LECTURE 2

GRA 6035

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REVIEW: LECTURE I, PROBLEM SHEET I

After LI, PSI you should know:

- how to recognize a linear system
- how to solve a linear system
- Gaussian elimination
- how to find pivot positions
- how to compute the rank of a matrix

Rank:

= number of

A $m\times n$-matrix: $\text{rk}\, A = \#$ pivot positions in A
$= \#$ pivots in echelon form

Ex:

$$A = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 7 & 2 & -1 & 1 \\ 4 & 2 & 1 & 2 \end{array}\right) \to \cdots \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -12 & -22 & -27 \\ 0 & 0 & 0 & -1/2 \end{pmatrix} \qquad \text{rk}\, A = \underline{\underline{3}}$$

(3x4)   (echelon form)

Notice: $\text{rk}\, A \le m$, $\text{rk}\, A \le n \Rightarrow$ maximal rank is $\min\{m,n\}$

$\text{rk}\, A = 0$ when $A = 0$, otherwise $\text{rk}\, A > 0$

possible values of $\text{rk}\, A = 0, 1, \ldots, \min\{m,n\}$

A is 3x4: possible values of rk A is 0, 1, 2, 3

Proposition:

We look at a $m\times n$ linear system with coeff. matrix A and augmented matrix $\hat{A}$. $(A|b)$

Then we have:

1) $\text{rk}\, A < \text{rk}\, \hat{A}$: no solutions
2) $\text{rk}\, A = \text{rk}\, \hat{A}$ and $\text{rk}\, A = n$: one solution
3) $\text{rk}\, A = \text{rk}\, \hat{A}$ and $\text{rk}\, A < n$: infinitely many solutions ($n - \text{rk}\, A$ = degrees of freed.)

Ex:

$$x_1 + x_2 + x_3 = 3$$
$$x_1 + 2x_2 + 4x_3 = 7$$
$$x_1 + 3x_2 + 9x_3 = 13$$

$n = 3$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \quad \hat{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \qquad \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

one solution $\Longleftarrow$ $\mathrm{rk}\, A = 3$ $\quad$ $\mathrm{rk}\, \hat{A} = 3$

PLAN FOR LECTURE 2:

1. MATRIX ALGEBRA
2. DETERMINANTS
3. MINORS AND RANK

$\Big\}$ [FMEA] 1.1, 1.3, 1.4, (1.9)

1. MATRIX ALGEBRA:

An $m \times n$-matrix is a rectangular table of numbers with m rows and n columns.

Ex: $A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \end{pmatrix}$ is a 2×3-matrix

$a_{11} = 2 \quad a_{12} = 3 \quad a_{13} = 5$

$a_{21} = 1 \quad a_{22} = 4 \quad a_{23} = 7$

row 2
column 1

Matrix Operations:

i) Addition/subtraction.

- position by position
- defined if the matrices have the same size

Ex:

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is not defined

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$

ii) Scalar multiplication

- position by position
- scalar = number

Ex:

$2 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$

iii) Multiplication.

- $A \cdot B$ is defined if the sizes are compatible

$$\underset{m\times n}{A}\ ;\ \underset{n\times p}{B} \longrightarrow \underset{m\times p}{AB}$$

- the entries in the product are computed as

Ex:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \overset{1\cdot 1 + 2\cdot 0}{1} & 4 \\ 3 & 10 \end{pmatrix}$$

$2\times 2 = 2\times 2$

$$\underbrace{(a_{i1}\ a_{i2} \ldots a_{in})}_{\text{row } i \text{ of } A} \cdot \underbrace{\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}}_{\text{column } j \text{ of } B} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}}_{\text{entry in position } (i,j) \text{ in } AB}$$

- $A \cdot I = I \cdot A = A$

Identity matrix

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\vdots$

iv) **Transpose**

- A^T is obtained by making the rows in A the colums in A^T $\quad (A^T = A')$

Ex:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Inverse of $2 = 2^{-1} = \frac{1}{2}$ $\quad \{ 2 \cdot \frac{1}{2} = \frac{1}{2} \cdot 2 = 1$

v) **Inverse**

- An inverse of A is a matrix A^{-1} such that $A \cdot A^{-1} = A^{-1} \cdot A = I$
- A matrix has at most one inverse
- If A has an inverse, A is called invertible
- When $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 2x2:

If $ad-bc=0$, then A is not invertible
If $ad-bc \neq 0$, then

$$\boxed{A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Ex:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{1} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^{-1}$ does not exist so $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ not invertible

Note: When A is square ($n \times n$-matrix), we can compute powers

$$A^2 = A \cdot A$$
$$A^3 = A \cdot A \cdot A$$
$$\vdots$$
$$A^n = \underbrace{A \cdot A \cdot A \cdot \ldots \cdot A}_{n \text{ copies}}$$

$$\left(\begin{array}{l} A^0 = I \\ A^1 = A \end{array} \right)$$

Matrix Laws:

$\begin{cases} A, B, C & \text{- matrices} \\ r & \text{- a number} \end{cases}$

Whenever the operations are defined, we have:

$$A+B = B+A$$
$$(A+B)+C = A+(B+C)$$
$$A+0 = A$$
$$A+(-A) = 0$$
$$r\cdot(A+B) = rA + rB$$

$$(AB)C = A(BC)$$
$$A\cdot(B+C) = AB + AC$$
$$(A+B)\cdot C = AC + BC$$
$$(rA)\cdot B = A\cdot(rB) = r\cdot(AB)$$
$$A\cdot I = IA = A$$

But: $AB \neq BA$

Ex: $(A+B)^2 = (A+B)\cdot(A+B)$

$$= A^2 + \underbrace{A\cdot B + B\cdot A}_{\neq 2AB} + B^2$$

$$(A+B)^T = A^T + B^T$$
$$(rA)^T = rA^T$$
$$(AB)^T = B^T A^T$$
$$(A^T)^T = A$$

$$\left.\begin{array}{l} A\cdot A^{-1} = A^{-1}\cdot A = I \\ (rA)^{-1} = r^{-1}\cdot A^{-1} \\ (AB)^{-1} = B^{-1}A^{-1} \\ (A^T)^{-1} = (A^{-1})^T \\ (A^{-1})^{-1} = A \end{array}\right\}$$

when A, B are invertible matrices and $r \neq 0$

Special matrices:

$$0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

zero matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

identity matrix

Diagonal matrix:

$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$$

Upper / lower triangular matrix:

$$U = \begin{pmatrix} d_1 & * & * & \cdots & * \\ 0 & d_2 & * & \cdots & * \\ 0 & 0 & d_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$$

upper

$$L = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ * & d_2 & 0 & \cdots & 0 \\ * & * & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & d_n \end{pmatrix}$$

lower

Ex: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ upper triangular

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ — ″ —

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ diagonal

Note: A square echelon form is upper triangular

Symmetric:

$A = A^T$

Ex: $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 7 \\ -1 & 7 & 4 \end{pmatrix} = A$

$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 7 \\ -1 & 7 & 4 \end{pmatrix} = A^T$

Vector:

$\underline{v}$ $m \times 1$-matrix (column vector)

$\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

matrix with one column

(2) Determinants

If A is an $n \times n$-matrix (square), then we can compute the determinant of A

$$\det(A) = |A|$$

The result is a number.

The case $n=2$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underline{ad - bc}$$

The case $n=1$:

$$A = (a) \qquad |A| = |a| = \underline{a}$$

The general case:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Cotacter expansion:

$$|A| = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + \cdots + a_{1n} \cdot C_{1n}$$

(along the first row)

Cofactors:

$$C_{ij} = \underbrace{(-1)^{i+j}}_{\text{sign}} \cdot \underbrace{M_{ij}}_{\text{minor}}$$

(determinant of a submatrix)

signs

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & + & \cdots \\ \vdots & \vdots & & & \end{pmatrix}$$

M_{ij} = determinant of the matrix you get when you delete row i and column j from A

$$= \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cancel{a_{21}} & \cdots & \cancel{a_{2n}} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ (3x3)

$$|A| = 1 \cdot \underset{6}{\underset{\parallel}{C_{11}}} + 1 \cdot \underset{-5}{\underset{\parallel}{C_{12}}} + 1 \cdot \underset{1}{\underset{\parallel}{C_{13}}}$$

$$= 6 - 5 + 1 = \underline{\underline{2}}$$

$$C_{11} = + \begin{vmatrix} \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{1} & 2 & 4 \\ \cancel{1} & 3 & 9 \end{vmatrix} = + \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} = 2 \cdot 9 - 3 \cdot 4 = \underline{6}$$

$$C_{12} = - \begin{vmatrix} \cancel{1} & \cancel{1} & \cancel{1} \\ 1 & \cancel{2} & 4 \\ 1 & \cancel{3} & 9 \end{vmatrix} = - \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} = -(9-4) = \underline{-5}$$

$$C_{13} = + \begin{vmatrix} \cancel{1} & \cancel{1} & \cancel{1} \\ 1 & 2 & \cancel{4} \\ 1 & 3 & \cancel{9} \end{vmatrix} = + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = \underline{1}$$

You can compute $|A|$ using cofactor expansion along any row or column.

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{vmatrix} = 1 \cdot 2 = 2$$

Alternative method:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{matrix} \\ -1 \\ -1 \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix} \begin{matrix} \\ \\ -2 \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} = E$$

* $\det(E) = 1 \cdot 1 \cdot 2 = 2$
* $\det(A) = 2$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = 1 \cdot (1 \cdot 2 - \cancel{0 \cdot 3}) = 1 \cdot 1 \cdot 2$$

Fact:

- The determinant of an upper (or lower) triangular matrix is the product of the diagonal entries. Echelon form is upper triangular.
- When you do row operations,
  * add a multiple of one row to another row } determinant does not change
  * multiply a row by $c \neq 0$ } determinant change by a factor of c
  * interchange two rows } determinant changes by a factor of -1.

