BI NORWEGIAN BUSINESS SCHOOL

Department of Economics

Solutions:	**GRA 60352 Mathematics**
Examination date:	30.09.2011, 14:00 – 15:00
Permitted examination aids:	Bilingual dictionary BI-approved exam calculator: Texas Instruments BA II Plus™
Answer sheets:	Answer sheet for multiple choice examinations
Total number of pages:	2

Correct answers: B-B-C-D-C-A-D-A

Question 1.

We reduce the augmented matrix to echelon form:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -4 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right) \dashrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & 0 & 5 \end{array}\right) \dashrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -1 & 1 & 1 & 3 \\ 0 & 0 & 2 & 0 & 5 \\ 0 & 0 & 0 & 2 & -1 \end{array}\right)$$

From the pivot positions, we see that the system is consistent with a unique solution, and the correct answer is alternative **B**.

Question 2.

We form the matrix A with the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as columns, and reduce A to an echelon form:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 16 \\ 7 & 3 & 32 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & 7 \\ 0 & -11 & 11 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

From the pivot positions, we see that $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent and that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$. We see this since the solutions of the vector equation

$$x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{0}$$

is given by $x + 2y + 3z = 0$ and $-7y + 7z = 0$ with z as a free variable, or $x = -5z$, $y = z$, z is free. So $z = -1$ gives

$$5\mathbf{v}_1 - 1\mathbf{v}_2 - 1\mathbf{v}_3 = \mathbf{0} \quad \Rightarrow \quad \mathbf{v}_3 = 5\mathbf{v}_1 - \mathbf{v}_2$$

Hence the correct answer is alternative **B**.

Question 3.

We reduce the matrix A to an echelon form:

$$A = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 2 & 1 & -1 & 2 \\ 7 & 8 & -8 & h \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & h-7 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & h-7 \end{pmatrix}$$

We see that the rank of A is two if $h = 7$ and three if $h \neq 7$, and the correct answer is alternative **C**.

Question 4.

The characteristic equation of A is $\lambda^2 - 10\lambda + 21 = 0$, and therefore the eigenvalues are $\lambda = 3$ and $\lambda = 7$. The correct answer is alternative **D**.

Question 5.

The eigenvalues of A are $\lambda = 1$ (with multiplicity two) and $\lambda = 2$, since we have

$$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & h & h^2 \\ 0 & 1-\lambda & h+4 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (1-\lambda)^2(2-\lambda) = 0$$

We compute the eigenvectors of $\lambda = 1$, the eigenvalue of multiplicity 2, by reducing the matrix $A - I$ to the reduced echelon form:

$$\begin{pmatrix} 0 & h & h^2 \\ 0 & 0 & h+4 \\ 0 & 0 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & h & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that there is one degree of freedom if $h \neq 0$, and two degrees of freedom if $h = 0$. Therefore, A is diagonalizable if and only if $h = 0$, and the correct answer is alternative **C**.

Question 6.

The symmetric matrix of the quadratic form $Q(x_1, x_2) = 3x_1^2 - 24x_1x_2 + 48x_2^2$ is

$$A = \begin{pmatrix} 3 & -12 \\ -12 & 48 \end{pmatrix}$$

The principal minors are $\Delta_1 = 3, 48$ and $\Delta_2 = 0$. Therefore A is positive semidefinite but not positive definite, and the correct answer is alternative **A**.

Question 7.

We compute the Hessian matrix of $f(x_1, x_2, x_3) = x_1x_2x_3$: First, we compute the first order partial derivatives

$$f_1' = x_2x_3, \; f_2' = x_1x_3, \; f_3' = x_1x_2$$

and then we compute the second order partial derivatives and form the Hessian matrix

$$H(f) = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}$$

The first principal minors are $\Delta_1 = 0, 0, 0$ and $\Delta_2 = -x_3^2, -x_2^2, -x_1^2$. It is not true that $-x_3^2, -x_2^2, -x_1^2 \geq 0$ for all points (x_1, x_2, x_3), and therefore f is neither convex nor concave. The correct answer is alternative **D**.

Question 8.

The set $S = \{(x, y) : 3x^2 - 12xy + 48y^2 \leq 12\}$ of $\mathbb{R}^2$ is shown as the shaded region in the figure. We see that it is closed, since the boundary is the points that satisfy the equation $3x^2 - 12xy + 48y^2 \leq 12$, and these boundary points are included in the set. We also see that the set is bounded and convex. The correct answer is alternative **A**.