LECTURE 5

GRA 6035

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REVIEW: Eigenvalues, Eigenvectors and Diagonalization

Facts: A $n \times n$-matrix

1. A diagonalizable $(P^{-1}AP = D)$ $\Longleftrightarrow$
   (a) A has n eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily all different; that is, counted with multiplicity)
   and
   (b) A has n linearly independent eigenvectors $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ (with eigenvalues $\lambda_1, \ldots, \lambda_n$) (that is, if $\lambda = \lambda_i$ has multiplicity m, then $(A - \lambda_i I)\underline{x} = \underline{0}$ has m degrees of freedom)

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix} \qquad P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \cdots \,\middle|\, \underline{v_n} \right)$$

2. If A is symmetric, then A is diagonalizable.
3. If A has n different eigenvalues, then A is diagonalizable.

Ex: 1) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: $\lambda^2 + 1 = 0$, no eigenvalues $\Rightarrow$ not diagonalizable

2) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: $\lambda^2 - 2\lambda + 1 = 0$

$\lambda_1 = 1$, $\lambda_2 = 1$

($\lambda = 1$ mult. 2)

$$\lambda = \frac{2 \pm \sqrt{4 - 4 \cdot 1}}{2}$$

$$(\lambda - 1)^2 = 0$$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \right)$, $\underline{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, no $\underline{v_2}$

$\Downarrow$

<u>not diagonalizable</u>

$\underline{\lambda = 1}$: $\underbrace{\begin{pmatrix} 1-1 & 1 \\ 0 & 1-1 \end{pmatrix}}_{A - \lambda I} \cdot \underline{x} = \underline{0}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x = x$ (free)
$y = 0$

$$\underline{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{= \underline{v_1}}$$

3) $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$

(see next page)

3) $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$

<u>Eigenvalues:</u>

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = (3-\lambda)\cdot\Big((2-\lambda)(1-\lambda)+1\Big) - 1\Big((1-\lambda)+1\Big) + 1\Big(-1+(2-\lambda)\Big)$$

$$= (3-\lambda)(2-\lambda)(1-\lambda) + 3 \cancel{-\lambda} - 2 \cancel{+\lambda} + 1 - \lambda$$

$$= (3-\lambda)(2-\lambda)(1-\lambda) + (2-\lambda)$$

$$= (2-\lambda)\Big((3-\lambda)(1-\lambda)+1\Big) = (2-\lambda)(\lambda^2-4\lambda+4) = (2-\lambda)\cdot(\lambda-2)^2 = 0$$

$\lambda_1 = 2,\ \lambda_2 = 2,\ \lambda_3 = 2$

($\lambda = 2$ mult. 3)

<u>Eigenvectors:</u>

$A - \lambda I$

$\lambda = 2$: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \underline{x} = \underline{0}$

$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\downarrow$

$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\left.\begin{array}{l} x = -z \\ y = 0 \\ z \text{ free} \end{array}\right\} \quad \underline{x} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z\cdot\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

<u>A not diagonalizable</u>

$\left\{\begin{array}{ll} \text{mult. } \lambda = 2: & 3 \\ \text{degrees of freedom } \lambda = 2: & 1 \end{array}\right.$

Plan:

1. Intro to optimization
2. Quadratic forms and defiteness
3. Bordered Hessian for Quadratic forms

Reading:

[FMEA] 1.7 (1.8)

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1. Intro to optimization

Function $f(x_1, x_2, \dots, x_n) = f(\underline{x})$ $\quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

unconstrained optimization: $\max/\min f(\underline{x})$

constrained optimization: $\max/\min f(\underline{x})$ subject to some constraints

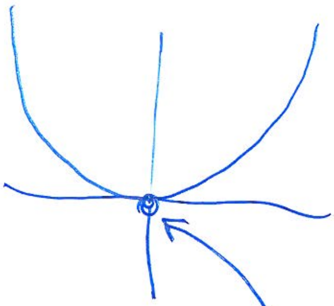
Ex: $\max/\min \overbrace{x_1^2 - 7x_1x_2 + 3x_2^2}^{f(x_1, x_2)}$

