ECTURE 4

GRA 6035

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REVIEW OF LECTURE 3:

The vectors $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ ~~in~~ $\mathbb{R}^m$ are <u>linearly independent</u> if

$$c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_n\underline{v}_n = \underline{0}$$

has only the trivial solution $c_1 = c_2 = \dots = c_n = 0$. This will be the case if and only if

$$\mathrm{rk}\left(\underline{v}_1 \,\vdots\, \underline{v}_2 \,\vdots\, \dots \,\vdots\, \underline{v}_n\right) = n$$

<u>Ex:</u>

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \underline{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$\mathrm{rk}\, A = 2$ since $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$, or alternatively

$$-4\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{pmatrix} \qquad \begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ -3c_2 - 6c_3 &= 0 \end{aligned} \qquad \boxed{\begin{aligned} c_1 &= c_3 \\ c_2 &= -2c_3 \\ c_3 &\text{ free} \end{aligned}}$$

<u>Conclusion:</u> $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ <u>not</u> linearly independent

$\{\underline{v}_1, \underline{v}_2\}$ <u>are</u> ——"——

and $1 \cdot \underline{v}_1 - 2\underline{v}_2 + 1 \cdot \underline{v}_3 = \underline{0} \Rightarrow \boxed{\underline{v}_3 = -\underline{v}_1 + 2\underline{v}_2}$ $\quad c_1 = 1, c_2 = -2, c_3 = 1$

OFFICE HOURS:

New office hours: $\boxed{\text{Mon } 12\text{–}14}$ at B4-032 (my office).

Plan Lecture 4

Reading:

[FMEA] 1.5–1.6

① Eigenvalues, eigenvectors
② Diagonalization

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① Eigenvalues and eigenvectors

A : $n \times n$-matrix

Defn: An eigenvalue of A is a number λ such that

$$A\underline{x} = \lambda \underline{x}$$

has a non-zero solution $\underline{x} \neq \underline{0}$. (a vector)

If λ is an eigenvalue for A, then all the vectors $\underline{x} \in \mathbb{R}^n$ ($\underline{x}$ is an n-vector) such that

$$A\underline{x} = \lambda \underline{x}$$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad$ $\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$: $\quad A\underline{x} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$$\lambda \cdot \underline{x} = \lambda \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \end{pmatrix}$$

Conclusion: $\lambda = 3$, $\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution

$\lambda = 3$ eigenvalue

$\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvector

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: $\underline{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$A\underline{x} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\lambda\underline{x} = \lambda \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda \end{pmatrix}$$

no solutions of $A\underline{x} = \lambda\underline{x}$

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda \end{pmatrix} \quad \begin{array}{l} \lambda = 4 \\ 2\lambda = 5 \\ 8 = 5 \text{ impossible.} \end{array}$$

$\underline{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is <u>not</u> eigenvector

<u>Standard method:</u> Finding eigenvalues and eigenvectors for A.

① <u>Find eigenvalues</u>

$A\underline{x} = \lambda\underline{x}$ has non-trivial solutions

$A\underline{x} - \lambda\underline{x} = \underline{0}$ —— " ——

$n \times n$ matrix $\rightarrow (A - \lambda I)\underline{x} = \underline{0}$ —— " ——

$$\boxed{\det(A - \lambda I) = 0} \longleftarrow$$

Characteristic equation

If we look at the linear system

$C \cdot \underline{x} = \underline{0}$

$\det C = 0$: free variables non-zero sol's.

$\det C \neq 0$: $C^{-1} \cdot C\underline{x} = C^{-1}\underline{0}$

$\underline{x} = \underline{0}$

<u>Ex:</u> $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

<u>Char. eqn:</u> $\det(A - \lambda I) = 0$

$$\det\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda) - 1 \cdot 1 = 0$$

$$4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0 \qquad \underline{\lambda = 3, \lambda = 1}$$

<u>Conclusion:</u>

Two eigenvalues for A:

$\lambda_1 = 3, \; \lambda_2 = 1$

Facts:

(1) A $n \times n$-matrix $\Rightarrow$ $|A - \lambda I| = 0$ is an n'th order polynomial eqn. in λ.

(2) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\Rightarrow$ $|A - \lambda I| = 0$ can be written as

$$\lambda^2 - trA \cdot \lambda + \det(A) = 0$$

$(tr\, A = a + d \ , \ \det A = ad - bc)$

(3) If A is a symmetric matrix, then: $|A - \lambda I| = (-1)^n \cdot (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$ where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues.

