LECTURE 6

GRA 6035

MATHEMATICS

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(*) Midterm: Fri OCT 12TH (NEXT WEEK) 14.00-15.00

MULTIPLE CHOICE – SEE PREVIOUS EXAMS

MATERIAL:
- LECTURE 1–6
- PROBLEM SHEET 1–6
- PROBLEMSESSION 1

BORDERED HESSIANS NOT FOR MIDTERM (SECT. 1.8)

REVIEW: LECTURE 5

A: symmetric $n \times n$-matrix

leading principal minors

A positive definite $\Leftrightarrow$ $D_1, D_2, D_3, \ldots, D_n$ positive

A negative –"– $\Leftrightarrow$ D_1 negative, D_2 positive, D_3 negative, ...

(or $(-1)^i \cdot D_i > 0$)

A positive semidefinite $\Leftrightarrow$ $\Delta_1, \Delta_2, \ldots, \Delta_n \geq 0$ for all principal minors

A negative –"– $\Leftrightarrow$ $\Delta_1 \leq 0$, $\Delta_2 \geq 0$, $\Delta_3 \leq 0, \ldots$ for all principal minors

(or $(-1)^i \Delta_i \geq 0$)

A indefinite $\Leftrightarrow$ All other cases

Ex: $Q(x,y,z) = 6xy + 12y^2 - 10xz - 6yz + 3z^2$

$$A = \begin{pmatrix} 0 & 3 & -5 \\ 3 & 12 & -3 \\ -5 & -3 & 3 \end{pmatrix}$$

$D_1 = 0$

$D_2 = -9$

$D_3 =$

$\Rightarrow$ $\left.\begin{matrix} Q \\ A \end{matrix}\right\}$ indefinit

$\Delta_1 = 0, 12, 3$

$\Delta_2 = -9, 27, -25$

PLAN:

1. SETS and Topology. CONVEX SETS. [FMEA] 2.2 - 2.3
2. Convex and CONCAVE FUNCTIONS
3. HESSIAN MATRICES AND CONVEXITY

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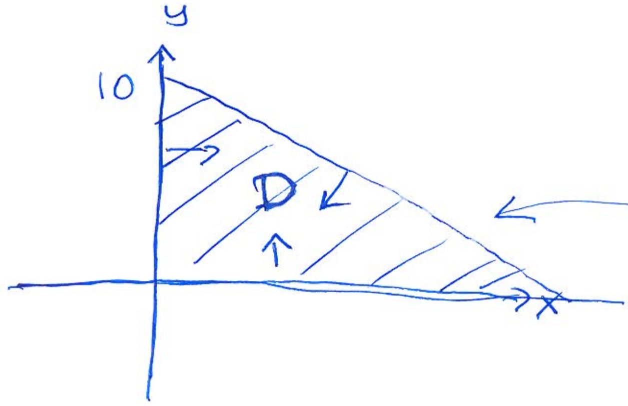
1. Sets. Topology. Convex Sets

Ex: max $f(x,y) = x^2 + 3y^2$ subject to $\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + 3y \leq 30 \end{cases}$



$2x + 3y = 30$

$3y = 30 - 2x$

$y = 10 - \frac{2}{3}x$

$D = D_f = \{ (x,y) : x \geq 0, y \geq 0, 2x + 3y \leq 30 \}$

domain of definition of f

all points (x,y) such that $x \geq 0, y \geq 0, 2x + 3y \leq 30$

Ex: max $1 - x^2 - y^2 = f(x,y)$

no constraints

$D = D_f = \{ (x,y) \} = \mathbb{R}^2$

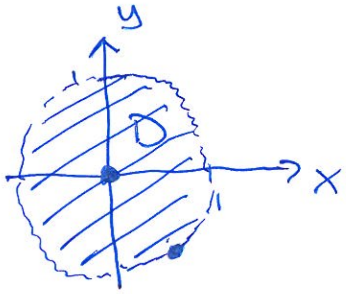
Notions for sets:

(D is a set of points in $\mathbb{R}^n$)

- open sets
- closed sets
- bounded sets
- convex sets

(boundary points)

Ex: $D = \{(x,y) : x^2+y^2 < 1\}$



$x^2+y^2=1$

circle with radius 1 and center (0,0)

Boundary points:

A boundary point for D is a point (x,y) such that there are points "very close to" (x,y) that are both included in D and outside of D.

