LECTURE 3

GRA 6035

MATHEMATICS

EIVIND ERIKSEN

SEP 06 2012

REMINDER:

* NO LECTURE NEXT WEEK
* TWO LECTURES THE WEEK AFTER:

TUE	18/09	AT 17.00 – 19.45	C1-060
THU	20/09	08.00 – 10.45	C1-010

REVIEW:

- You should know how to compute with matrices, especially
  * matrix multiplication * determinants * minors/cofactors (* inverses)
- Two ways of computing rank:
  * rank is the number of pivot positions (via Gauss/row operations)
  * rank is the maximal order of a nonzero minor (via minors/determinants)
- If A is $n\times n$-matrix: $\begin{cases} |A|\neq 0 \Rightarrow \operatorname{rk} A = n \\ |A| = 0 \Rightarrow \operatorname{rk} A < n \end{cases}$

- Connections between 1) Linear systems 2) Minor and rank
  * When a linear system has coefficient matrix A and augmented matrix $\hat{A} = (A \vdots \underline{b})$ we have:

$\operatorname{rk} A = \operatorname{rk} \hat{A}$: at least one solution $\longrightarrow$ $\begin{cases} \operatorname{rk} A = n: & \text{one solution } (n = \#\text{vars.}) \\ \operatorname{rk} A < n: & \text{infinitely many solutions} \\ & \text{number of degrees of freedom: } n - \operatorname{rk} A \\ & (= \text{free variables}) \end{cases}$

$\operatorname{rk} A < \operatorname{rk} \hat{A}$: no solutions

Ex:
$$\begin{aligned} x+2y+3z&=4 \\ 2x+3y-z&=5 \end{aligned} \qquad A = \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \end{pmatrix} \end{array} \quad \hat{A} = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 3 & -1 & 5 \end{array}\right) \quad n=3 \text{ (no. of variables)}$$

$\operatorname{rk} A = 2$ (since $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3-4 = -1 \neq 0$) and $\operatorname{rk}\hat{A} = 2$ (same reason)

$\Rightarrow$ at least one solution $\underset{(\operatorname{rk} A = \operatorname{rk}\hat{A})}{}$, $\underset{(n - \operatorname{rk} A)}{3-2=1}$ degree of freedom, z free variable (the minor used x, y columns)

$$\boxed{\begin{aligned} x+2y &= 4-3z \\ 2x+3y &= 5+z \end{aligned}} \rightarrow \text{We can solve for } x \text{ and } y \text{ in terms of } z$$

- Result: $\text{rk}\, A = \text{rk}\, A^T$ (since $\det M = \det M^T$ when M is a $k\times k$ submatrix)

Ex: $\text{rk}\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix} = \text{rk}\begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix}$

Plan:

1. Vectors
2. Linear independence
3. Rank and linear independence

Readings

[FMEA] 1.1 – 1.3

1. Vectors

A column vector is a $m\times 1$-matrix ($n=1$, 1 column). It is also called an m-vector.

Ex:

$\underline{v} = \begin{pmatrix}2\\1\end{pmatrix}$ 2-vector $\qquad \underline{w} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ 3-vector

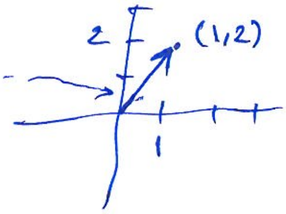
The collection of all m-vectors is called $\mathbb{R}^m$.

We can compute with vectors using some of the matrix operations:

- addition / subtraction of vectors: $\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$
- scalar multiplication: $5\cdot\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}10\\5\end{pmatrix}$

We can ~~a~~ interpret vectors and vector operations geometrically.

