LECTURE 9

GRA 6035

MATHEMATICS

EIVIND ERIKSEN

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REVIEW: Envelope thms

Case A: Unconstrained optimization

$f(\underline{x};a) = f(x_1,\dots,x_n;a)$ $\rightsquigarrow$ Problem: $\boxed{\max/\min_{\underline{x}} f(\underline{x};a)}$

(a: parameter)

Solution: $\underline{x}^*(a) = x_1^*(a), x_2^*(a), \dots, x_n^*(a)$

(depends on a)

max/min value: $f^*(a) = f(\underline{x}^*(a))$

= optimal value function

Envelope thm:

$$\frac{d}{da} f^*(a) = \left.\frac{\partial f(\underline{x};a)}{\partial a}\right|_{\underline{x}=\underline{x}^*(a)}$$

Case B: Constrained optimization – Lagrange problem

Problem: max/min $f(\underline{x};a)$ subject to $\begin{cases} g_1(\underline{x};a)=0 \\ g_2(\underline{x};a)=0 \\ \vdots \\ g_m(\underline{x};a)=0 \end{cases}$

Solution: $\underline{x}^*(a) = (x_1^*(a),\dots,x_n^*(a))$ with

Lagrange multipliers $\underline{\lambda}^*(a) = (\lambda_1^*(a),\dots,\lambda_m^*(a))$

Optimal value function: $f^*(a) = f(\underline{x}^*(a))$ (max/min value)

Envelope thm:

$$\frac{d}{da} f^*(a) = \left.\frac{\partial L(\underline{x},\underline{\lambda};a)}{\partial a}\right|_{\underline{x}=\underline{x}^*(a),\ \underline{\lambda}^*=\underline{\lambda}^*(a)}$$

Plan:

1. Bordered Hessians
2. Kuhn - Tucker problems

[FMEA] 3.4–3.6
(not Thm 3.6.2 and quasi-concave progr.)

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1. Bordered Hessians

Lagrange problem:

$$\max/\min \quad f(\underline{x}) = f(x_1, \dots, x_n) \quad \text{subject to} \quad \begin{cases} g_1(\underline{x}) = b_1 \\ \vdots \\ g_m(\underline{x}) = b_m \end{cases}$$

Write $\mathcal{L} = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \lambda_2 g_2(\underline{x}) - \dots - \lambda_m g_m(\underline{x})$
and solve Lagrange conditions:

$$\text{FOC: } \boxed{\begin{matrix} \mathcal{L}'_{x_1} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{matrix}} \quad + \quad \text{C: } \boxed{\begin{matrix} g_1(\underline{x}) = b_1 \\ \vdots \\ g_m(\underline{x}) = b_m \end{matrix}}$$

The solutions are candidates for max/min.

To solve problem
(find global max/min = maximal/minimal value among all admissible pts)

Method I: For cand. $(\underline{x}^*; \underline{\lambda}^*)$
Check if $\mathcal{L}(\underline{x}; \underline{\lambda}^*)$ is:

concave $\Rightarrow \underline{x}^*$ max
convex $\Rightarrow \underline{x}^*$ min

Method 2:
- check that there is a solution (for instance if constraint set is bounded)
- find pts that are admissible where NDCQ fails
- compare points and their values

To check if a specific solution (to Lagrange cond.) is local max/min:

Use bordered Hessian matrix

$$B = \left(\begin{array}{c|c} 0 & - \\ \hline | & \mathcal{L}'' \end{array}\right)$$

Check sign of certain leading principal minors at B

Bordered Hessian:

$$\max/\min \ f(\underline{x}) \quad \text{s.t.} \quad \begin{cases} g_1(\underline{x}) = b_1 \\ \vdots \\ g_m(\underline{x}) = b_m \end{cases}$$

General Lagrange problem.

$$\mathcal{L}(\underline{x}; \underline{\lambda}) = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \dots - \lambda_m g_m(\underline{x})$$

Point: $(\underline{x}^*, \underline{\lambda}^*)$ that satisfies $\boxed{\text{FOC}} + \boxed{\text{C}}$

