LECTURE 8

GRA 6035

MATHEMATICS

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PLAN:

① LAGRANGE PROBLEMS (REVIEW & CONTINATION)

② ENVELOPE THEOREMS

③ BORDERED HESSIANS ← No time for bordered Hessians today.

READING:

[FMEA] 3.1, 3.3 - 3.4

Midterm, Problem 7:

$$f(x,y,z) = 2x^2 + hy^3 + z^4$$

$\underline{h=1}$: $f = 2x^2 + y^3 + z^4$

$$f'' = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & 12z^2 \end{pmatrix}$$

$D_1 = 4$ ← positive

$D_2 = 24y$ ← can be positive or negative

$\Downarrow$

$y \geq 0$: f'' pos. semidefn.

$y < 0$: f'' ~~neg. semidefn.~~ indefinite (since $D_2 = 24y < 0$)

$\longrightarrow$ f not convex (not concave)

Similar for other values of $h \neq 0$.

Remember:

f convex $\iff$ $f''(x,y,z)$ pos. semidefn. for all (x,y,z)

1 LAGRANGE PROBLEMS

<u>Standard form:</u> $\underline{x} = (x_1, x_2, \ldots, x_n)$

max/min $f(\underline{x})$ subject to
$$\begin{cases} g_1(\underline{x}) = b_1 \\ g_2(\underline{x}) = b_2 \\ \vdots \qquad \vdots \\ g_m(\underline{x}) = b_m \end{cases}$$

Langrange function = Lagrangian : $\begin{cases} \underline{x} = x_1, x_2, \ldots, x_n \leftarrow \text{variables} \\ \underline{\lambda} = \lambda_1, \lambda_2, \ldots, \lambda_m \leftarrow \text{Lagrange multipliers} \end{cases}$

$$L(\underline{x}, \underline{\lambda}) = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \lambda_2 \cdot g_2(\underline{x}) - \ldots - \lambda_m g_m(\underline{x})$$

<u>Lagrange conditions:</u> $\begin{cases} m+n \text{ equations} \\ m+n \text{ unknowns} \end{cases}$

FOC = first order conditions
$$n\left\{\boxed{\begin{array}{c} \frac{\partial L}{\partial x_1} = 0 \\ \frac{\partial L}{\partial x_2} = 0 \\ \vdots \\ \frac{\partial L}{\partial x_n} = 0 \end{array}}\right.$$

Constraints = C
$$m\left\{\boxed{\begin{array}{c} g_1(\underline{x}) = b_1 \\ g_2(\underline{x}) = b_2 \\ \vdots \\ g_m(\underline{x}) = b_m \end{array}}\right.$$

<u>Method:</u> Write down and solve the <u>Lagrange conditions</u>. For any solution $\underline{(\underline{x}^*, \underline{\lambda}^*)}$, the point $\underline{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is a <u>candidate</u> for max/min.

<u>Tip:</u>
- Write down <u>all</u> eqn's before you start solving them
- solve the <u>easiest</u> eqn's first

Ex: max $f(x,y) = x+3y$ subj. to $x^2+y^2=10$

$n=2$
$m=1$

Lagrangian: $\mathcal{L}(x,y,\lambda) = \underbrace{x+3y}_{f} - \lambda\underbrace{(x^2+y^2)}_{g}$

$$\left(\begin{array}{ccc} g(x,y) & = & b \\ \shortparallel & & \shortparallel \\ x^2+y^2 & & 10 \end{array} \right)$$

In some texts, one prefers to write $x^2+y^2-10=0$ and
$\mathcal{L} = x+3y-\lambda(x^2+y^2-10)$
The result will be the same FOC's.

FOC:
$$\begin{cases} \frac{\partial L}{\partial x} = 1-\lambda\cdot 2x = 0 & (1) \\ \frac{\partial L}{\partial y} = 3-\lambda\cdot 2y = 0 & (2) \end{cases}$$

C: $x^2+y^2 = 10$ (3)

Solve: (1) $1-\lambda\cdot 2x = 0$

$1 = \lambda\cdot 2x$ ← $\lambda=0$ is no solution ok to divide by λ

$x = 1/2\lambda$

(2) $3-\lambda\cdot 2y=0$ ← $\lambda=0$ is not solution ok to divide by λ.