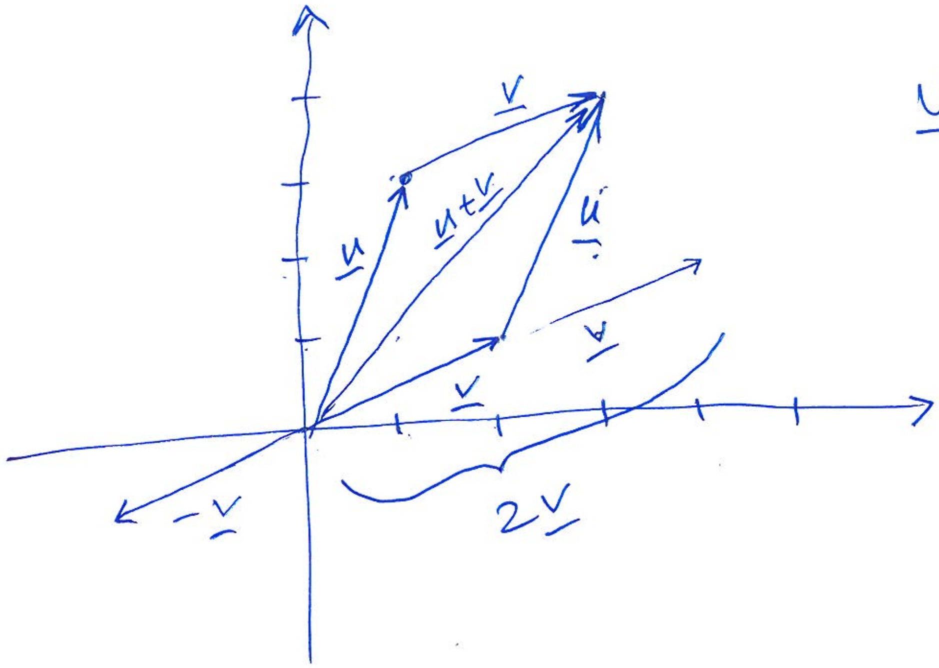
$\underline{v} = \begin{pmatrix}1\\2\end{pmatrix}$



The vector $\underline{v} = \begin{pmatrix}v_1\\ \vdots\\ v_m\end{pmatrix}$ is interpreted as the arrow starting at $(0,\dots,0)$ and ending in $(v_1, v_2, \dots, v_m)$.

Ex: $\underline{u} = \begin{pmatrix}1\\3\end{pmatrix}$ $\underline{v} = \begin{pmatrix}2\\1\end{pmatrix}$

think of vectors as displacements



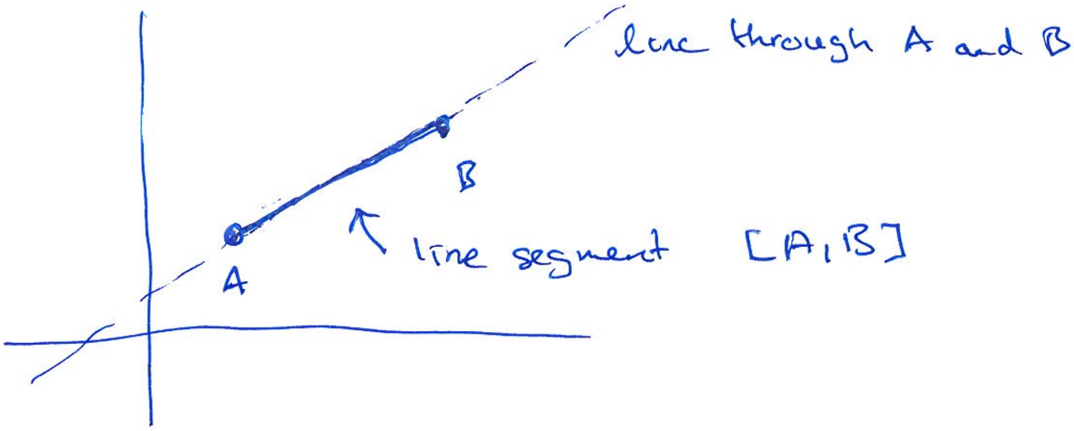
$$\underline{u} + \underline{v} = \underline{v} + \underline{u} = \begin{pmatrix}3\\4\end{pmatrix}$$

$$2\underline{v} = 2 \cdot \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$$

$$-\underline{v} = -1 \cdot \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-2\\-1\end{pmatrix}$$

Lines and line segments

Let A, B be two points in m-dimensional space



$A = (a_1, a_2, \ldots, a_m)$ $\quad$ $B = (b_1, b_2, \ldots, b_m)$

$\updownarrow$ $\quad$ $\updownarrow$

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Equations for $[A, B]$:

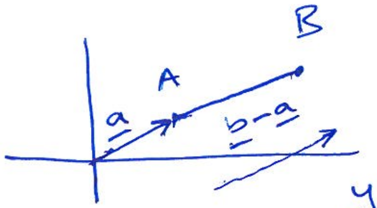
$$[A, B] = \{ (1-t)\cdot \underline{a} + t \cdot \underline{b} \text{ where } 0 \le t \le 1 \}$$

= all points of the form

$(1-t)\cdot \underline{a} + t\cdot \underline{b}$ where $0 \le t \le 1$

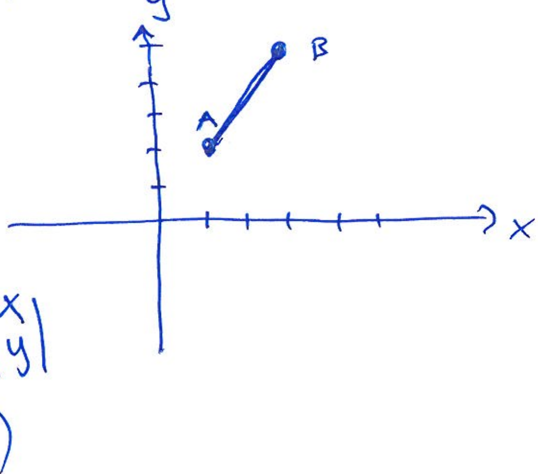
$$\underline{a} + (-t)\cdot \underline{a} + t\underline{b}$$

$$= \underline{a} + t\cdot(\underline{b} - \underline{a})$$



Ex: $A = (1,2)$ $\quad \underline{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$B = (3,5)$ $\quad \underline{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$



$$(1-t)\cdot a + t\underline{b} = (1-t)\cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\cdot \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - t + 3t \\ 2 - 2t + 5t \end{pmatrix} = \begin{pmatrix} 2t+1 \\ 3t+2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(0 \le t \le 1)$$

② Linear Independence

Let $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ be m-vectors.

__Defn:__

A __linear combination__ of $\{v_1, v_2, \dots, v_n\} = \{\underline{v}_i\}$ is an expression of the form

$$a_1 \cdot \underline{v}_1 + a_2 \underline{v}_2 + \dots + a_n \underline{v}_n$$

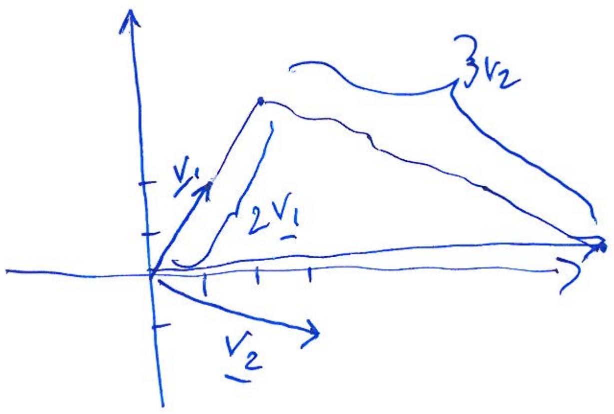
where $a_1, a_2, \dots, a_n$ are numbers (scalars).

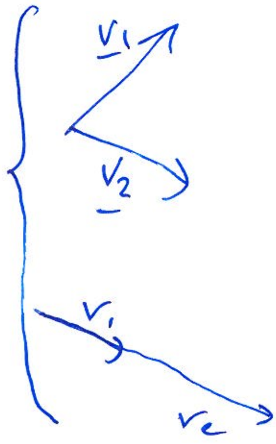
__Ex:__ $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\{\underline{v}_1, \underline{v}_2\}$

$$2 \cdot \underline{v}_1 + 3 \cdot \underline{v}_2 = 2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3 \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 9 \\ -3 \end{pmatrix} = \underline{\begin{pmatrix} 11 \\ 1 \end{pmatrix}}$$

$$3 \underline{v}_1 - 1 \underline{v}_2 = 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \underline{\begin{pmatrix} 0 \\ 7 \end{pmatrix}}$$

$$a_1 \underline{v}_1 + a_2 \underline{v}_2 = a_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} a_1 + 3a_2 \\ 2a_1 - a_2 \end{pmatrix}$$





$\underline{v}_1, \underline{v}_2$ not in the same direction

$\underline{v}_1, \underline{v}_2$ in the same direction

$(\underline{v}_1 = c \cdot \underline{v}_2)$

or

$(\underline{v}_2 = c \cdot \underline{v}_1)$

Ex: Is $\begin{pmatrix}5\\3\end{pmatrix}$ a linear combination of $\underline{v}_1 = \begin{pmatrix}1\\2\end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix}3\\-1\end{pmatrix}$?