Use Bordered Hessian to check if $\underline{x}^*$ is local max/min

Bordered Hessian matrix:

$$B = \left(\begin{array}{c|c} 0 & C \\ \hline C^T & \mathcal{L}''(\underline{x}; \underline{\lambda}^*) \end{array} \right), \text{ where}$$

$$C = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}$$

same matrix used in NDCQ

and

Hessian matrix of $\mathcal{L}(\underline{x}; \underline{\lambda}^*)$

$$\mathcal{L}''(\underline{x}; \underline{\lambda}^*) = \begin{pmatrix} \mathcal{L}''_{x_1 x_1}(\underline{x}; \underline{\lambda}^*) & \cdots & \mathcal{L}''_{x_1 x_n}(\underline{x}; \underline{\lambda}^*) \\ \vdots & & \\ \mathcal{L}''_{x_1 x_n}(\underline{x}; \underline{\lambda}^*) & \cdots & \mathcal{L}''_{x_n x_n}(\underline{x}; \underline{\lambda}^*) \end{pmatrix}$$

In general, B is an $(m+n) \times (m+n)$-matrix.

n = # variables
m = # constraints

Compute the last $\underline{n-m}$ leading principal minors of B.

<u>Result:</u> Compute the last $n-m$ leading principal minors of B.

D_i's have alternating signs, and the last D_{n+m} has the same sign as $(-1)^n$
$\Rightarrow$ $\underline{x}^*$ is a <u>local max</u>

D_i's have the same sign, equal to the sign of $(-1)^m$
$\Rightarrow$ $\underline{x}^*$ is a <u>local min</u>

<u>Special case:</u> $n-m=1$

$|B| = D_{n+m}$ same sign as $(-1)^n$ $\quad$ $\underline{x}^*$ local max

$|B| = D_{n+m}$ same sign as $(-1)^m$ $\quad$ $\underline{x}^*$ local min

<u>Special case:</u> $n-m=2$

$|B| = D_{n+m}$: sign $(-1)^n$
D_{n+m-1} : sign $(-1)^{n+1}$
$\quad$ $\underline{x}^*$ local max

$|B| = D_{n+m}$: sign $(-1)^m$
D_{n+m-1} : sign $(-1)^m$
$\quad$ $\underline{x}^*$ local min

Alternative description (used in [FMEA])

$$(-1)^m \cdot D_i > 0 \quad \text{for } i = m+(m+1), m+(m+2), \dots, m+n \Rightarrow \underline{x}^* \ \underline{\text{local min}}$$

$$(-1)^{m+i} D_i > 0 \quad \text{for } i = m+(m+1), m+(m+2), \dots, m+n \Rightarrow \underline{x}^* \ \underline{\text{local max}}$$

This means:

Sign of D_i = Sign of $(-1)^m$ $\quad i = 2m+1, \dots, m+n \Rightarrow$ local min

Sign of D_i = Sign of $(-1)^{m+i}$ $\quad i = 2m+1, \dots, m+n \Rightarrow$ local max

Ex: $n=3, m=1$

$D_3 < 0, D_4 < 0 \Rightarrow$ local min

$D_3 > 0, D_4 < 0 \Rightarrow$ local max

Ex: max/min $x+3y$ subj. to $x^2+y^2=10$

$$\mathcal{L} = x+3y - \lambda(x^2+y^2)$$

$$\begin{array}{ll} \text{FOC} & \left\{ \begin{array}{l} \mathcal{L}'_x = 1-\lambda\cdot 2x = 0 \\ \mathcal{L}'_y = 3-\lambda\cdot 2y = 0 \end{array} \right. \\ \text{C} & \quad x^2+y^2 = 10 \end{array} \xrightarrow[\substack{\text{Lagrange} \\ \text{Conditions} \\ \text{(see Lect. 8)}}]{\text{solve}} (x,y;\lambda) = \begin{array}{l} \underline{(1,3;\tfrac{1}{2})} \\ \underline{(-1,-3;-\tfrac{1}{2})} \end{array}$$

$f(1,3) = \underline{10}$

$f(-1,-3) = \underline{-10}$

Is $(x,y;\lambda) = (1,3;1/2)$ a local max?

$$B = \begin{pmatrix} 0 & 2x & 2y \\ 2x & -1 & 0 \\ 2y & 0 & -1 \end{pmatrix}_{x=1,y=3} \leftarrow \left\{ \begin{array}{l} C = (2x \ \ 2y) \\ \mathcal{L}'' = \begin{pmatrix} -2\lambda & 0 \\ 0 & -2\lambda \end{pmatrix}_{\lambda=1/2} \\ \quad = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array} \right.$$

$$\boxed{B = \begin{pmatrix} 0 & 2 & 6 \\ 2 & -1 & 0 \\ 6 & 0 & -1 \end{pmatrix}}$$

