EXTRA PROBLEM SESSION

GRA 6035

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DEC 10^{TH} 2012

MATHEMATICS

MOCK EXAM: DECEMBER 2012

1. $A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

a) $|A| = 4\cdot(15) - 1\cdot(3) + 1\cdot(-3) = 60 - 3 - 3 = \underline{\underline{54}}$

$\text{rk}\, A = \underline{\underline{3}}$ (since $|A| \neq 0$)

b) $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = 0$

$(4-\lambda)\cdot\left[(4-\lambda)^2 - 1\right] - 1\cdot(4-\lambda-1) + 1\cdot(1-(4-\lambda))$

$(4-\lambda)((4-\lambda)^2-1) - (3-\lambda) + (-3+\lambda)$

$= (4-\lambda)((4-\lambda)^2-1) + \underbrace{2\lambda - 6}_{2(\lambda-3)} = 0$

$= (4-\lambda)\cdot(\lambda^2 - 8\lambda + 15) + 2(\lambda-3) = 0$

$= (4-\lambda)(\lambda-3)(\lambda-5) + 2(\lambda-3) = 0$

$\underline{(\lambda-3)}\cdot\left[(4-\lambda)(\lambda-5) + 2\right] = 0$

$\underline{\lambda = 3}$ or $-\lambda^2 + 9\lambda - 18 = 0$

$\underline{\lambda = 3}, \underline{\lambda = 6}$

Eigenvalues:

$\underline{\underline{\lambda_1 = 3, \lambda_2 = 3, \lambda_3 = 6}}$

It is not possible to find eigenvalues by first using row operations on A and then solve the characteristic equation of the echelon form.

You can however do this:

$$\begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} \xrightarrow[\text{row operations}]{} \cdots \rightarrow \cdots$$

It is possible to see directly that $\lambda = 3$ is an eigenvalue.

$$\begin{vmatrix} 4-3 & 1 & 1 \\ 1 & 4-3 & 1 \\ 1 & 1 & 4-3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Then find remaining eigenvalues.

Is A diagonalizable? Yes, since A is symmetric.

Alternatively: $\underbrace{\lambda_1 = 3, \lambda_2 = 3}, \lambda_3 = 6$

$$(A-3I) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

free free

two free var $\Rightarrow$ A diagonalizable

2. $f(x,y;h) = hx^4 + y^4 + 4x^2 - (6+h)xy + 4y^2 - 3h$

a) $f'_x = 4hx^3 + 8x - (6+h)y$

$f'_y = 4y^3 - (6+h)x + 8y$

$$f'' = \begin{pmatrix} 12hx^2 + 8 & -(6+h) \\ -(6+h) & 12y^2 + 8 \end{pmatrix}$$

h=0: $f'' = \begin{pmatrix} 8 & -6 \\ -6 & 12y^2+8 \end{pmatrix}$

$D_1 = 8 > 0$

$D_2 = 8(12y^2+8) - 36$

$= 96y^2 + \underbrace{64 - 36}$

$= 96y^2 + 28 > 0$

Since $D_1, D_2 > 0$ for all (x,y), f is convex for h=0.

General h: f'' for general h is given above

$D_1 = 12hx^2 + 8$ if $h \geq 0$ then $D_1 > 0$ for all (x,y)

$D_2 = (12hx^2+8)(12y^2+8) - (6+h)^2$

$= \underbrace{144hx^2y^2}_{\substack{\geq 0 \\ \text{if } h \geq 0}} + \underbrace{96hx^2}_{\substack{\geq 0 \\ \text{if } h \geq 0}} + \underbrace{96y^2}_{\geq 0} + \underbrace{64 - (6+h)^2}_{}$

$64 - (6+h)^2 \geq 0$

$\Updownarrow$

$h \leq 2$

h = −0.1:

$D_1 = -1.2x^2 + 8$ is not always positive

Conclusion:

f is convex $\Leftrightarrow$ $0 \leq h \leq 2$

b) $\underline{h=0}$: f is convex

(x,y) global min $\Leftrightarrow$ (x,y) is stationary pt.