$3 = \lambda\cdot 2y$

$y = 3/2\lambda$

(3) $x^2+y^2=10$

$\left(\frac{1}{2\lambda}\right)^2+\left(\frac{3}{2\lambda}\right)^2=10 \quad | \cdot 4\lambda^2$

$\frac{1}{\cancel{4\lambda^2}}\cdot\cancel{4\lambda^2} + \frac{9}{\cancel{4\lambda^2}}\cdot\cancel{4\lambda^2} = 10\cdot 4\lambda^2$

$10 = 10\cdot 4\lambda^2$

$\lambda^2 = \frac{10}{40} = 1/4 \qquad \lambda = \pm 1/2$

Two solutions:

$\lambda = 1/2$: $x=1$, $y=3$ ⟶ $(x^*, y^*; \lambda^*) = (1, 3; 1/2)$ ← $f=10$

$\lambda = -1/2$: $x=-1$, $y=-3$ ⟶ $(x^*, y^*; \lambda^*) = (-1, -3; -1/2)$ ← $f=-10$

Given the list of solutions to the Lagrange conditions, how do find max/min?

Method 1: Using convexity / concavity.

In the Ex:

Best candidate for max: $(x^*, y^*; \lambda^*) = (1, 3; 1/2)$ (since $f=10$ is max.)

Compute the Hessian of $\mathcal{L}(\underline{x}, y; \lambda^*)$

$$\mathcal{L} = x + 3y - \lambda^* \cdot (x^2 + y^2)$$

We only consider $\mathcal{L}$ as a function of (x, y) and put in the special value of λ from the pt: $\lambda^* = \frac{1}{2}$

$$\boxed{\mathcal{L}(x, y; 1/2) = x + 3y - \frac{1}{2}(x^2 + y^2)}$$

$$\mathcal{L}'_x(x, y; 1/2) = 1 - \frac{1}{2} \cdot 2x = 1 - x$$

$$\mathcal{L}'_y(x, y; 1/2) = 3 - \frac{1}{2} \cdot 2y = 3 - y$$

$$\mathcal{L}''(x, y; 1/2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \left.\begin{array}{l} D_1 = -1 \\ D_2 = 1 \end{array}\right\} \quad \mathcal{L}(x, y; 1/2) \text{ is } \underline{\text{concave}}$$

Conclusion: $\mathcal{L}(x, y; 1/2)$ concave $\Rightarrow$ $(1, 3; 1/2)$ is max

(i.e. global max within constraint)

General result:

If $(\underline{x}^*, \underline{\lambda}^*) = (x_1^*, x_2^*, \ldots, x_n^*; \lambda_1^*, \ldots, \lambda_m^*)$ is a solution to the Lagrange conditions, then we have:

$\mathcal{L}(\underline{x}; \underline{\lambda}^*) = \mathcal{L}(x_1, \ldots, x_n; \lambda_1^*, \ldots, \lambda_m^*)$ concave $\Rightarrow$ $(\underline{x}^*; \underline{\lambda}^*)$ is max.

$\mathcal{L}(\underline{x}; \lambda^*)$ — " — convex $\Rightarrow$ $(\underline{x}^*; \underline{\lambda}^*)$ is min.

If $\mathcal{L}(\underline{x}; \lambda^*)$ is not convex/concave, this method is inconclusive.

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In[4]:= Plot3D[{x + 3 y, 0}, {x, -4, 4}, {y, -4, 4},
          RegionFunction → Function[{x, y, z}, x^2 + y^2 ≤ 10]]
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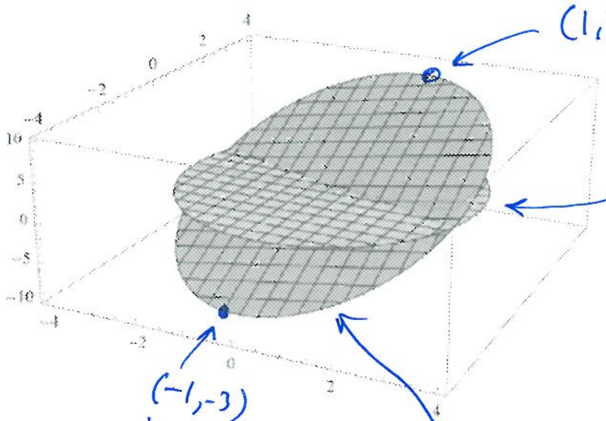
Out[4]=