(4) If A has n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then:

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \dots \cdot \lambda_n$$
$$tr\,(A) = \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n$$

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ non-symmetric

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$\underline{\lambda^2 + 1 = 0}$ $\quad \lambda^2 = -1$

<u>No eigenvalues</u>

Ex: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ diagonal

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \cdot (2-\lambda)(1-\lambda) = 0$$

$\underline{\lambda_1 = 1}, \ \underline{\lambda_2 = 2}, \ \underline{\lambda_3 = 1}$

$\lambda = 1$ mult. 2
$\lambda = 2$ mult. 1

$$\det(A) = 1 \cdot 2 \cdot 1 = 2$$
$$tr\,(A) = 1 + 2 + 1 = 4$$

Ex: $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\cdot \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\cdot(\lambda^2-4\lambda+3) = 0$$

$\underline{\lambda_1 = 1}$ $\quad \underline{\lambda_2 = 3}, \underline{\lambda_3 = 1}$

Standard method: Eigenvectors; $A\underline{x} = \lambda \underline{x}$

(2) If λ is an eigenvalue: (λ found in (1))

$A\underline{x} = \lambda \underline{x}$

$(A-\lambda I)\underline{x} = \underline{0}$ $\leftarrow$ linear system; we solve it for $\underline{x}$.

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ In step (1) we found that $\underline{\lambda_1 = 3}$ and $\underline{\lambda_2 = 1}$ are the eigenvalues.

$\underline{\lambda = 3}$: $\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix} \underline{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = \underline{\underline{y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}}} \Leftarrow \quad \begin{matrix} -x+y=0 \Rightarrow x=y \\ y \text{ free} \end{matrix}$$

Eigenvectors for $\lambda = 3$: $\underline{\underline{y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$, y free

Eigenvectors for $\lambda = 1$:

$$\begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix} \cdot \underline{x} = \underline{0} \quad \longleftarrow \quad (A - \lambda I)\underline{x} = \underline{0}$$

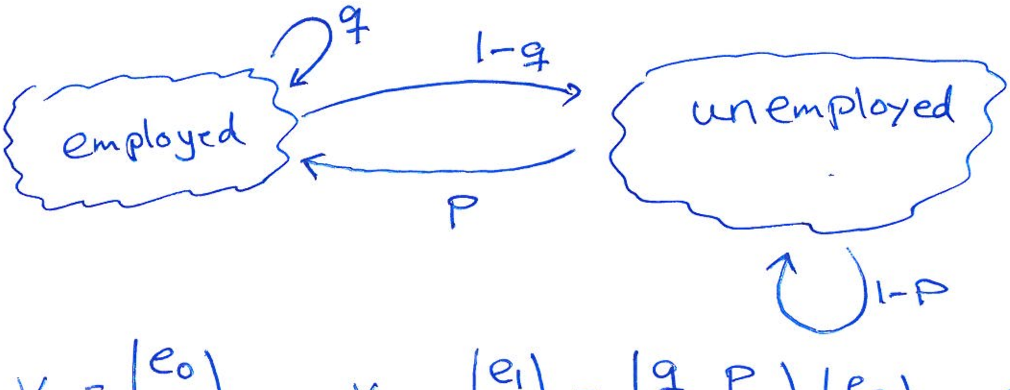
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} x+y=0 \\ y \text{ free} \end{matrix} \qquad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Facts:

① For each eigenvalue λ for A, the equation $(A - \lambda I)\underline{x} = \underline{0}$ is a linear system with at least one degree of freedom.

② If λ has multiplicity m, then $(A - \lambda I)\underline{x} = \underline{0}$ has at <u>most</u> m degrees of freedom.

Motivation: Linear dynamical system



$$\underline{v}_0 = \begin{pmatrix} e_0 \\ u_0 \end{pmatrix} \qquad \underline{v}_1 = \begin{pmatrix} e_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} q & p \\ 1-q & 1-p \end{pmatrix} \cdot \begin{pmatrix} e_0 \\ u_0 \end{pmatrix} = A \cdot \underline{v}_0$$

state vector

$$e_1 = q e_0 + p u_0$$
$$u_1 = (1-q) e_0 + (1-p) u_0$$

$$A = \begin{pmatrix} q & p \\ 1-q & 1-p \end{pmatrix}$$
transition matrix

$$\underline{v}_1 = A \underline{v}_0$$
$$\underline{v}_2 = A \cdot \underline{v}_1 = A \cdot A \underline{v}_0 = A^2 \underline{v}_0$$
$$\vdots$$
$$\underline{v}_T = A^T \underline{v}_0$$

Ex: $\underline{v_0} = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}$ $\qquad A = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix}$

$$\underline{v_1} = A \cdot \underline{v_0} = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix} \cdot \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} = ?$$

$$\underline{v_2} = A^2 \underline{v_0} = \begin{pmatrix} 0.998 & -- \\ -- & -- \end{pmatrix}^2 \cdot \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} = ?$$

$$\vdots \qquad \qquad \vdots$$

$$v_{30} = A^{30} \cdot \underline{v_0} = \begin{pmatrix} 0.998 & -- \\ -- & -- \end{pmatrix}^{30} \cdot \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} = ?$$