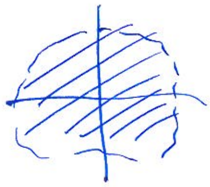
In this case, the boundary points are the points on the circle $x^2+y^2=1$.

Definition: A set is <u>open</u> if no boundary points are included in the set

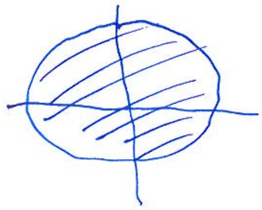
A set is <u>closed</u> if all boundary points are included in the set.

Typical examples:

$\{(x,y) : x^2+y^2 < 1\}$ is <u>open</u>



$\{(x,y) : x^2+y^2 \leq 1\}$ is closed

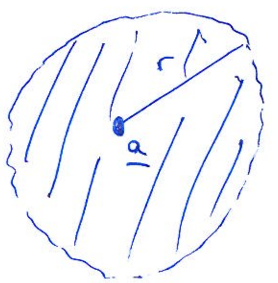


Definitions can be stated more precisely using <u>open balls</u>.

An <u>open ball</u> in $\mathbb{R}^n$ with center in $\underline{a} = (a_1, a_2, \dots, a_n)$ and radius $r > 0$ is given by

$$B(\underline{a}, r) = \{ (x_1, x_2, \dots, x_n) : \text{distance from } (x_1, \dots, x_n) \text{ to } (a_1, \dots, a_n) \text{ is less than } r \}$$

$$= \{ (x_1, \dots, x_n) : (x_1 - a_1)^2 + \dots + (x_n - a_n)^2 < r^2 \}$$

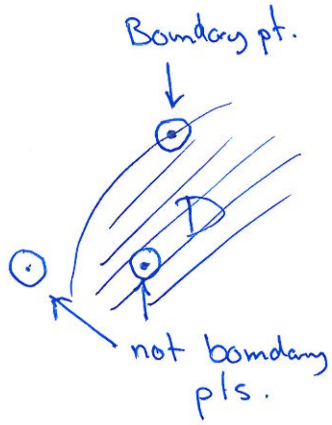


$B(\underline{a}, r)$

For $n=2$, $B(\underline{a}, r)$ is the disk bounded by a circle. 

For $n=3$, $B(\underline{a}, r)$ is the ball bounded by a sphere. 

<u>Defn:</u> Let D be a set in $\mathbb{R}^n$.



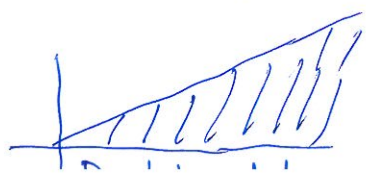
a) $\underline{x}$ is a <u>boundary point</u> of D if any ball $B(\underline{x}, r)$ with center $\underline{x}$ contains both points in D and points outside D.

b) D is <u>closed</u> if <u>all</u> boundary pts of D are in D

D is <u>open</u> if <u>no</u> boundary pts of D are in D

c) D is <u>bounded</u> if there is a ball $B(\underline{a}, r)$ (with a finite radius r) that contains all pts in D

D bounded

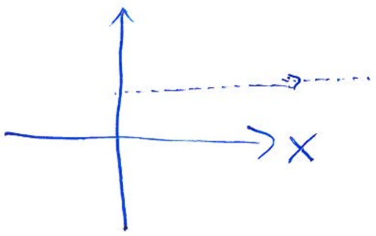


Thm:

If f is a <u>continuous</u> function defined on a set D that is <u>closed</u> and <u>bounded</u>, then f has a max and a min.

