LECTURE 7

GRA 6035

EIVIND ERIKSEN   OCT 11TH 2012   MATHEMATICS

MIDTERM:

FRI OCT 12TH AT 14.00–15.00

MULTIPLE CHOICE – SEE PREVIOUS EXAMS

MATERIAL:
- LECTURE 1–6 (NOT LECTURE 7)
- PROBLEM SHEET 1–6 (NOT PROBL. SHEET 5, PB. 6–7)
- PROBLEM SESSION 1

NOT BORDERED HESSIANS

(BUT HESSIANS are of course relevant)

REVIEW LECTURE 6:

* Sets and topology: (see separate note on It's Learning)
  - determine when a subset D in $\mathbb{R}^n$ is open/closed, convex, bounded
  - to determine this, it is useful to be able to sketch the set D

* Hessians, convex/concave functions
  - Compute the Hessian matrix $H(f) = f''$ of a function $f(\underline{x})$ defined on a subset D in $\mathbb{R}^n$
  - determine if f is convex/concave

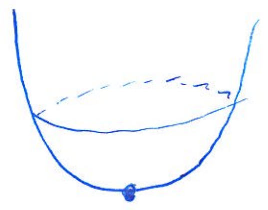
f convex $\iff$ $f''(\underline{x})$ positive semidefinite for all $\underline{x}$ in D

(f strictly convex $\Leftarrow$ $f''(\underline{x})$ positive definite — " —)

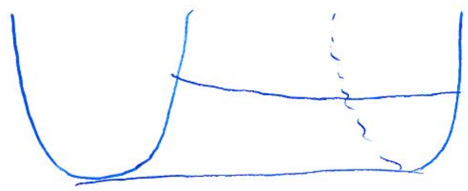
f concave $\iff$ $f''(\underline{x})$ negative semidefinite for all $\underline{x}$ in D

(f strictly concave $\Leftarrow$ $f''(\underline{x})$ negative definite — " —)

strictly convex



convex, not strictly convex



Ex: $f(x,y,z) = xyz$, $D = \mathbb{R}^3$.

i) Compute the Hessian.

$f'_x = yz$

$f'_y = xz$

$f'_z = xy$

$$\boxed{\begin{matrix} f''_{xx} = 0 & f''_{xy} = z & f''_{xz} = y \\ f''_{yx} = z & f''_{yy} = 0 & f''_{yz} = x \\ f''_{zx} = y & f''_{zy} = x & f''_{zz} = 0 \end{matrix}}$$

Hessian: $$f''(x,y,z) = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix} = H(f)(x,y,z)$$

ii)

	pos. semidefn.	neg. semidefn.
$D_1 = 0$	≥ 0	≤ 0
$D_2 = 0 - z^2 = -z^2$	≥ 0	≥ 0
	no.	no.
	not convex	not concave

not the case for all (x,y,z) in D

Plan: Lecture 7

① Extremal points

② Lagrange problems

[FMEA] 3.1 – 3.3
(except Envelope Thm)

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① Extremal points = max/min-points

Let $f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$ be a function in n variables, defined on a subset $D = D_f$ of $\mathbb{R}^n$.

Ex: $f(x,y,z) = xyz$,
$D = \mathbb{R}^3$
(all points (x,y,z))

$f(x,y,z) = z\sqrt{x^2+y^2-1}$,
$D = \{(x,y,z) : x^2+y^2 \geq 1\}$

Defn: Let $\underline{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ be a point in D.

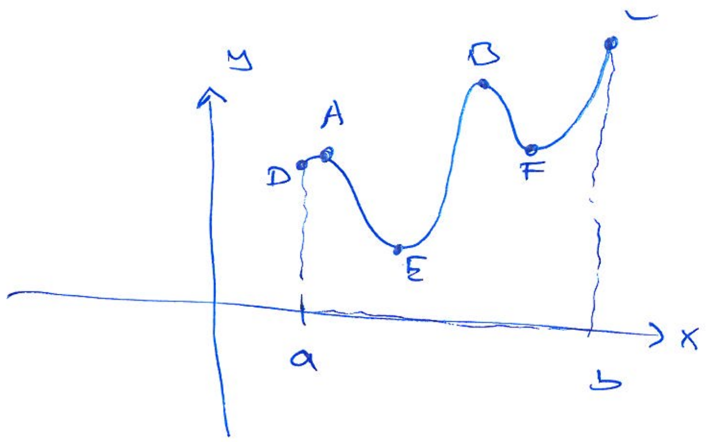
$\underline{x}^*$ is a maximum for f (global maximum) if
$$f(\underline{x}^*) \geq f(\underline{x}) \text{ for all } \underline{x} \text{ in } D$$

$\underline{x}^*$ is a minimum for f (global minimum) if
$$f(\underline{x}^*) \leq f(\underline{x}) \text{ for all } \underline{x} \text{ in } D$$