(1,3) is max

the circle $x^2+y^2=10$

(−1,−3) is min.

the graph of $z = x+3y$ over (restricted to) the circle

Method 2: Using elimination

Thm:

If $\underline{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is a solution to the Lagrange problem (max/min) then we have:

> If $\underline{x}^*$ satisfy NDCQ condition, then there are $\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*$ such that $(\underline{x}^*, \underline{\lambda}^*)$ satisfy the Lagrange conditions.

① We must check that ~~the~~ Lagrange problem has a solution:

> **Extreme Value Thm:**
>
> If f is continuous, and defined on a closed and bounded set, then f has a max and a min.

This means: If the constraint-set is bounded, then there is max/min.

$$\{\underline{x} : g_1(\underline{x}) = b_1, \; g_2(\underline{x}) = b_2, \ldots, g_m(\underline{x}) = b_m\}$$

② We must check if there are points in the constraint set that doesnot satisfy NDCQ.

NDCQ: non-degenerate constraint qualification

> Points in the constraint set that do not satisfy NDCQ, can be max even if they do not satisfy first order conditions (FOC's)

NDCQ:
$$\mathrm{rk}\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix} = m$$

Ex: Constraint: $x^2+y^2=10$ $\quad g_1 = x^2+y^2$, $m=1$

NDCQ: $\boxed{\mathrm{rk}\begin{pmatrix} 2x & 2y \end{pmatrix} = 1}$

Check: If $(x,y) \neq (0,0)$, then $\mathrm{rk}=1$ (NDCQ satisfied)
If $(x,y)=(0,0)$, then $\mathrm{rk}=0$. This point does not satisfy $x^2+y^2=10$.

$\Downarrow$

Conclusion:

① There is a solution. $\leftarrow$ $\{(x,y) : x^2+y^2=10\}$ is bounded.

② Make an extended list:

All points satisfy Lagrange conditions $\leftarrow$ $\boxed{\begin{matrix}(1,3;\ 1/2) \\ (-1,-3;\ -1/2)\end{matrix}}$

+

All points that satisfy constraints, but do not satisfy NDCQ $\leftarrow$ $\boxed{\text{no pts.}}$

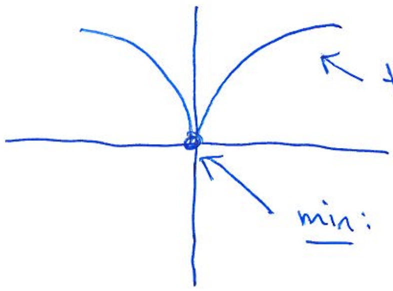
Max/min is the point in the extended list with the largest/smallest value of f.

$f(1,3)=10$
$f(-1,-3)=-10$
$(1,3)$ is max

Strange example:

$$\boxed{\min \; y \quad \text{subj. to} \quad x^2 - y^3 = 0}$$

Graphical: picture



← the curve $x^2 - y^3 = 0$

min: $y = 0$ at $(0,0)$

$y^3 = x^2$

$y = \sqrt[3]{x^2}$

See that $y \geq 0$ for all x (since $x^2 \geq 0$) and $(0,0)$ is on the graph. Hence $\boxed{\min \; y \text{ subj. to } x^2 - y^3 = 0}$ has solution $(0,0)$, where $\underline{y = 0}$.

But: If we compute FOC's + C:

$$\mathcal{L} = y - \lambda(x^2 - y^3)$$

FOC: $\begin{cases} \mathcal{L}'_x = -\lambda \cdot 2x = 0 \\ \mathcal{L}'_y = 1 - \lambda \cdot (-3y^2) = 0 \end{cases}$

C: $x^2 - y^3 = 0$

$$\boxed{\begin{array}{ll} (1) & -2\lambda x = 0 \\ (2) & 1 + 3\lambda y^2 = 0 \\ (3) & x^2 - y^3 = 0 \end{array}}$$

(1) $\lambda = 0$ or $x = 0$

If $\lambda = 0$, then (2) gives $1 = 0$, a contradiction. So $\underline{x = 0}$. Then (3) gives $\underline{y = 0}$. But again (2) gives $1 = 0$, a contradiction.

