LECTURE 12

GRA 6035

MATHEMATICS

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NOV 15^{TH} 2012

PLAN:

① Second order linear differential equations

- review of homogeneous case
- inhomogeneous case
- Euler's diff. eqn.

② Difference equations

Reading:

[FMEA] 6.1 – 6.4
11.1 – 11.4

Next week: (Last lecture)

* Finish difference equations
* Review of course / exam problems

1 Second order linear differential equations

General form: $\boxed{\ddot{y} + a\dot{y} + by = f(t)}$

We only consider the case of constant coefficients, ie that a, b are constants → a, b constants

$f(t)$ function in t

Two cases:

i) Homogeneous case: $f(t) = 0$

ii) Inhomogeneous case: $f(t) \neq 0$

i) Homogeneous case (review from last week)

$$\ddot{y} + a\dot{y} + by = 0$$

Char. eqn: $r^2 + ar + b = 0 \rightsquigarrow$ Solutions: (char. roots) $\begin{cases} r_1 \neq r_2 \\ r \text{ double root} \\ \text{no solns} \end{cases}$

$$\left(r = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \right)$$

General solution:

A) $a^2 - 4b > 0$: $r_1 \neq r_2$ $\qquad y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

B) $a^2 - 4b = 0$: r double root $\qquad y = C_1 e^{rt} + C_2 t e^{rt} = (C_1 + C_2 t) e^{rt}$

C) $a^2 - 4b < 0$: no (real) solutions $\qquad y = e^{\alpha t} \cdot \left(C_1 \cos(\beta t) + C_2 \sin(\beta t) \right)$

where $\alpha = -a/2$, $\beta = \sqrt{4b - a^2}/2$

Ex: $y'' - 4y' + 4y = 0$

Char. eqn: $r^2 - 4r + 4 = 0$

$$r = \frac{4 \pm \sqrt{16 - 4 \cdot 4}}{2} = 2 \pm 0$$

$r = 2$ double root

General solution: $y = C_1 \cdot e^{2t} + C_2 t e^{2t}$

Ex: $y'' - 2y' + 5y = 0$

Char. eqn: $r^2 - 2r + 5 = 0$

$$r = \frac{2 \pm \sqrt{4 - 4 \cdot 5}}{2} = 1 \pm \frac{\sqrt{-16}}{2}$$

no real roots; $\alpha = 1$, $\beta = \frac{\sqrt{16}}{2} = 2$

General solution:
$$y = e^{\alpha t} \cdot \left(C_1 \cos(\beta t) + C_2 \sin(\beta t) \right)$$
$$= e^{t} \cdot \left(C_1 \cos(2t) + C_2 \sin(2t) \right)$$

Inhomogeneous case : $y'' + ay' + by = f(t)$

Ex: $y'' - 6y' + 8y = 8$

Superposition principle:

The general solution of $y'' + ay' + by = f(t)$ is given by

$$\boxed{y = y_h + y_p}$$

$\left(\begin{array}{l} y_p : \text{particular} \\ y_h : \text{homogeneous} \end{array} \right)$

where

* y_h is the general solution of the corresponding homogeneous diff. eqn.

$$y'' + ay' + by = 0$$

* y_p is a particular solution of the diff. eqn.

$$y'' + ay' + by = f(t)$$

Ex: $y'' - 6y' + 8y = 8$

General solution: $y = y_h + y_p = \underline{C_1 e^{2t} + C_2 e^{4t} + 1}$

i) y_h: $y'' - 6y' + 8y = 0$

$r^2 - 6r + 8 = 0$

$r = \dfrac{6 \pm \sqrt{36 - 4 \cdot 8}}{2} = \dfrac{6 \pm 2}{2}$

$r_1 = 2, \quad r_2 = 4$

$\Downarrow$

$y_h = \underline{C_1 e^{2t} + C_2 e^{4t}}$

ii) y_p: $y'' - 6y' + 8y = 8$

Guess: $y = A$ (A constant)

$y' = 0$

$y'' = 0$

Check: $0 - 6 \cdot 0 + 8A = 8$

$8A = 8 \quad \underline{A = 1}$

$\underline{y_p = 1}$

Ex: $y' - 2y = t^2$ linear first order diff. eqn. with constant coeffs.