$\underline{n-m = 2-1 = 1}$: 1 last leading principal minor $\}$ $D_3 = |B|$

$$|B| = \begin{vmatrix} 0 & 2 & 6 \\ 2 & -1 & 0 \\ 6 & 0 & -1 \end{vmatrix} = -2\cdot(-2) + 6\cdot(6) = \underline{40} \text{ (positive)}$$

$(-1)^n = (-1)^2 = +1$

$(-1)^m = (-1)^1 = -1$

$$\underbrace{|B| = 40 \quad (-1)^n = +1 \text{ (positive)}}$$

$$\Downarrow$$

$(1,3)$ is a $\underline{\text{local max}}$

Extra example:

max/min $x^2y^2z^2$ subj. to $x^2+y^2+z^2=3$

$\mathcal{L} = x^2y^2z - \lambda(x^2+y^2+z^2)$

FOC: $\boxed{\begin{matrix} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \\ \mathcal{L}'_z = 0 \end{matrix}}$ C: $\boxed{x^2+y^2+z^2=3}$ $\Big\}$ One solution is $(x,y,z;\lambda) = (1,1,1;1)$

Is this local max or local min?

$$B = \left(\begin{array}{c|ccc} 0 & 2x & 2y & 2z \\ \hline 2x & 2y^2z^2-2\lambda & 4xyz^2 & 4xzy^2 \\ 2y & 4xyz^2 & 2x^2z^2-2\lambda & 4yzx^2 \\ 2z & 4xzy^2 & 4yzx^2 & 2x^2y^2-2\lambda \end{array}\right)_{(x,y,z;\lambda)=(1,1,1;1)} = \left(\begin{array}{ccc|c} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ \hline 2 & 4 & 4 & 0 \end{array}\right)$$

$n-m = 3-1 = 2$: Compute D_3, D_4

$$D_3 = \begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 4 \\ 2 & 4 & 0 \end{vmatrix} = -2(-8) + 2\cdot 8 = \underline{32} > 0$$

$$D_4 = \begin{vmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ 2 & 4 & 4 & 0 \end{vmatrix} = (-2)\cdot\begin{vmatrix} 2 & 2 & 2 \\ 0 & 4 & 4 \\ 4 & 0 & 4 \end{vmatrix} + 4\cdot\begin{vmatrix} 0 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 0 & 4 \end{vmatrix} + (-4)\cdot\begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 4 \\ 2 & 4 & 4 \end{vmatrix}$$

$$= (-2)\cdot(2\cdot 16 + 4\cdot 0) + 4\big((-2)\cdot 8 + 2\cdot 0\big) + (-4)\cdot\big((-2)\cdot 0 + 2\cdot 8\big)$$

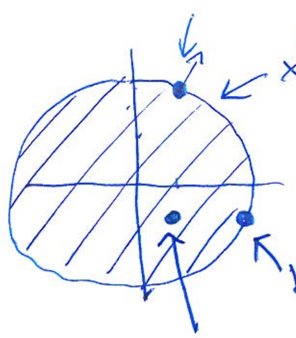
$$= -4\cdot 16 + 4\cdot(-16) + (-4)\cdot 16 = -3\cdot 4\cdot 16 = -3\cdot 64 = \underline{-192} < 0$$

$\left.\begin{matrix} D_3 > 0 \\ D_4 < 0 \end{matrix}\right\}$ Alt. signs

Last sign $= (-1)^n = (-1)^3 = -1$ ok $\Longrightarrow$ $(x,y,z) = (1,1,1)$ is local max

2 Kuhn-Tucker problems

Ex: $\max\ x+3y$ subj. to $x^2+y^2 \leq 10$

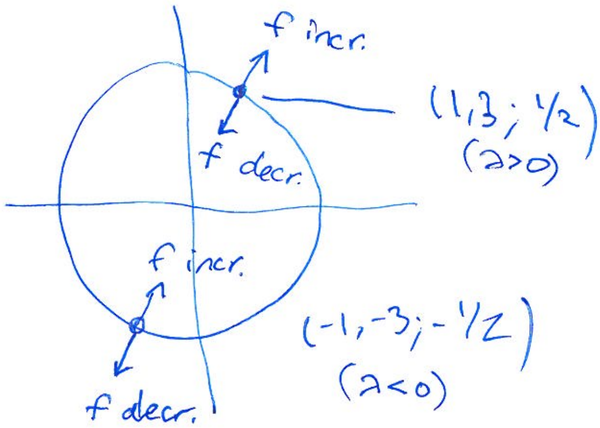


Admissible pts:

Pts that satisfy the constraints.