$\underline{x}^*$ is a local maximum for f if
$$f(\underline{x}^*) \geq f(\underline{x}) \text{ for all } \underline{x} \text{ in } D \text{ close to } \underline{x}^*$$

$\underline{x}^*$ is a local minimum for f if
$$f(\underline{x}^*) \leq f(\underline{x}) \text{ for all } \underline{x} \text{ in } D \text{ close to } \underline{x}^*$$

Ex:



$y = f(x)$, $x \in [a,b] = D$

A, B, C : local max
C : global max = max

D, E, F : local min
E : global min = min

Defn:

A <u>stationary point</u> for f is a point $\underline{x}^*$ in D_f such that

$$f'_{x_1}(\underline{x}^*) = 0, \; f'_{x_2}(\underline{x}^*) = 0, \; \dots, \; f'_{x_n}(\underline{x}^*) = 0$$

A <u>critical point</u> for f is a point $\underline{x}^*$ in D_f s.t.

either i) $\underline{x}^*$ is a stationary pt.
or ii) $f'_{x_1}(\underline{x}^*), f'_{x_2}(\underline{x}^*), \dots, f'_{x_n}(\underline{x}^*)$ is not defined

We only consider C^2 functions, so ii) cannot happen. Hence, in this course, $\boxed{\text{stationary pt} = \text{critical pt}}$

Ex: $f(x,y,z) = x^2 + y^2 - z^2 + 1$, $D_f = \mathbb{R}^3$

$$\left.\begin{aligned} f'_x &= 2x = 0 \\ f'_y &= 2y = 0 \\ f'_z &= -2z = 0 \end{aligned}\right\}$$

Stationary pts:

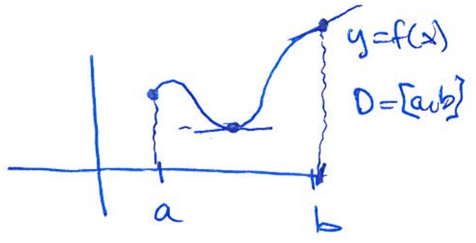
$x=0, y=0, z=0$

$(x,y,z) = (0,0,0)$

Thm If $\underline{x}^*$ is a local max/min for f, and if $\underline{x}^*$ is <u>not</u> a boundary point, then $\underline{x}^*$ is a stationary pt.

In other words, all local max/min are either

$$\boxed{\begin{array}{l}\text{i) Stationary pts} \\ \text{ii) Boundary pts.}\end{array}}$$



Ex: $f(x,y) = 2x + 4y \quad , D_f = \{(x,y) : x^2 + y^2 \leq 1\}$

i) <u>Stationary pts:</u>

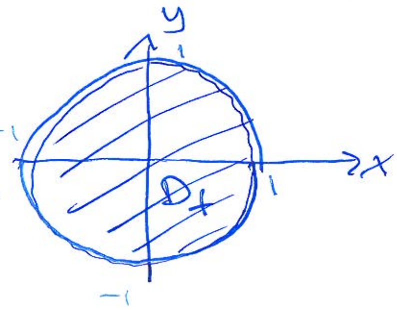
$f'_x = 2 = 0$

$f'_y = 4 = 0$

no sol'ns

$\Rightarrow$ no stationary pts.

$z = 2x + 4y$ gives 3-dim picture



ii) <u>Boundary pts</u>

Boundary of D_f :

$D_f = \{(x,y) : x^2 + y^2 \leq 1\}$

<u>Boundary:</u>

The circle $x^2 + y^2 = 1$

Convex / concave functions

If $f(\underline{x})$ is convex and $\underline{x}^*$ is a stationary pt, then $\underline{x}^*$ is a global min.

If $f(\underline{x})$ is concave and $\underline{x}^*$ is a stationary pt, then $\underline{x}^*$ is a global max.

Ex: $f(x,y) = 2x - y - x^2 + xy - y^2$, $D_f = \mathbb{R}^2$

$f'_x = 2 - 2x + y$ $\quad f''_{xx} = -2$ $\quad f''_{xy} = 1$

$f'_y = -1 + x - 2y$ $\quad f''_{yx} = 1$ $\quad f''_{yy} = -2$

$$f'' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

Convex/concave: $\left.\begin{array}{l} D_1 = -2 < 0 \\ D_2 = 3 > 0 \end{array}\right\}$ f'' is negative defn. for all $\underline{x}$

$\Downarrow$

f is concave

(strictly concave)

Stationary pts: $\begin{array}{l} f'_x = 0 \\ f'_y = 0 \end{array}$ $\quad \begin{array}{l} 2 - 2x + y = 0 \\ -1 + x - 2y = 0 \end{array}$

$$\begin{array}{r} -2x + y = -2 \\ x - 2y = 1 \end{array} \qquad \left(\begin{array}{cc|c} -2 & 1 & -2 \\ 1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & -3 & 0 \end{array}\right)$$

$\underline{x} = (1,0)$ is global max

$x = 1$
$y = 0$

$x - 2y = 1$
$-3y = 0$

$f(1,0) = 2 - 1 = 1$ max. value

Classification of stationary pts

Let $f(\underline{x})$ be a function defined on D_f, and let $\underline{x}^* = (x_1^*, x_2^*, \dots, x_n^*)$ be a stationary pt. of f.