Ex:

① max $x = f(x,y)$ on $D = \mathbb{R}^2$



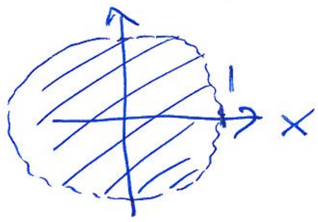
no solution

$f(x,y) \to \infty$

not bounded

② max $f(x,y) = x$ on $D = \{(x,y) : x^2+y^2 < 1\}$

$x^2+y^2<1$



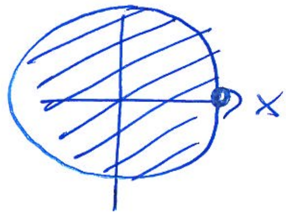
no solution

$f(x,y) \to 1$

not closed

③ max $f(x,y) = x$ on $D = \{(x,y) : x^2+y^2 \leq 1\}$

$x^2+y^2 \leq 1$



max $= 1$

closed bounded

Definition:

A set D is <u>bounded</u> if there is a ~~disk~~ ball that contains D.

A <u>ball</u> is a set $B(\underline{a}, r) = \{(x_1, x_2, \dots, x_n) : d(\underline{x}, \underline{a}) < r\}$.

Ex: $B((0,0),1) = \{(x,y) : x^2+y^2 < 1\}$

* Sets defined by $\leq$, $\geq$, $=$ are typically <u>closed</u> sets
  Sets defined by $<$, $>$ are typically <u>open</u>.

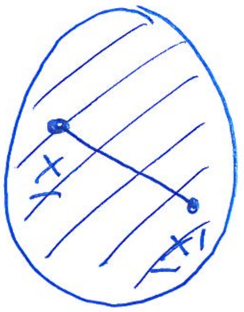
* Sets are <u>bounded</u> if it is enclosed by a circle / ~~ball~~ sphere with <u>finite</u> radius.

Convex sets

A set D is <u>convex</u> if the following condition holds:

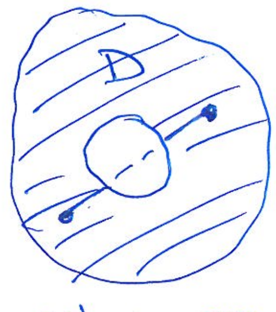
If $\underline{x}, \underline{x}'$ are points in D then the line segment $[\underline{x}, \underline{x}']$ is in D

<u>Typical convex set:</u>

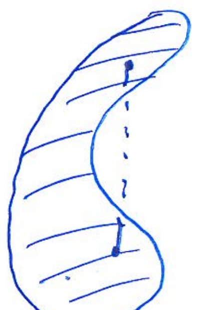


$$D = \{(x,y) : x^2 + y^2 \leq 1\}$$

<u>Typically non-convex sets</u>



not convex

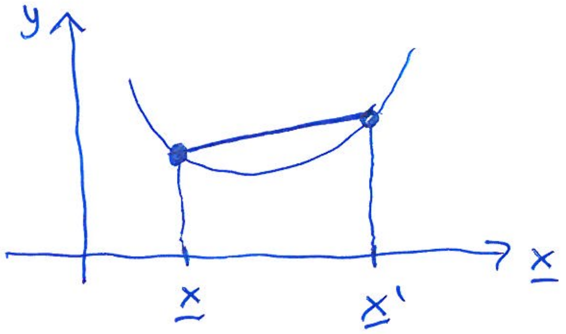


not convex

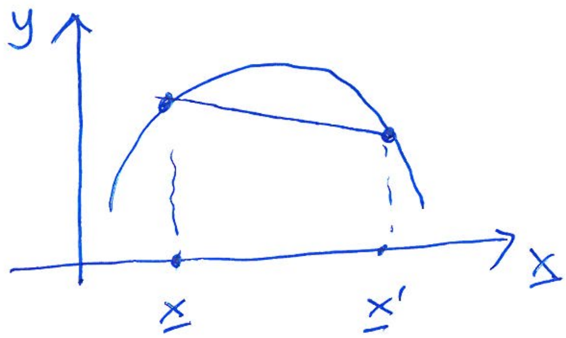
(2) Convex and concave functions

Let $f(\underline{x}) = f(x_1, x_2, \dots, x_n)$ be defined on a convex set D_f.