$x^2+y^2=10$ or $x^2+y^2<10$

(constraint is binding) (constraint is non-binding)



Interpretation of λ:

For a point on the circle to solve the max problem, we must have

$$\boxed{\lambda \geq 0}$$

$x^2+y^2=b$ (with $b=10$):

$\lambda>0$ means $\frac{d}{db}f^*(b)=\lambda>0$

($f^*(b)$ increases when b increases)

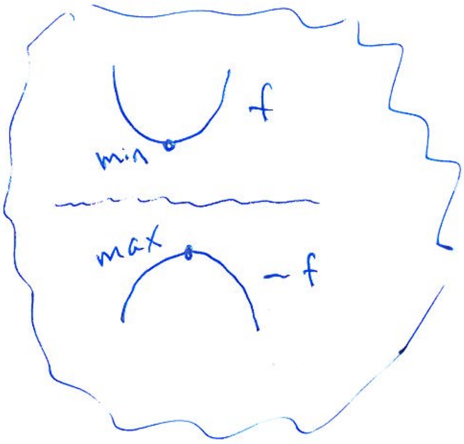
Kuhn-Tucker problem in standard form:

$$\boxed{\max f(\underline{x}) \quad \text{subject to} \quad \begin{cases} g_1(\underline{x}) \leq b_1 \\ g_2(\underline{x}) \leq b_2 \\ \vdots \\ g_m(\underline{x}) \leq b_m \end{cases}}$$

Ex: $\min\ x+3y$ s.t. $x^2+y^2 \leq 10$ (Kuhn-Tucker pbl.)

$$=$$

$$\boxed{\max\ -(x+3y) \quad \text{s.t.} \quad x^2+y^2 \leq 10}$$ Std. form.



If the constraint is

$$x+y+z \geq 1 \quad | \cdot (-1)$$

$$\Updownarrow$$

$$-(x+y+z) \leq -1$$

Method: $\left\{ \begin{array}{l} \underline{\text{When problem is in standard form:}} \\ \mathcal{L} = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \lambda_2 g_2(\underline{x}) - \dots - \lambda_m g_m(\underline{x}) \end{array} \right.$

Kuhn-Tucker conditions: (CSC = complementary slackness cond.)

FOC: $\boxed{\begin{array}{l} \mathcal{L}'_{x_1} = 0 \\ \mathcal{L}'_{x_2} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{array}}$

C: $\boxed{\begin{array}{l} g_1(\underline{x}) \leq b_1 \\ g_2(\underline{x}) \leq b_2 \\ \vdots \\ g_m(\underline{x}) \leq b_m \end{array}}$

CSC: $\lambda_i \geq 0$ and if $g_i(\underline{x}) < b_i$ then $\lambda_i = 0$ (for all i)

Solve KT-conditions and get candidates for max.

Ex: $\max\ x+3y$ s.t. $x^2+y^2 \le 10$ (std)

$$\mathcal{L} = x+3y-\lambda(x^2+y^2)$$

FOC: $\boxed{\begin{aligned} \mathcal{L}'_x &= 1-\lambda\cdot 2x = 0 \\ \mathcal{L}'_y &= 3-\lambda\cdot 2y = 0 \end{aligned}}$

C: $\boxed{x^2+y^2 \le 10} \iff \boxed{\begin{aligned} &a)\ x^2+y^2 = 10 \\ &\quad \text{or} \\ &b)\ x^2+y^2 < 10 \end{aligned}}$

CSC: $\boxed{\begin{aligned} &\lambda \ge 0 \\ &\text{and} \\ &\text{if } x^2+y^2<10 \text{ then } \lambda = 0 \end{aligned}} \iff \boxed{\begin{aligned} &a)\ \lambda \ge 0 \\ &b)\ \lambda = 0 \end{aligned}}$

Case a):

$$\left.\begin{aligned} x^2+y^2 &= 10 \\ 1-\lambda\cdot 2x &= 0 \\ 3-\lambda\cdot 2y &= 0 \\ \lambda &\ge 0 \end{aligned}\right\}$$

First three eqn's: (see Lect. 8)

$(x,y;\lambda) = (1,3;1/2)$

$\cancel{(-1,-3;-1/2)}$

does not satisfy $\lambda \ge 0$

One solution in case a): $(1,3;1/2)$

Case b):

$$\left.\begin{aligned} x^2+y^2 &< 10 \\ 1-\lambda\cdot 2x &= 0 \\ 3-\lambda\cdot 2y &= 0 \\ \lambda &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &1-0\cdot 2x = 0 \\ &\Downarrow \\ &1 = 0 \text{ not possible} \end{aligned}$$

no sol'n. in case b).