Thm:

$H(f)(\underline{x}^*)$ pos. defn. $\Longrightarrow$ $\underline{x}^*$ local min.

$H(f)(\underline{x}^*)$ neg. defn. $\Longrightarrow$ $\underline{x}^*$ local max

$H(f)(\underline{x}^*)$ indefinite $\Longleftrightarrow$ $\underline{x}^*$ saddle pt (i.e not local max, not local min)

If $H(f)(\underline{x}^*)$ is i) pos. semidefn, but not pos defn.
ii) neg. semi-defn. but not neg. defn.

Then the test is inconclusive.

Ex: $f = x^2y - xy^2 - x + y$

Stationary pts:

$f'_x = 2xy - y^2 - 1 = 0$ $\qquad 2xy = 1 + y^2$

$f'_y = x^2 - 2xy + 1 = 0$ $\qquad x^2 - (\cancel{1} + y^2) + \cancel{1} = 0$

$x^2 - y^2 = 0 \qquad x^2 = y^2 \Rightarrow \boxed{\begin{matrix} x = y \\ \text{or} \\ x = -y \end{matrix}}$

i) $x = y$: $2x^2 = 1 + x^2$
$x^2 = 1 \qquad x = \pm 1$

$(x,y) = (1,1)$
$(-1,-1)$

stationary pts

ii) $x = -y$: $-2y^2 = 1 + y^2$
$-3y^2 = 1$ no soln

$(x,y) = (1,1)$:

$$f'' = \begin{pmatrix} 2y & 2x-2y \\ 2x-2y & -2x \end{pmatrix} \qquad f''(1,1) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$D_1 = 2$
$D_2 = -4$
indefinite

(1,1) is saddle pt

$(x,y) = (-1,-1)$:

$$f''(-1,-1) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

$\left.\begin{array}{l} D_1 = -2 \\ D_2 = -4 \\ \text{indefinite} \end{array}\right\}$ (−1,−1) is saddle pt.

Thm:

If f is continuous and D_f is closed and bounded, then f has a global max and a global min.

In this case, you know:

~~Alg~~ – there is a global max $\underline{x}^*$

– $\underline{x}^*$ is also local max

– hence $\underline{x}^*$ is a stationary pt that is local max or a boundary pt.

– Compute $f(\underline{x})$ for the list of possible global max and compare

② Lagrange Problems

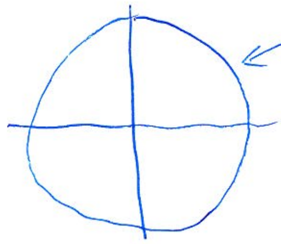
Std form:

$$\boxed{\max/\min \; f(\underline{x}) = f(x_1, x_2, \ldots, x_n) \quad \text{subject to} \quad \begin{cases} g_1(x_1 \ldots x_n) = b_1 \\ g_2(x_1 \ldots x_n) = b_2 \\ \vdots \\ g_m(x_1, \ldots, x_n) = b_m \end{cases}}$$

Ex: max $x + 3y$ subj. to. $x^2 + y^2 = 10$

$x + 3y = f(x,y)$ (objective fn.)

$x^2 + y^2 = g(x,y)$, $10 = b$



$x^2 + y^2 = 10$

$$\mathcal{L} = x + 3y - \lambda(x^2 + y^2)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 1 - \lambda \cdot 2x = 0 \\ \mathcal{L}'_y &= 3 - \lambda \cdot 2y = 0 \end{aligned}} \quad \text{FOC}$$

Lagrange conditions: $\mathcal{L} = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \lambda_2 \cdot g_2(\underline{x}) \ldots$

First order condition: (FOC)

$$\boxed{\begin{aligned} \mathcal{L}'_{x_1} &= 0 \\ \mathcal{L}'_{x_2} &= 0 \\ &\vdots \\ \mathcal{L}'_{x_n} &= 0 \end{aligned}}$$

Constraints:

$$\boxed{\begin{aligned} g_1(\underline{x}) &= b_1 \\ g_2(\underline{x}) &= b_2 \\ &\vdots \\ g_m(\underline{x}) &= b_m \end{aligned}}$$

} m+n equations in m+n variables

Thm

If $\underline{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is a solution to the Lagrange problem (max/min), then either i) there are Lagrange multipliers $\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*$ such that $(\underline{x}^*, \underline{\lambda}^*)$ satisfy the Lagrange conditions, or ii) $\underline{x}^*$ does not satisfy NDCQ.

Comment:

1) NDCQ is a technical condition (non-degenerate constraint qualification) that I will talk about next week. It is usually satisfied.

2) When we solve the Lagrange conditions, we get candidates for min/max.