Concl: No solutions to Lagrange cond.

Explanation: NDCQ!

NDCQ: $\text{rk}\begin{pmatrix} 2x & -3y^2 \end{pmatrix} = 1$

NDCQ not satisfied when $x = 0, y = 0$, and this point satisfies constraints.

So: $(0,0)$ can be min even if it does not satisfy FOC's.

It is the min!

② Envelope thms

"Smooth dependence of parameters"

<u>Ex:</u>

$$\max_{x,y} \left(13x + qy - C(x,y)\right) = \pi(x,y) \qquad \left(\begin{matrix}\text{unconstrained}\\ \text{problem}\end{matrix}\right)$$

$$\pi(x,y) = 13x + qy - C(x,y),$$

$$\left(\text{where } C(x,y) = 0.04x^2 - 0.01xy + 0.01y^2 + 4x + 2y + 500\right)$$

q: parameter, the price of product #2.

<u>Solution:</u>

$$\pi'_x = 13 - 0.08x + 0.01y - 4 = 0$$

$$\pi'_y = q + 0.01x - 0.02y - 2 = 0$$

$$\boxed{\begin{aligned} x &= \frac{320}{3} + \frac{20}{3}\cdot q \\ y &= \frac{160}{3}q - \frac{140}{3} \end{aligned}} \leftarrow \underline{\text{stationary pt.}}$$

(one pt. for each value of q)

$$\pi'' = \begin{pmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{pmatrix}$$

$D_1 = -0.08 < 0$

$D_2 = 0.0016 - 0.0001 > 0$

neg. defn. $\Rightarrow \pi$ <u>concave</u>

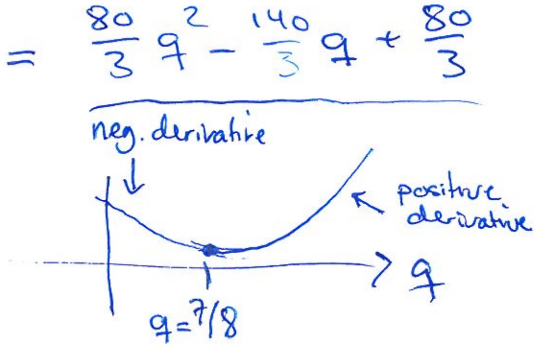
For any q, the pt. $(x^*(q), y^*(q))$ maximizes profit π

<u>Solution:</u> $$\left(x^*(q), y^*(q)\right) = \left(\frac{20}{3}q + \frac{320}{3},\ \frac{160}{3}q - \frac{140}{3}\right)$$

<u>Optimal value function:</u>

$$f^*(q) = f\left(x^*(q), y^*(q)\right) = \frac{80}{3}q^2 - \frac{140}{3}q + \frac{80}{3}$$

$$\frac{d}{dq} f^*(q) = \frac{160}{3}q - \frac{140}{3}$$



Envelope thm (Unconstrained version)

The optimal value function of the unconstrained optimization problem

$$\boxed{\max \ f(x_1, \dots, x_n; a)}$$

satisfy:

$$\frac{d}{da} f^*(a) = \left.\frac{\partial f}{\partial a}\right|_{x_1 = x_1^*(a),\, x_2 = x_2^*(a), \dots,\, x_n = x_n^*(a)}$$

$f^*(a) = f(x_1^*(a), x_2^*(a), \dots, x_n^*(a))$
(optimal value function)

In the example:

$$\max \ \underbrace{13x + qy - C(x,y)}_{= \ \pi(x,y;q)}$$

$$\frac{d\pi^*(q)}{dq} = y \Big|_{x = x^*(q),\, y = y^*(q)} = y^*(q) = \frac{160}{3}q - \frac{140}{3}$$

Envelope thm (Lagrange problem case)

$$\boxed{\text{max/min } f(\underline{x}) \text{ subj. to } \begin{cases} g_1(\underline{x}) = 0 \\ g_2(\underline{x}) = 0 \\ \vdots \\ g_m(\underline{x}) = 0 \end{cases}}$$

Lagrange problem

$\downarrow$

$$\boxed{\text{max/min } f(\underline{x}; a) \text{ subj. to } \begin{cases} g_1(\underline{x}; a) = 0 \\ g_2(\underline{x}; a) = 0 \\ \vdots \\ g_m(\underline{x}; a) = 0 \end{cases}}$$

Lagrange problem with parameter a.