$\max/\min \underbrace{xyz}_{f(x,y,z)}$ subject to $\underbrace{x^2+y^2+z^2}_{g(x,y,z)} \leq \underbrace{1}_{a}$

or

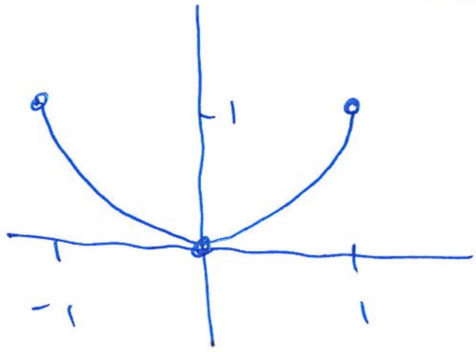
$D_f = \{(x,y,z) : x^2+y^2+z^2 \leq 1\}$

Ex: $f(x) = x^2$



no max
min at $x=0$

Ex: $f(x) = x^2$, $x \in [-1,1]$



max at $x=1$ and $x=-1$
min at $x=0$

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② Quadratic forms

A quadratic form is a polynomial function

$$Q(x_1, \ldots, x_n) = c_{11}x_1^2 + c_{12}x_1x_2 + \ldots$$

such that each term has degree two.

Ex: $n=3$

$$Q(x_1,x_2,x_3) = c_{11}x_1^2 + c_{12}x_1x_2 + c_{13}x_1x_3 + c_{22}x_2^2 + c_{23}x_2x_3 + c_{33}x_3^2$$

$$Q(x_1,x_2,x_3) = x_1^2 + 2x_1x_2 + x_2^2 + x_3^2$$

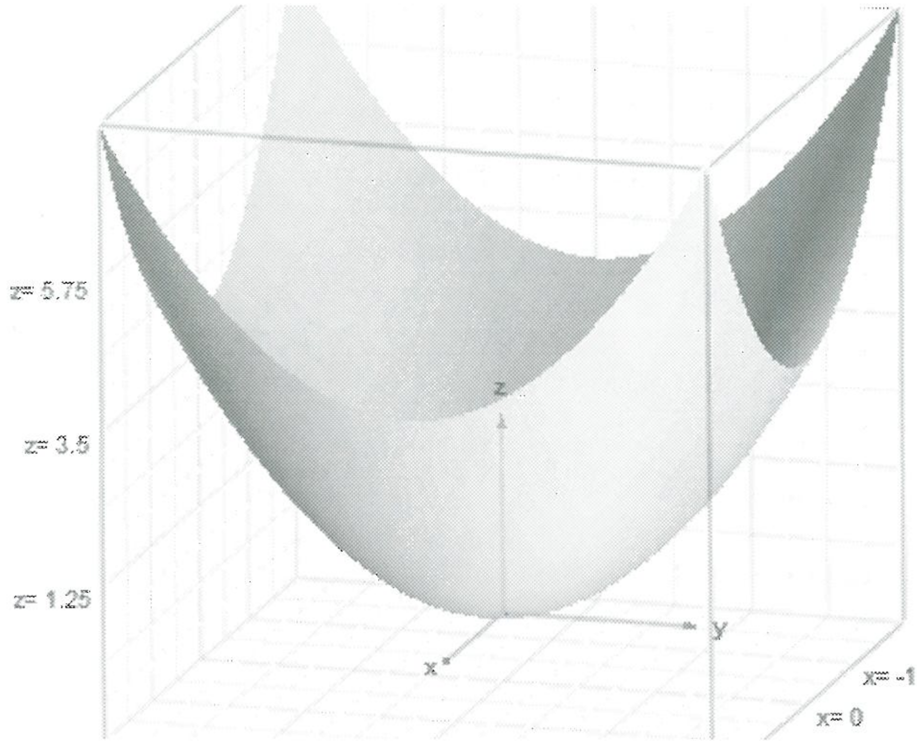
$$Q_1' = 2x_1 + 2x_2 \quad Q_2' = 2x_1 + 2x_2 \quad Q_3' = 2x_3$$

Facts:

① $Q(0,0,\ldots,0) = 0$

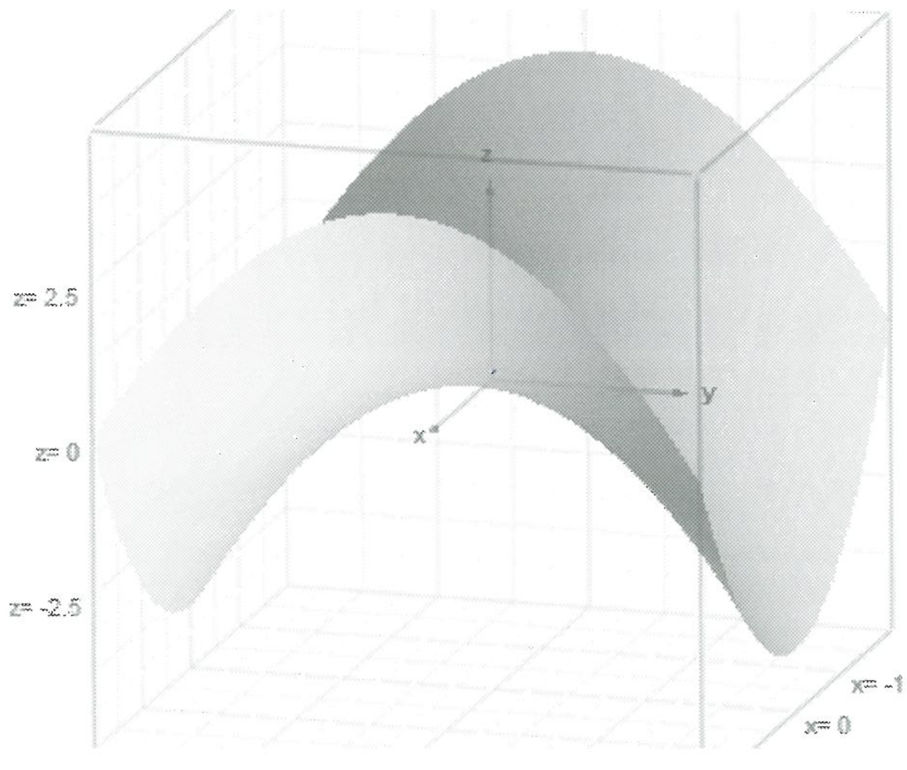
② $\frac{\partial Q}{\partial x_1}(0) = \frac{\partial Q}{\partial x_2}(0) = \ldots = \frac{\partial Q}{\partial x_n}(0) = 0$