Solutions of KT conditions: $(1,3;1/2)$ $f=10$

(We gather solutions from all the different cases)

Given the list of solutions to the Kuhn–Tucker conditions, how do we find the (global) max?

Method I:

Compute $f(\underline{x})$ for each solution to Kuhn-Tucker cond. } Point with highest value of $f(\underline{x})$ is best candidate

$(\underline{x}^*; \lambda^*)$

Compute $\mathcal{L}(\underline{x}; \underline{\lambda}^*)$, a function of $\underline{x} = (x_1, \dots, x_n)$, and its Hessian.

Result:

$$\mathcal{L}(\underline{x}; \lambda^*) \text{ concave} \Longrightarrow \underline{x}^* \text{ is max}$$

Method 2: If Method I does not work, argue by elimination:

(i) Establish that there is a max

If $\{\underline{x} : g_1(\underline{x}) \le b_1, g_2(\underline{x}) \le b_2, \dots, g_m(\underline{x}) \le b_m\}$ is a bounded ~~bounded~~ set, then there is a max by extreme value thm.

(ii) Given that there is a max, the max must be attained at a point where either the Kuhn-Tucker conditions hold (ie one of the solutions computed earlier) or in an admissible point where NDCQ fails

NDCQ in Kuhn-Tucker case:

If $\underline{x}$ is admissible, then some constraints are binding and others are non-binding. Form the matrix C with rows

$$\begin{pmatrix} \frac{\partial g_i}{\partial x_1} & \frac{\partial g_i}{\partial x_2} & \cdots & \frac{\partial g_i}{\partial x_n} \end{pmatrix}$$

but only include rows corresponding to binding constraints.

Then NDCQ is given by

$$\boxed{\text{rk } C = \# \text{ rows in } C}$$

The NDCQ in the Kuhn–Tucker case is easiest to write down and check case by case (according to which constraints are binding).

Conclusion:

If $\{\underline{x} : g_i(\underline{x}) \le b_i\}$ is bounded, or the KT has a max for some other reason, then:

* Form an extended list of candidates

$$\boxed{\text{Solutions of KT-conditions}} + \boxed{\text{Admissible pts where NDCQ fails}}$$

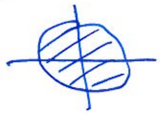
* Compute $f(\underline{x})$ for all pts in extended list. Highest value of $f(\underline{x})$ = maximum value.

Method I:

Ex: $\mathcal{L}(\underline{x}, \lambda^*) = x + 3y - \frac{1}{2}(x^2+y^2)$ Concave?

$$\mathcal{L}'' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \left.\begin{matrix} D_1 = -1 \\ D_2 = 1 \end{matrix}\right\} \quad \begin{matrix} \mathcal{L} \text{ concave} \\ \Downarrow \\ (1,3) \text{ max} \end{matrix}$$

Method 2: $\{(x,y) : x^2+y^2 \le 10\}$ is bounded $\longrightarrow$



NDCQ in KT case: $\begin{pmatrix} 2x & 2y \end{pmatrix}$

Case a): $\operatorname{rk}(2x \;\; 2y) = 1$

Case b): no condition

use only the rows from the matrix

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}$$

corresponding to binding constraints.

no. of rows in the matrix.

Admissible pts where NDCQ fails:

Case a) $\operatorname{rk}(2x \;\; 2y) < 1$

$\Rightarrow x = y = 0$

But $(x,y) = (0,0)$ does not satisfy $x^2+y^2=10$.

No pts.

b) No condition to check $\Rightarrow$ never fails

No pts.

Extended List: $\underbrace{(1,3;1/2)}_{\substack{\text{solution}\\ \text{of KT}\\ \text{conditions}}}$, $\underbrace{\text{none}}_{\substack{\text{adm. pts.}\\ \text{where NDCQ}\\ \text{fails}}}$

$f(1,3) = 10$

must be max