b = 9/2$, the system has infinitely many solutions (one degree of freedom). When $a = 7$, we compute $3-(b-3)(a+2) = 30-9b$. So when $a = 7$ and $b \neq 30/9 = 10/3$, the system is inconsistent, and when $a = 7$, $b = 10/3$, the system has infinitely many solutions (one degree of freedom).

9 Pivot positions

We redude the matrix to an echelon form using row elementary row operations:

$$\left(\begin{array}{ccccc} 1 & 3 & 4 & 1 & 7 \\ 3 & 2 & 1 & 0 & 7 \\ -1 & 3 & 2 & 4 & 9 \end{array}\right) \Rightarrow \left(\begin{array}{ccccc} 1 & 3 & 4 & 1 & 7 \\ 0 & -7 & -11 & -3 & -14 \\ 0 & 6 & 6 & 5 & 16 \end{array}\right) \Rightarrow \left(\begin{array}{ccccc} 1 & 3 & 4 & 1 & 7 \\ 0 & -7 & -11 & -3 & -14 \\ 0 & 0 & -24/7 & * & * \end{array}\right)$$

We have not computed the entries marked $*$ since they are not needed to find the pivot positions. The pivot positions in the matrix are marked with a box:

$$\left(\begin{array}{ccccc} \boxed{1} & 3 & 4 & 1 & 7 \\ 3 & \boxed{2} & 1 & 0 & 7 \\ -1 & 3 & \boxed{2} & 4 & 9 \end{array}\right)$$

10 Basic and free variables

We find the augmented matrix and reduce it to an echelon form using elementary row operations:

$$\left(\begin{array}{cccc|c} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cccc|c} 1 & 6 & -7 & 3 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & -3 & -1 & 1 & 4 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cccc|c} 1 & 6 & -7 & 3 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{array}\right)$$

We see that the system has infinitely many solutions and one degree of freedom (z is a free variable and x, y, w are basic variables). To express x, y, w in terms of z, we find the reduced echelon form:

$$\begin{pmatrix} 1 & 6 & -7 & 3 & | & 1 \\ 0 & 3 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & 2 & | & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 6 & -7 & 3 & | & 1 \\ 0 & 1 & 1/3 & 1/3 & | & 1/3 \\ 0 & 0 & 0 & 1 & | & 5/2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -9 & 0 & | & -7/2 \\ 0 & 1 & 1/3 & 0 & | & -1/2 \\ 0 & 0 & 0 & 1 & | & 5/2 \end{pmatrix}$$

We see that $x - 9z = -7/2$, $y + z/3 = -1/2$ and $w = 5/2$. This means that the solution is given by $x = 9z - 7/2$, $y = -z/3 - 1/2$, $w = 5/2$ (z is a free variable).

11 Linear Systems

a) We find the augmented matrix of the linear system and reduce it to an echelon form:

$$\begin{pmatrix} 1 & -3 & 6 & | & -1 \\ 2 & -5 & 10 & | & 0 \\ 3 & -8 & 17 & | & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -3 & 6 & | & -1 \\ 0 & 1 & -2 & | & 2 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

Back substitution gives the solution $x = 5$, $y = 6$, $z = 2$.

b) We find the augmented matrix of the linear system and reduce it to an echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 12 & 2 & -3 & | & 5 \\ 3 & 4 & 1 & | & -4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 1 & -2 & | & -4 \\ 0 & 0 & -35 & | & -35 \end{pmatrix}$$

Back substitution gives the solution $x = 1$, $y = -2$, $z = 1$.

12 Rank of a matrix I

a) We find an echelon form of the matrix:

$$\begin{pmatrix} 1 & 2 \\ 8 & 16 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

We see that the rank of A is 1 since there is one pivot position.

b) We find an echelon form of the matrix:

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 4 \\ 0 & -6 & -7 \end{pmatrix}$$

We see that the rank of A is 2 since there are two pivot positions.

c) We find an echelon form of the matrix:

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that the rank of A is 2 since there are two pivot positions.

13 Rank of a matrix II

a) We find an echelon form of the matrix:

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & -4 & 2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

We see that the rank of A is 3 by counting pivot positions.
b) We find an echelon form of the matrix:

$$\begin{pmatrix} 2 & 1 & 3 & 7 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 11 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 & 3 & 7 \\ 0 & 4.5 & 4.5 & 4.5 \\ 0 & 0.5 & 0.5 & 0.5 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 & 3 & 7 \\ 0 & 4.5 & 4.5 & 4.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that the rank of A is 2 by counting pivot positions.
c) We find an echelon form of the matrix:

$$\begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ -2 & -5 & -2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 5 & 3 & 0 \\ 0 & -1 & -2 & -2 \\ 0 & -9 & -4 & 2 \end{pmatrix}$$

We interchange the two middle rows to get easier computations:

$$\begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 5 & 3 & 0 \\ 0 & -9 & -4 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & -7 & -10 \\ 0 & 0 & 14 & 20 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & -7 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that the rank of A is 3 by counting pivot positions. T

14 Homogeneous linear systems

Let A be the 4×6 coefficient matrix of the homogeneous linear system. Then $n = 6$ (there are 6 variables) while $\operatorname{rk} A \leq 4$ (there cannot be more than one pivot position in each row). So there are at least two degrees of freedom, and the system has non-trivial solutions.

15 Linear system with parameters

We find the coefficient matrix A and the augmented matrix $\hat{A}$ of the system:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & -2 & p \end{pmatrix}, \quad \hat{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 2 & -3 & 2 & 4q \\ 3 & -2 & p & q \end{array} \right)$$

Then we compute an echelon form of $\hat{A}$ (which contains an echelon form of A as the first three columns):

$$\hat{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 2 & -3 & 2 & 4q \\ 3 & -2 & p & q \end{array} \right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & 0 & p-3 & -5q \end{array} \right)$$

By counting pivot positions, we see that the ranks are given by

$$\operatorname{rk} A = \begin{cases} 3 & p \neq 3 \\ 2 & p = 3 \end{cases} \qquad \operatorname{rk} \hat{A} = \begin{cases} 3 & p \neq 3 \text{ or } q \neq 0 \\ 2 & p = 3 \text{ and } q = 0 \end{cases}$$

The linear system has one solution if $p \neq 3$, no solutions if $p = 3$ and $q \neq 0$, and infinitely many solutions (one degree of freedom) if $p = 3$ and $q = 0$.

16 Midterm Exam in GRA6035 24/09/2010, Problem 3

We compute an echelon form of A using elementary row operations, and get

$$A = \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & -2 & 4 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & 7 \\ 0 & 0 & 0 & 0 & -6 \end{pmatrix}$$

Hence A has rank 3.

17 Mock Midterm Exam in GRA6035 09/2010, Problem 3

We compute an echelon form of A using elementary row operations, and get

$$A = \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence A has rank 3.

18 Midterm Exam in GRA6035 24/05/2011, Problem 3

We compute an echelon form of A using elementary row operations, and get

$$A = \begin{pmatrix} 2 & 10 & 6 & 8 \\ 1 & 5 & 4 & 11 \\ 3 & 15 & 7 & -2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 5 & 4 & 11 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence A has rank 2.