$\left\{ \right.$ $\underline{x} = \underline{0}$ is a stationary point for Q



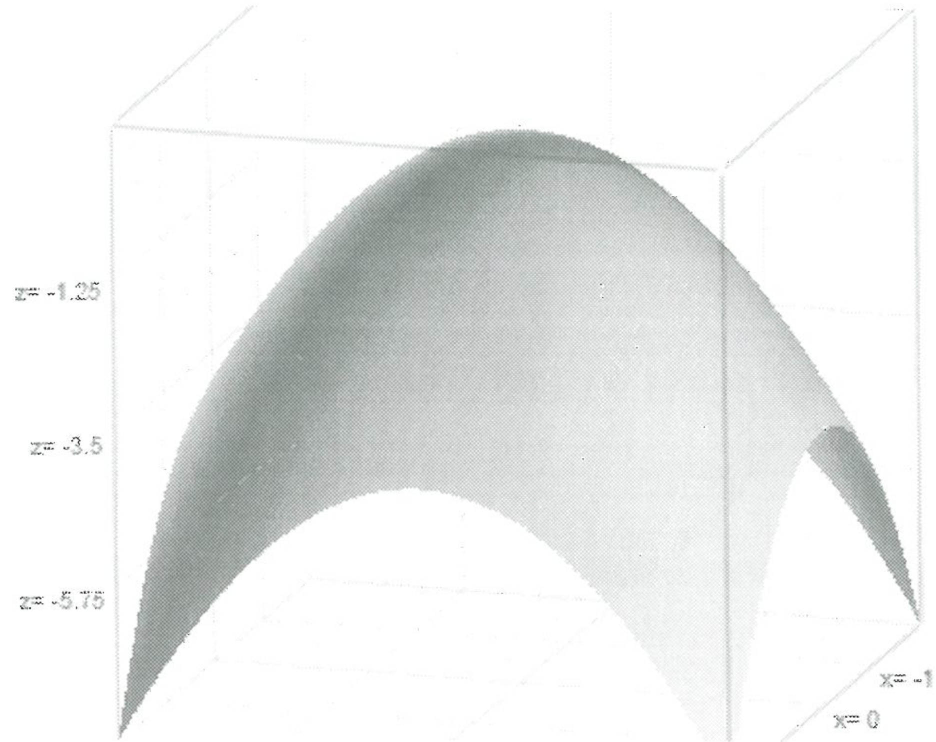
$z = x^2 + y^2$

positive definite



$z = x^2 - y^2$

indefinite



$z = -x^2 - y^2$

negative definite

Let $Q(\underline{x}) = Q(x_1, \dots, x_n)$ be a quadratic form.

Defn:

Q is positive definite if $Q(\underline{x}) > 0$ for all $\underline{x} \neq \underline{0}$

negative definite " $Q(\underline{x}) < 0$ — " —

Q is positive semidefinite if $Q(\underline{x}) \geq 0$ for all $\underline{x}$

negative semidefinite " $Q(\underline{x}) \leq 0$ — " —

Q is indefinite if $Q(\underline{x})$ takes both positive and negative values

Remark:

Q positive definite (semidefinite) $\iff$ $\underline{x} = \underline{0}$ is a minimum

Q negative definite (semidefinite) $\iff$ $\underline{x} = \underline{0}$ is a maximum

Q indefinite $\iff$ $\underline{x} = \underline{0}$ is a saddle point

Matrix form:

$$Q(x_1, x_2) = c_{11} x_1^2 + c_{12} x_1 x_2 + c_{22} x_2^2$$

$$= (x_1 \ x_2) \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{x}^T \cdot A \cdot \underline{x}$$

$$= \begin{pmatrix} x_1 a_{11} + x_2 a_{21} & x_1 a_{12} + x_2 a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= x_1^2 a_{11} + x_1 x_2 a_{21} + x_1 x_2 a_{12} + x_2^2 a_{22}$$

$a_{11} = c_{11}$

$a_{21} + a_{12} = c_{12}$

$a_{22} = c_{22}$

Ex: $Q(x_1,x_2) = 3x_1^2 - 4x_1x_2 + 7x_2^2$

$$= \underline{x}^T \cdot \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix} \underline{x} = \underline{x}^T A \underline{x}, \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$Q(x_1,x_2,x_3) = x_1^2 + 2x_1x_2 + x_2^2 + x_3^2$

$$= \underline{x}^T \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underline{x}, \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Fact: Any quadratic form $Q(\underline{x})$ in n variables can be written in matrix form as

$$Q(x) = \underline{x}^T A \underline{x}$$

for a unique symmetric $n \times n$-matrix A.

Explicitly: $A = (a_{ij})$ with $\begin{cases} a_{ii} = c_{ii} \\ a_{ij} = a_{ji} = \dfrac{c_{ij}}{2}, & i \neq j \end{cases}$

Thm: Let $Q(\underline{x})$ be a quadratic form, and let A be its symmetric matrix, such that $Q(\underline{x}) = \underline{x}^T A \underline{x}$. ~~Then:~~ Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A. Then:

Q positive definite $\Leftrightarrow \lambda_1, \lambda_2, \ldots, \lambda_n > 0$

Q negative definite $\Leftrightarrow \lambda_1, \lambda_2, \ldots, \lambda_n < 0$

Q positive semidefinite $\Leftrightarrow \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$

Q negative " $\Leftrightarrow \lambda_1, \lambda_2 \ldots \lambda_n \leq 0$

Q indefinite $\Leftrightarrow$ both positive and negative eigenvalues

Method:

quadratic form $Q(\underline{x})$ $\rightsquigarrow$ symmetric matrix A $\longrightarrow$ eigenvalues of A $\lambda_1, \lambda_2, \dots, \lambda_n$