Convex function



Concave function



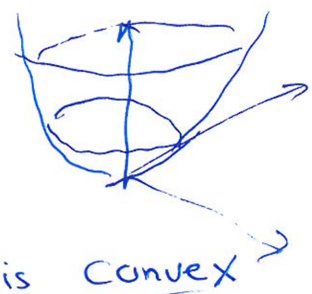
Defn:

f is **convex** if the following condition holds:

For any points $\underline{x}, \underline{x}'$ in D_f, the straight line segment from the point at $\underline{x}$ to the point at $\underline{x}'$ on the graph of f lies **on or over** the graph

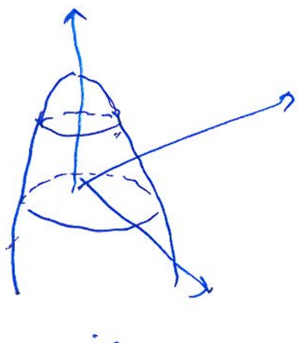
Concave → **on or under**

Ex: $f(x,y) = x^2 + y^2$ on $D_f = \mathbb{R}^2$



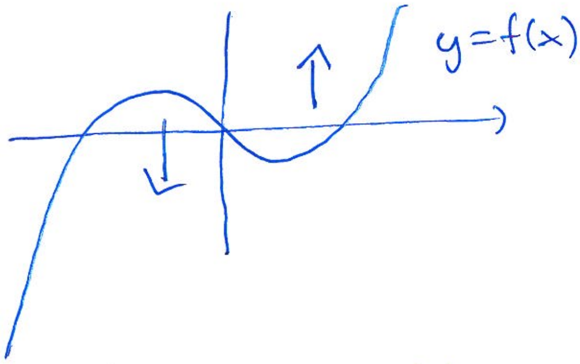
f is **convex** function

Ex: $g(x,y) = 1 - x^2 - y^2$



g is **concave**

Ex: $f(x) = x^3 - x$



neither convex nor concave

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(3) Hessian matrix and convexity

Ex: $f(x,y,z) = xy - z^2$

$f'_x = y$

$f'_y = x$

$f'_z = -2z$

$$\boxed{\begin{matrix} f''_{xx} = 0 & f''_{xy} = 1 & f''_{xz} = 0 \\ f''_{yx} = 1 & f''_{yy} = 0 & f''_{yz} = 0 \\ f''_{zx} = 0 & f''_{zy} = 0 & f''_{zz} = -2 \end{matrix}}$$

Hessian matrix:

$$H(f) = f'' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Ex: $f(x,y) = e^{x+y}$

$f'_x = e^{x+y} \cdot 1 = e^{x+y}$ $\quad f''_{xx} = e^{x+y} \cdot 1$ $\quad f''_{xy} = e^{x+y} \cdot 1$

$f'_y = e^{x+y} \cdot 1 = e^{x+y}$ $\quad f''_{yx} = e^{x+y} \cdot 1$ $\quad f''_{yy} = e^{x+y} \cdot 1$

$$H(f) = f'' = \begin{pmatrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{pmatrix}$$

A function $f(\underline{x})$ is called C^2 if all the second order partial derivatives of f exists and are continuous.

All functions that we consider are C^2.

<u>Result:</u>

If $f(\underline{x})$ is a C^2, then $H(f)$ is a <u>symmetric</u> matrix. That is,
$$\begin{cases} f''_{xy} = f''_{yx} \\ f''_{xz} = f''_{zx} \\ \vdots \end{cases}$$

<u>Result:</u> Let $f(x_1, \dots, x_n)$ be a function defined on an open and convex set D_f. Then:

f is <u>convex</u> $\iff$ $H(f)$ is positive semidefinite for all points (x,y) in D_f.

f is <u>concave</u> $\iff$ $H(f)$ is negative semidefinite for all points (x,y) in D_f.