Equilibrium: $A \cdot \underline{v} = \underline{v}$ $\left\{ \begin{array}{l} \underline{v} \text{ eigenvector for } A \\ \text{with } \lambda = 1. \end{array} \right.$

$(A - 1 \cdot I)\underline{v} = \underline{0}$ $\qquad \begin{pmatrix} -0.002 & 0.136 \\ 0.002 & -0.136 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$-0.002\, v_1 + 0.136\, v_2 = 0$$

$$v_1 = \frac{0.136}{0.002} v_2 = 68 v_2$$

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 68 v_2 \\ v_2 \end{pmatrix} = v_2 \cdot \begin{pmatrix} 68 \\ 1 \end{pmatrix}$$

$$v_2 = \frac{1}{69} : \quad \underline{v} = \begin{pmatrix} 68/69 \\ 1/69 \end{pmatrix}$$

Fact: With this kind of set-up, there is a unique equilibrium state $\underline{v}$, and $\lim_{T \to \infty} A^T \cdot \underline{v_0} = \underline{v}$ for any $\underline{v_0}$.

2 Diagonalization

A: $n \times n$-matrix

Defn: A is diagonalizable if there is a diagonal matrix D and an invertible matrix P such that

$$P^{-1} \cdot A \cdot P = D$$

Motivation:

If $P^{-1}AP = D$, then $\boxed{A = PDP^{-1}}$.

This means:

$$A^n = \underbrace{A \cdot A \cdot A \cdot \ldots \cdot A}_{n} = \underbrace{(PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\ldots(\cancel{P}DP^{-1})}_{n}$$

$$\boxed{A^n = PD^nP^{-1}}$$

$$D = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -0.2 \end{pmatrix}$$

$$D^n = \begin{pmatrix} 0.1^n & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & (-0.2)^n \end{pmatrix}$$

(easy to compute D^n when D is diagonal, difficult to compute A^n)

How to find a diagonalization of A: (if it is possible)

* find all eigenvalues of A

If there are n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we put

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

* find all eigenvectors of A for $\lambda_1, \ldots, \lambda_n$:

If there are n linearly independent eigenvectors, then we put

$$P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \cdots \,\middle|\, \underline{v_n} \right)$$

and A is diagonalizable. If not, A is not diagonalizable.

Fact: If $\lambda = \lambda_i$ has m_i degrees of freedom, i.e if $(A - \lambda_i I)\underline{x} = 0$ has m_i degrees of freedom then $\lambda = \lambda_i$ will contribute with m_i linearly independent eigenvectors.

Ex:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$\lambda_1 = 1$
$\lambda_2 = 3$
$\lambda_3 = 1$

$$\left.\begin{aligned} D &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \\ P &= \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{aligned}\right\}$$

Yes, A is diagonalizable.

$$P^{-1}AP = D$$
$$\Downarrow$$
$$A = PDP^{-1}$$
$$\Downarrow$$
$$A^n = PD^nP^{-1}$$

$$A^n = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^n \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$$

$\lambda = 1$ (mult. 2)
$\lambda = 3$ (mult 1)

$\lambda = 3$:

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \underline{x} = \underline{0}$$

$$\underline{x} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \underset{\underset{v_3}{\parallel}}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

$\lambda = 1$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ \cancel{1} & \cancel{0} & \cancel{1} \end{pmatrix} \underline{x} = \underline{0}$$

$$\underline{x} = \begin{pmatrix} -z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix}$$

$$= y \cdot \underset{\underset{v_1}{\parallel}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + z \cdot \underset{\underset{v_2}{\parallel}}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

Some facts:

1) If a linear system $A \cdot \underline{x} = 0$ has m degrees of freedom, then there are m free variables $s_1, s_2, \ldots, s_m$, and the solutions can be written in standard form as

$$\underline{x} = s_1 \cdot \underline{v_1} + s_2 \underline{v_2} + \ldots + s_m \underline{v_m}$$

use Gaussian elimination to find solutions.

Then $\{\underline{v_1}, v_2, \ldots, \underline{v_m}\}$ are linearly independent vectors

2) When you combine eigenvectors corresponding to different eigenvalues, you get linearly independent vectors.

3) A $n \times n$-matrix:

A diagonalizable $\Longleftrightarrow$ A has n linearly independent eigenvectors

$\Updownarrow$

i) A has n (real) eigenvalues, counted with multiplicity

and

ii) For each eigenvalue λ, the multiplicity of λ equals the number of degrees of freedom of $(A - \lambda I)\underline{x} = \underline{0}$.

4) If A is symmetric, then A is diagonalizable.