<u>Stationary pts:</u>

$f'_x = 4hx^3 + 8x - (6+h)y = 0$ (1) $8x - 6y = 0$

$f'_y = 4y^3 - (6+h)x + 8y = 0$ (2) $4y^3 - 6x + 8y = 0$

(1) $x = \frac{6y}{8} = \underline{\frac{3}{4}y}$

(2) $4y^3 - 6 \cdot \left(\frac{3}{4}y\right) + 8y = 0$

$4y^3 - 4.5y + 8y = 0$

$y(4y^2 + 3.5) = 0$

$\underline{y=0}$ or $4y^2 + 3.5 = 0$ no solutions $\Rightarrow$ <u>Conclusion!</u>

$(x,y) = \underline{(0,0)}$

$y=0$, $x=0$

is global min.

c) $f^*(h) = f(x^*(h), y^*(h))$

$f^*(0) = f(0,0) = -3h = \underline{0}$

$f = hx^4 + \dots$
$+ (6+h)xy$
$+ \dots$
$-3h$

$\frac{df^*(h)}{dh} = \frac{\partial f}{\partial h}\Big|_{x=x^*(h),\, y=y^*(h)}$

Env. Thm

$= x^4 - xy - 3 \Big|_{x=x^*(h),\, y=y^*(h)}$

$\frac{df^*(0)}{dh} = \underbrace{x^*(0)^4}_{0^4} - \underbrace{x^*(0)\, y^*(0)}_{0 \cdot 0} - 3 = \underline{-3}$

The minimum value $f^*(h)$ will <u>decrease</u> when h changes from $h=0$ to $h>0$, since

$$-3 = \frac{df^*(h)}{dh} < 0$$

3.

a) $y_{t+2} - 5y_{t+1} + 4y_t = 2^t$

difference eqn, sec. order inhom.

$$y_t = y_t^h + y_t^p = \underline{\underline{C_1 + C_2 \cdot 4^t - 2^{t-1}}}$$

$\underline{y_t^h}$: $\quad y_{t+2} - 5y_{t+1} + 4y_t = 0$

$$r^2 - 5r + 4 = 0$$

$$\underline{r=1}, \ \underline{r=4} \quad \Rightarrow y_t^h = C_1 \cdot 1^t + C_2 \cdot 4^t$$

$$= \underline{C_1 + C_2 \cdot 4^t}$$

$\underline{y_t^p}$: $\quad f_t = 2^t \quad \left.\begin{aligned} f_{t+1} &= 2^{t+1} = 2 \cdot 2^t \\ f_{t+2} &= 2^{t+2} = 4 \cdot 2^t \end{aligned}\right\}$ $y_t = A \cdot 2^t$

Try in eqn: $\quad y_{t+2} - 5y_{t+1} + 4y_t = 2^t$

$$4A \cdot 2^t - 5(2A \cdot 2^t) + 4(A \cdot 2^t) = 2^t$$

$$2^t(4A - 10A + 4A) = 2^t$$

$$-2A = 1 \qquad A = \underline{-1/2}$$

$$y_t^p = \underline{-\frac{1}{2} \cdot 2^t = -2^{t-1}}$$

b) $y' = t\cdot(y-1)^2$, $y(0)=3$ $(t=0, y=3)$

differential equation, first order
<u>seperable</u>

$\frac{1}{(y-1)^2}y' = t$

$\int \frac{1}{(y-1)^2}\,dy = \int t\,dt$

$\frac{1}{(y-1)^2} = (y-1)^{-2}$

$-\frac{1}{y-1} = \frac{1}{2}t^2 + C$ implicit solution

$\int \frac{1}{(y-1)^2}\,dy = \frac{1}{-2+1}(y-1)^{-2+1} + C$
$= -(y-1)^{-1} + C$

$-\frac{1}{2} = \frac{1}{2}\cdot 0^2 + C \Rightarrow \underline{C = -\frac{1}{2}}$