Matrix laws for determinants

A, B are $n \times n$-matrix, r is a number

$$|AB| = |A| \cdot |B|$$

$$|A^T| = |A|$$

$$|A^{-1}| = 1/|A| \quad \text{when } A^{-1} \text{ exists}$$

$$|rA| = r^n \cdot |A|$$

Applications:

a) Inverse matrices

A $n \times n$-matrix: If $|A| = 0$, then A^{-1} does not exist

If $|A| \neq 0$, then A^{-1} does exist.

Moreover,

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

$$\operatorname{adj}(A) = \operatorname{Cof}(A)^T = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T$$

b) Linear systems

$$\begin{pmatrix} x+y+z \\ x-y+z \\ x+2y+4z \end{pmatrix}$$

Ex: $x+y+z=1$
$x-y+z=4$
$x+2y+4z=7$

3x3 linear system

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}}_{\underline{b}}$$

$\boxed{A \cdot \underline{x} = \underline{b}}$ matrix form

If the linear system is nxn:

$A\underline{x} = \underline{b}$

$|A| \neq 0$

$A\underline{x} = \underline{b} \quad | A^{-1}\cdot$

$A^{-1} \cdot (A\underline{x}) = A^{-1} \cdot \underline{b}$

$I\underline{x} = A^{-1}\underline{b}$

$\underline{x} = A^{-1}\underline{b}$ one solution

$|A| = 0$

either no solutions
or
infinitely many solutions

Ex:

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$$

③ Minors and rank

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & -1 & 2 & 1 \\ 1 & 7 & 7 & 3 \end{pmatrix}$$

(3×4)

Defn:

A minor of an $m \times n$-matrix A of order k is the determinant of a $k \times k$-submatrix of A, obtained by deleting $m-k$ rows and $n-k$ columns.

Fact:

$$\boxed{\operatorname{rk} A = \text{maximal order of a non-zero minor}}$$

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & -1 & 2 & 1 \\ 1 & 7 & 7 & 3 \end{pmatrix}$$

3×4

$k=3$: $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 1 & 7 & 7 \end{vmatrix} = 0$

$\begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 1 \\ 1 & 7 & 3 \end{vmatrix} \neq 0$

$\Downarrow$

$\operatorname{rk} A = 3$

Result: If A is an $n \times n$-matrix, then

$$\boxed{\operatorname{rk} A = n \iff |A| \neq 0}$$

Relationship with linear systems:

Let A and $\hat{A}$ be the coeff. matrix and augmented matrix of a linear system.

$\operatorname{rk} A < \operatorname{rk} \hat{A}$: no solution

$\operatorname{rk} A = \operatorname{rk} \hat{A} = n$: one solution

$\operatorname{rk} A = \operatorname{rk} \hat{A} < n$: infinitely many solutions
$(n - \operatorname{rk} A)$ deg. of freedom.

Ex:

$$\begin{aligned} x + 2y + 3z + 5w &= 0 \\ 2x - y + 2z + w &= 0 \\ x + 7y + 7z + 3w &= 0 \end{aligned}$$

A as in the example above

$\operatorname{rk} A = 3 \quad \operatorname{rk} \hat{A} = 3 \implies$ inf. many solutions
$4 - 3 = \underline{1}$ deg. of freedom.

$$\begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 1 \\ 1 & 7 & 3 \end{vmatrix} \neq 0$$

z-column deleted $\implies$ z is free.

it may or may not happen that it is possible to choose other var's as free
$\Updownarrow$
there are other 3×3 minors that are $\neq 0$

Ex: Computation of 5×5-determinant

> From Final Exam 12/2011
> this determinant was used in the solutions
> - other methods are also possible to use

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} \quad -1/4 \quad -1/4$$

$$=$$

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & -1 & 2 & 1/4 & 1/4 \\ 0 & 1 & 0 & 1/4 & 1/4 \end{vmatrix} \quad 1/2 \quad -1/2$$

$$=$$

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & 0 & 2 & 3/4 & -1/4 \\ 0 & 0 & 0 & -1/4 & 3/4 \end{vmatrix} \quad -1/3$$

$$=$$

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & 0 & 0 & 3/4+2/3 & -1/4 \\ 0 & 0 & 0 & -1/4 & 3/4 \end{vmatrix} = 4\cdot 2\cdot 6\cdot\left(\left(3/4+2/3\right)\cdot\left(\frac{3}{4}\right)-\frac{1}{16}\right)$$

$$= 48\cdot\left(\frac{17}{12}\cdot\frac{3}{4}-\frac{1\cdot 3}{16\cdot 3}\right)$$

$$= 17\cdot 3 - 1\cdot 3 = 16\cdot 3 = \underline{\underline{48}}$$