If I can solve the equation and find x_1, x_2, then the answer is yes.

$$x_1 \cdot \underline{v}_1 + x_2 \cdot \underline{v}_2 = \begin{pmatrix}5\\3\end{pmatrix}$$ ← vector equation

$$x_1 \cdot \begin{pmatrix}1\\2\end{pmatrix} + x_2 \cdot \begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix}$$

$$\boxed{\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 - x_2 &= 3 \end{aligned}}$$ ← linear system

↓ matrix form

$$\begin{pmatrix}1 & 3\\2 & -1\end{pmatrix} \cdot \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix}$$ ← matrix equation

$$A \cdot \underline{x} = \underline{b}$$

$$|A| = \begin{vmatrix}1 & 3\\2 & -1\end{vmatrix} = -1 - 6 = -7 \neq 0 \Rightarrow A^{-1} \text{ exists}$$

$$A^{-1} = \frac{1}{-7}\begin{pmatrix}-1 & -3\\-2 & 1\end{pmatrix}$$

$$\Rightarrow \underline{x} = A^{-1} \cdot \underline{b}$$

$$= -\frac{1}{7}\begin{pmatrix}-1 & -3\\-2 & 1\end{pmatrix} \cdot \begin{pmatrix}5\\3\end{pmatrix} = -\frac{1}{7}\begin{pmatrix}-14\\-7\end{pmatrix}$$

$$\underline{x} = \begin{pmatrix}2\\1\end{pmatrix} \Rightarrow \begin{pmatrix}5\\3\end{pmatrix} = 2\underline{v}_1 + 1 \cdot \underline{v}_2$$

Notice:

vector equation $\Longleftrightarrow$ linear system $\Longleftrightarrow$ matrix equation

Example:

Let $\underline{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$. What are the linear combinations of $\{\underline{v_1}, \underline{v_2}\}$? Are all 2-vectors lin. comb.?

$$x_1 \cdot \overset{\underline{v_1}}{\overset{\shortparallel}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}} + x_2 \cdot \overset{\underline{v_2}}{\overset{\shortparallel}{\begin{pmatrix} -2 \\ 2 \end{pmatrix}}} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{aligned} 2x_1 - 2x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \end{aligned}$$

$$A = \begin{array}{c} \begin{array}{cc} \underline{v_1} & \underline{v_2} \\ \downarrow & \downarrow \end{array} \\ \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} \end{array} \qquad |A| = 4+2 = 6 \neq 0$$

$$A \cdot \underline{x} = \underline{b} \implies \underline{x} = A^{-1}\underline{b}$$

Explicitly: $\underline{x} = \frac{1}{6}\begin{pmatrix} 2 & 2 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 2b_1 + 2b_2 \\ -b_1 + 2b_2 \end{pmatrix}$

In general, if we look at $x_1\underline{v_1} + x_2\underline{v_2} + \dots + x_n\underline{v_n} = \underline{0}$, this gives $A \cdot \underline{x} = \underline{0}$, where

$$A = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \dots \,\middle|\, \underline{v_n} \right)$$

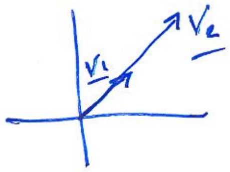
The vectors $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$ are linearly dependant if at least one of the vectors can be written as a linear combinations of the others; for example

$$\underline{v_1} = a_2 \underline{v_2} + a_3 \underline{v_3} + \ldots + a_n \underline{v_n}$$

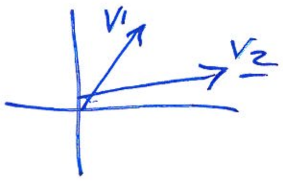
If this is not possible, then the vectors are linearly independant.

Ex: $\{\underline{v_1}, \underline{v_2}\}$

If $v_1 = c\underline{v_2}$ or $\underline{v_2} = c\underline{v_1}$, then $\{\underline{v_1}, \underline{v_2}\}$ linearly dependant.



If not, $\{\underline{v_1}, \underline{v_2}\}$ linearly independant.