$-\frac{1}{y-1} = \frac{1}{2}t^2 - \frac{1}{2} = \frac{t^2-1}{2}$

$\frac{1}{y-1} = -\frac{(t^2-1)}{2}$

$y-1 = -\frac{2}{t^2-1} = \frac{-2}{t^2-1}$

$y = \frac{-2}{t^2-1} + 1 = \frac{-2+t^2-1}{t^2-1} = \underline{\underline{\frac{t^2-3}{t^2-1}}}$

c) $(2y-e^t)y' = (ye^t + 2e^{2t})$, $y(0)=2$

$(-ye^t - 2e^{2t}) + (2y-e^t)y' = 0$

differential eqn. first order
<u>exact</u>

$$\boxed{\begin{aligned} (1)\quad & \frac{\partial h}{\partial t} = -ye^t - 2e^{2t} \\ (2)\quad & \frac{\partial h}{\partial y} = 2y - e^t \end{aligned}}$$

(2) $\Rightarrow$ $\underline{h = y^2 - e^t y + C(t)}$

(1) $\frac{\partial h}{\partial t} = -e^t y + C'(t)$
$= -ye^t - 2e^{2t}$

ok if $C(t) = -e^{2t}$

Conclusion:

The diff-eqn. is exact, with $h = y^2 - e^t \cdot y - e^{2t}$.

Solution:

$$y^2 - e^t \cdot y - e^{2t} = C \quad \text{(implicit form)}$$

$y(0) = 2$: $\quad 4 - e^0 \cdot 2 - e^0 = C$

$t=0, y=2$ $\qquad C = 1$

$$y^2 - e^t y - e^{2t} = 1$$

$$y^2 - e^t y - e^{2t} - 1 = 0 \qquad (a=1,\ b=-e^t,\ c=-e^{2t}-1)$$

$$y = \frac{e^t \pm \sqrt{(e^t)^2 - 4 \cdot 1 \cdot (-e^{2t}-1)}}{2}$$

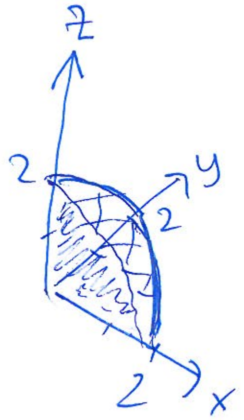
$$= \frac{1}{2}\left(e^t \pm \sqrt{e^{2t} + 4e^{2t} + 4}\right)$$

$$= \frac{1}{2}\left(e^t \pm \sqrt{5e^{2t} + 4}\right)$$

$t=0$: $\quad \frac{1}{2}\left(1 \pm \sqrt{5+4}\right) = 2 \quad$ so it must be $+$

$$y = \frac{1}{2}\left(e^t + \sqrt{5e^{2t} + 4}\right)$$

4. $\max \underset{f(x,y,z)}{\underset{\shortparallel}{x+2y+2z}}$ s.t. $\begin{cases} x^2+y^2+z^2 \le 4 \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{cases}$



Standard form:

$-x \le 0$

$-y \le 0$

$-z \le 0$

$$\mathcal{L} = x+2y+2z - \lambda(x^2+y^2+z^2) - v_1(-x) - v_2(-y) - v_3(-z)$$
$$= x+2y+2z - \lambda(x^2+y^2+z^2) + v_1 x + v_2 y + v_3 z$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 1 - \lambda \cdot 2x + v_1 = 0 \\ \mathcal{L}'_y &= 2 - \lambda \cdot 2y + v_2 = 0 \\ \mathcal{L}'_z &= 2 - \lambda \cdot 2z + v_3 = 0 \end{aligned}}$$

FOC

$\lambda \ge 0,\ v_1 \ge 0,\ v_2 \ge 0,\ v_3 \ge 0$ | $\boxed{\lambda, v_1, v_2, v_3 \ge 0}$

$$\left.\begin{aligned} &\text{if } x^2+y^2+z^2 < 4 \text{ then } \lambda = 0 \\ &\text{if } x > 0, \text{ then } v_1 = 0 \\ &\text{if } y > 0, \text{ then } v_2 = 0 \\ &\text{if } z > 0, \text{ then } v_3 = 0 \end{aligned}\right\} \iff \boxed{\begin{aligned} &\lambda \cdot (x^2+y^2+z^2-4) = 0 \\ &v_1 \cdot x = 0 \\ &v_2 \cdot y = 0 \\ &v_3 \cdot z = 0 \end{aligned}}$$

CSC

$$\boxed{\begin{aligned} &x^2+y^2+z^2 \le 4 \\ &x, y, z \ge 0 \end{aligned}}$$

C

Solve KT conditions:

(a) $x^2+y^2+z^2 < 4$

$\lambda=0$: FOC: $1+v_1=0$
$2+v_2=0$
$2+v_3=0$

$v_1=-1$
impossible

(b) $x^2+y^2+z^2=4$

$\lambda \geq 0$: FOC: $1-2\lambda x+v_1=0$
$2-2\lambda y+v_2=0$
$2-2\lambda z+v_3=0$

$x=0$: $1+v_1=0 \Rightarrow v_1=-1$ impossible
$\Rightarrow x>0,\ v_1=0$

$y=0$: $2+v_2=0 \Rightarrow v_2=-2$ imp.
$\Rightarrow y>0,\ v_2=0$

$z=0$: $2+v_3=0 \Rightarrow v_3=-2$ imp.
$\Rightarrow z>0,\ v_3=0$

$\Downarrow$

FOC: $1-2\lambda x=0 \Rightarrow x=\frac{1}{2\lambda}$
$2-2\lambda y=0 \Rightarrow y=\frac{2}{2\lambda}$
$2-2\lambda z=0$
$\Rightarrow z=\frac{2}{2\lambda}$

cannot have $\lambda=0$
(see case a)

$x^2+y^2+z^2=4$

$\left(\frac{1}{2\lambda}\right)^2+\left(\frac{2}{2\lambda}\right)^2+\left(\frac{2}{2\lambda}\right)^2=4$

$\frac{1+4+4}{4\lambda^2}=4$

$9=16\lambda^2$

$\lambda^2=\frac{9}{16} \Rightarrow \lambda=\frac{3}{4}$

$x=\frac{2}{3},\ y=\frac{4}{3},\ z=\frac{4}{3}$

$\lambda=\frac{3}{4},\ v_1=v_2=v_3=0$

solution to KT cond.

$\lambda_1 = b-a \qquad \lambda_2 = b-a \qquad \lambda_3 = b-a \qquad \lambda_4 = ?$

Use trace:

$$\operatorname{tr} A = \operatorname{tr}\begin{pmatrix} b & a & a & a \\ a & b & a & a \\ a & a & b & a \\ a & a & a & b \end{pmatrix} = b+b+b+b = 4b$$

$$\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 3(b-a) + \lambda_4$$

$$\Rightarrow \quad 4b = 3(b-a) + \lambda_4 \Rightarrow \lambda_4 = \underline{b+3a}$$

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 = \underline{\underline{(b-a)^3 \cdot (b+3a)}}$$

Is this max?

$$\mathcal{L}(x,y,z;\ 3/4,0,0,0) = x+2y+2z-\frac{3}{4}(x^2+y^2+z^2)$$

$$\mathcal{L}'' = \begin{pmatrix} -3/2 & 0 & 0 \\ 0 & -3/2 & 0 \\ 0 & 0 & -3/2 \end{pmatrix}$$

negative definite $\Rightarrow$ $\mathcal{L}$ concave

$D_1 = -3/2 < 0$

$D_2 = 9/4 > 0$

$D_3 = -27/8 < 0$

$\Downarrow$

$\left(\frac{2}{3},\frac{4}{3},\frac{4}{3}\right)$ is max

5. $$A = \begin{pmatrix} b & a & a & a \\ a & b & a & a \\ a & a & b & a \\ a & a & a & b \end{pmatrix}, \quad a \neq 0$$

Show that $\lambda = b-a$ is eigenvalue

$$A-\lambda I = A-(b-a)I = \begin{pmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{pmatrix} \quad \begin{pmatrix}\text{since} \\ b-(b-a)=a\end{pmatrix}.$$

$\Rightarrow \det(A-\lambda I)=0$, so $\lambda = b-a$ is eigenvalue

Find its multiplicity:

Number of free var's in $(A-\lambda I)\underline{x} = \underline{0}$ is $\underline{3}$:

$$\begin{pmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{pmatrix} \longrightarrow \begin{pmatrix} a & a & a & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\uparrow$ free $\uparrow$ free $\uparrow$ free

A is symmetric and therefore diagonalizable,

so $\text{mult}(b-a) = \underline{3}$