<u>Ex:</u> $f(x,y,z) = xy - z^2$

$$H(f) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\left.\begin{aligned} D_1 &= 0 \\ D_2 &= -1 \end{aligned}\right\} \text{ indefinite}$$

f is <u>not convex</u>, <u>not concave</u>

<u>Ex:</u> $f(x,y) = e^{x+y}$

$$H(f) = \begin{pmatrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{pmatrix}$$

$D_1 = e^{x+y}$ $\quad$ $\Delta_1 = e^{x+y}, e^{x+y} \geq 0$

$D_2 = 0$ $\quad$ $\Delta_2 = 0$

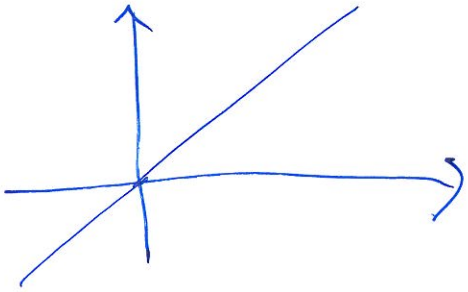
$H(f)$ positive semidefn. for all (x,y)

<u>f is convex</u> (but not concave)

If $f(x_1, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n + b$, then

$$H(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and f is both convex and concave. (This is the only type of fn. that is both convex and concave)



graph = straight line, plane, ...

Ex: $f(x,y) = \ln(x^2+y^2+1)$, $D_f = \mathbb{R}^2$

$$f'_x = \frac{1 \cdot 2x}{x^2+y^2+1} = \frac{2x}{x^2+y^2+1}$$

$$f'_y = \frac{1 \cdot 2y}{x^2+y^2+1} = \frac{2y}{x^2+y^2+1}$$

$$f''_{xx} = \left(\frac{2x}{x^2+y^2+1}\right)'_x = \frac{2\cdot(x^2+y^2+1) - 2x \cdot 2x}{(x^2+y^2+1)^2} = \frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2}$$

$$f''_{xy} = \left(\frac{2x}{x^2+y^2+1}\right)'_y = \frac{-2x \cdot 2y}{(x^2+y^2+1)^2} = \frac{-4xy}{(x^2+y^2+1)^2} = f''_{yx}$$

$$f''_{yy} = \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2}$$

$$H(f) = \begin{pmatrix} \frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2} & \frac{-4xy}{(x^2+y^2+1)^2} \\ \frac{-4xy}{(x^2+y^2+1)^2} & \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2} \end{pmatrix}$$

$$D_1 = \frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2} \begin{array}{l} \leftarrow \; x=2,\ y=0 \quad -6<0 \\ \; x=0,\ y=2 \quad 10>0 \\ \leftarrow \; \text{positive} \end{array}$$

Conclusion: f is not convex, not concave

Important note:

Convexity / concavity are global properties of the function f. For $f(\underline{x})$ to convex / concave, certain properties must satisfied for all points in D_f (that is, globally).

Ex: $Q(x,y,z) = 6xy + 12y^2 - 10xz - 6yz + 3z^2$
(quadratic form)

i) Symmetric matrix A for Q:

$$A = \begin{pmatrix} 0 & 3 & -5 \\ 3 & 12 & -3 \\ -5 & -3 & 3 \end{pmatrix}$$

ii) Hessian matrix of Q:

$$H(Q) = \begin{pmatrix} 0 & 6 & -10 \\ 6 & 24 & -6 \\ -10 & -6 & 6 \end{pmatrix}$$

We see that $\boxed{H(Q) = 2 \cdot A}$. This holds for all quadratic forms.
Therefore we can use either A or H(Q) to check definiteness:

A positive (semi) definite $\Longleftrightarrow$ H(Q) positive (semi) definiteness
" negative — " — $\Longleftrightarrow$ " negative — " —
" indefinite. $\Longleftrightarrow$ " indefinite