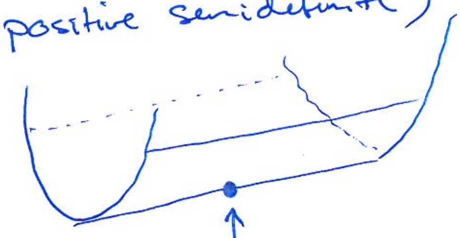
Ex: $Q(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + x_2^2 + x_3^2$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\cdot\left((1-\lambda)^2 - 1\right) = (1-\lambda)\cdot(\lambda^2 - 2\lambda) = 0$$

$\lambda = 1, \quad \lambda = 0, \lambda = 2$

$\lambda_1, \lambda_2, \lambda_3 \geq 0 \longrightarrow$ Q positive semidefinite
(A positive semidefinite)



$\underline{0} = (0,0,0)$

(Leading) Principal Minors

A symmetric matrix

Defn: The leading principal minor of order k, D_k, is the minor obtained from rows $1, 2, \dots, k$ and col's $1, 2, \dots, k$.

Ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$D_1 = 1$

$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

$D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

A principal minor of order k, Δ_k, is a minor obtained by choosing k rows and the same k columns.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Δ_1: $1,\ 1,\ 1$

Δ_2: $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0,\ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,\ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

Δ_3: $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$

Thm:

Q quadratic form, A its symmetric matrix. Let $D_1, \dots, D_n$ be the leading principal minors of A, and $\Delta_1, \dots, \Delta_n$ the principal minors of A.

$D_1, D_2, \dots, D_n > 0 \iff Q$ positive definite

$D_1 < 0, D_2 > 0, D_3 < 0, \dots \iff Q$ negative definite

(i.e. $(-1)^i D_i > 0 \quad i = 1, \dots, n$)

$\Delta_1, \Delta_2, \dots, \Delta_n \geq 0 \iff Q$ positive semidefinite

$\Delta_1 \leq 0, \Delta_2 \geq 0, \dots \iff Q$ negative semidefinite

(i.e. $(-1)^i \Delta_i \geq 0 \quad i = 1, \dots, n$)

All other cases $\iff Q$ indefinite

Ex: $Q(x_1,x_2,x_3) = x_1^2 + 2x_1x_2 + x_2^2 - x_3^2$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$D_1 = 1$
$D_2 = 0$
$D_3 = -1 \cdot 0 = 0$

$\Delta_1 = 1, 1, -1$
$\Delta_2 = 0, -1, -1$
$\Delta_3 = 0$

Δ_1 positive and negative values $\Rightarrow$ not positive semidefin.
not negative —"—
$\Downarrow$
Q indefinite

Note: If $D_2 < 0$, then Q is indefinite..

Patterns: $(n=3)$
$$\begin{cases} + & + & + \\ - & + & - \end{cases}$$

Ex: $Q(x_1,x_2,x_3) = x_1^2 + 2x_1x_2 - 3x_2^2 + 4x_2x_3 + 5x_3^2$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

$D_1 = 1$
$D_2 = -4 < 0$
indefinite

+ −
+ +
+ −

Ex: $Q(x_1,\ldots,x_n) = b_1x_1^2 + b_2x_2^2 + \ldots + b_nx_n^2$

$$A = \begin{pmatrix} b_1 & & 0 \\ & b_2 & \\ 0 & & \ddots & \\ & & & b_n \end{pmatrix}$$

Eigenvalues: $\lambda_1 = b_1$, $\lambda_2 = b_2$, $\vdots$, $\lambda_n = b_n$

Leading principal minors:
$D_1 = b_1$
$D_2 = b_1b_2$
$D_3 = b_1b_2b_3$
$\vdots$
$D_n = b_1b_2 \ldots b_n$

Positive definite: $b_1, b_2, \ldots, b_n > 0 \Leftrightarrow D_1, D_2, \ldots, D_n > 0$

negative definite: $b_1, b_2, \ldots, b_n < 0 \Leftrightarrow$
$D_1 = b_1 < 0$
$D_2 = b_1b_2 > 0$
$D_3 = b_1b_2b_3 < 0$
$\vdots$