Note: $g_i(\underline{x}; a) = 0$ (right side is zero)

Ex: $x^2 + y^2 = 10$

$x^2 + y^2 - 10 = 0$

Envelope thm:

$$\frac{d}{da} f^*(a) = \frac{\partial \mathcal{L}}{\partial a} \Big|_{x_1 = x_1^*(a), \ldots, x_n = x_n^*(a),\ \lambda_1 = \lambda_1^*(a), \ldots, \lambda_m = \lambda_m^*(a)}$$

Ex: $\boxed{\text{max } x + 3y \text{ subj. to } \cancel{x^2+y^2=b} \quad x^2 + y^2 - b = 0}$

$\mathcal{L} = x + 3y - \lambda \cdot (x^2 + y^2 - b)$

$$\frac{d}{db} f^*(b) = \lambda \Big|_{\lambda = \lambda^*(b)} = \lambda^*(b)$$

we can find $\lambda^*(b)$:

$$\left.\begin{array}{l} \mathcal{L}'_x = 1 - \lambda \cdot 2x = 0 \\ \mathcal{L}'_y = 3 - \lambda \cdot 2y = 0 \\ x^2 + y^2 - b = 0 \end{array}\right\} \quad \left.\begin{array}{l} x = \frac{1}{2\lambda},\ y = \frac{3}{2\lambda} \ \text{(as before)} \\ \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 - b = 0 \\ \frac{10}{4\lambda^2} = b \end{array}\right\} \Rightarrow \quad 10 = 4b\lambda^2$$

$b > 0$: $\lambda^2 = \frac{10}{4b}$

$\lambda = \pm\sqrt{\frac{10}{4b}}$

$b>0$ (continued):

$\lambda = \sqrt{\frac{10}{4b}} = \frac{1}{2}\sqrt{\frac{10}{b}}$: $x = \sqrt{\frac{b}{10}}$, $y = 3\sqrt{\frac{b}{10}}$, $\lambda = \frac{1}{2}\sqrt{\frac{10}{b}}$; $f = \sqrt{10b}$

$\lambda = -\frac{1}{2}\sqrt{\frac{10}{b}}$: $x = -\sqrt{\frac{b}{10}}$, $y = -3\sqrt{\frac{b}{10}}$, $\lambda = -\frac{1}{2}\sqrt{\frac{10}{b}}$; $f = -\sqrt{10b}$

Candidate for max: $(x^*(b), y^*(b); \lambda_i^*(b)) = \left(\sqrt{\frac{b}{10}}, 3\sqrt{\frac{b}{10}}; \frac{1}{2}\sqrt{10/b}\right)$

$\mathcal{L}(x,y;\lambda^*(b)) = x + 3y - \frac{1}{2}\sqrt{\frac{10}{b}}(x^2+y^2)$

$\mathcal{L}'' = \begin{pmatrix} -\sqrt{\frac{10}{b}} & 0 \\ 0 & -\sqrt{\frac{10}{b}} \end{pmatrix} \Rightarrow$ concave $\Rightarrow \left(\sqrt{\frac{b}{10}}, 3\sqrt{\frac{b}{10}}; \frac{1}{2}\sqrt{\frac{10}{b}}\right)$ is max

Optimal value for λ: $\lambda^*(b) = \frac{1}{2}\sqrt{\frac{10}{b}}$

Conclusion: (in case $b>0$)

$$\boxed{\frac{d}{db} f^*(b) = \lambda^*(b) = \frac{1}{2}\sqrt{\frac{10}{b}}}$$

(when $b=10$ this is $\frac{1}{2}$)

Interpretation of Lagrange multipliers

$$\lambda_i^* = \frac{d}{db_i} f^*(\underline{b})$$

λ_i^*: marginal rate of change in the optimal value function with respect to b_i, the constant in the constraint

$g_i(x_1, \dots, x_n) = b_i$