$\underline{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$: $v_1 \neq c\underline{v_2}$ $\leftarrow$ $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

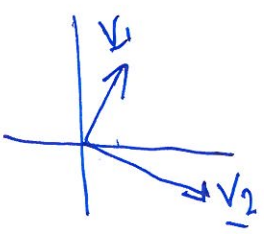
$v_2 \neq c\underline{v_1}$

$1 = 3c$
$2 = -c$

$c = 1/3$
$2 = -1/3$
not possible

$\Downarrow$

linealy independent



Method for determining linear dependence / independence

$\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$: m-vectors

Look at the vector equation:

$$x_1 \underline{v_1} + x_2 \underline{v_2} + \dots + x_n \underline{v_n} = \underline{0}$$

$$\Updownarrow$$

$$A \cdot \underline{x} = \underline{0} \quad , \quad A = \left(\underline{v_1} \,\vdots\, \underline{v_2} \,\vdots\, \dots \,\vdots\, \underline{v_n} \right)$$

Result:

$\underline{x} = \underline{0}$ is the only solution $\Longrightarrow$ $\{\underline{v_1}, \dots, \underline{v_n}\}$ linearly independent

there are solutions $\underline{x} \neq \underline{0}$ $\Longrightarrow$ $\{\underline{v_1}, \dots, \underline{v_n}\}$ linearly dependent

Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\underline{v_3} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$

Look at $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \cdot \underline{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_1 + 2x_2 + 4x_3 = 0 \end{cases}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 1 \cdot (-6) - 1(3) + 1 \cdot 3 = -6 \neq 0$$

Argument a) $|A| \neq 0 \Rightarrow A^{-1}$ exists $\Rightarrow$ $A\underline{x} = \underline{0}$

$x = A^{-1} \cdot \underline{0} = \underline{0}$

b) $|A| \neq 0 \Rightarrow \text{rk } A = 3$

$\Rightarrow \underset{\substack{\| \\ 3}}{\text{rk } A} = \underset{\substack{\| \\ 3}}{n} \Rightarrow$ one solution $\underline{x} = \underline{0}$

Conclusion: $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ linearly independent

Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \cdot \underline{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 1 \cdot (-3) - 1(\cdot 3) + 2 \cdot (3)$$

$$= -3 - 3 + 6 = \underline{0}$$

$|A| = 0 \Rightarrow \operatorname{rk} A < 3 \Rightarrow \#\text{free var's} = \underset{\overset{\|}{3}}{n} - \operatorname{rk} A > 0$

At least one free var. $\Rightarrow$ non-zero solution

$\Rightarrow \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ linearly dependent

Solve explicitly:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right) \xrightarrow{-1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{1/2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 0 \\ -2x_2 - 2x_3 &= 0 \\ x_3 & \text{ free} \end{aligned} \Rightarrow \begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned} \Rightarrow \underline{x} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

non-zero solutions for $x_3 \neq 0$

$x_3 = 1$: $\underline{x_1 = -1, x_2 = -1, x_3 = 1}$

$$(-1) \cdot \underline{v}_1 + (-1) \cdot \underline{v}_2 + 1 \cdot \underline{v}_3 = \underline{0}$$

$$\underline{v}_3 = \underline{v}_1 + \underline{v}_2$$

(or $\underline{v}_1 = -\underline{v}_2 + \underline{v}_3$

or $\underline{v}_2 = -\underline{v}_1 + \underline{v}_3$)

Result: If $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$ are n-vectors, i.e. if A is square, then

$|A| \neq 0 \Rightarrow \{\underline{v_1}, \ldots, \underline{v_n}\}$ linearly independent
$|A| = 0 \Rightarrow$ — " — dependent

3) Rank and linear independence

Let $A = (\underline{v_1} \mid \underline{v_2} \mid \ldots \mid \underline{v_n})$.

$\text{rk}\, A$ = maximal number of linearly independent vectors among $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$

Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\underline{v_3} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{matrix} \\ -1 \\ -1 \end{matrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \\ \\ 1/2 \end{matrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{rk}\, A = 2$: there are two pivot positions

$\{\underline{v_1}, \underline{v_2}\}$ are linearly independent since there are pivots in col. 1 & 2.

there are not three lin. independent vectors (if there were, $\text{rk}\, A$ would have been 3)