$$y = y_h + y_p = Ce^{2t} - \frac{1}{2}t^2 - \frac{1}{2}t - \frac{1}{4}$$ (Superposition principle)

y_h: $y' - 2y = 0$

Char. eqn.: $r - 2 = 0$

$r = 2 \Rightarrow y_h = Ce^{2t}$

y_p: $y' - 2y = t^2$

Guess: $y = At^2 + Bt + C$

$y' = 2A \cdot t + B$

Check: $(2At + B) - 2 \cdot (At^2 + Bt + C) = t^2$

$$\underbrace{(-2A)}_{=1} \cdot t^2 + \underbrace{(2A - 2B)}_{=0} t + \underbrace{(B - 2C)}_{=0} = t^2$$

$-2A = 1 \Rightarrow A = -1/2$

$2A - 2B = 0 \qquad B = -1/2$

$B - 2C = 0 \qquad -1/2 - 2C = 0 \Rightarrow C = -1/4$

$$y_p = -\frac{1}{2}t^2 - \frac{1}{2}t - \frac{1}{4}$$

How to find y_p in general:

Method: Guess y and verify

- The guess should include some <u>undetermined constants</u>
- To guess, start by looking at

  $f(t)$, $f'(t)$, $f''(t)$

  Choose the ~~guess~~ y such that it contains functions such as f, f', f''.

Ex: $f(t) = t$:

$$\left.\begin{aligned} f &= t \\ f' &= 1 \\ f'' &= 0 \end{aligned}\right\} \quad y = At + B$$

$f(t) = e^t$:

$$\left.\begin{aligned} f &= e^t \\ f' &= e^t \\ f'' &= e^t \end{aligned}\right\} \quad y = A \cdot e^t$$

$f(t) = e^{2t}$:

$$\left.\begin{aligned} f &= e^{2t} \\ f' &= 2e^{2t} \\ f'' &= 4e^{2t} \end{aligned}\right\} \quad y = A \cdot e^{2t}$$

- If the initial guess does not work, try to multiply the guess by t.

Ex: $y'' - 7y' + 12y = e^{2t}$ $\quad$ $r^2 - 7r + 12 = 0$, $r = 3, 4$

$$y = y_h + y_p = \underline{\underline{c_1 e^{3t} + c_2 e^{4t} + \frac{1}{2} e^{2t}}}$$

$\underline{y_p}$: $\left.\begin{array}{l} f = e^{2t} \\ f' = 2e^{2t} \\ f'' = 4e^{2t} \end{array}\right\}$ $\begin{array}{l} y = Ae^{2t} \\ y' = 2Ae^{2t} \\ y'' = 4Ae^{2t} \end{array}$

$$y'' - 7y' + 12y = e^{2t}$$

$$(4Ae^{2t}) - 7\cdot(2Ae^{2t}) + 12(Ae^{2t}) = e^{2t}$$

$$\cancel{e^{2t}}\cdot(4A - 14A + 12A) = \cancel{e^{2t}}$$

$$2A = 1 \Rightarrow \underline{A = 1/2}$$

$$\underline{y_p = \frac{1}{2} e^{2t}}$$

Euler equations:

$$\boxed{t^2 y'' + aty' + by = f(t)}$$

(not constant coefficients)

Ex: $t^2 \ddot{y} - 7t\dot{y} + 12y = 0$

Try: $\left.\begin{array}{l} y = t^r \\ y' = rt^{r-1} \\ y'' = r(r-1)t^{r-2} \end{array}\right\}$ $\begin{array}{l} t^2 \cdot r(r-1)t^{r-2} - 7t(rt^{r-1}) + 12t^r = 0 \\ t^r \cdot [r(r-1) - 7r + 12] = 0 \end{array}$

$$r(r-1) - 7r + 12 = 0$$

$$r^2 - 8r + 12 = 0$$

$$r = \frac{8 \pm \sqrt{64 - 48}}{2} = \frac{8 \pm 4}{2}$$

$$r = \underline{2}, \quad r = \underline{6}$$

General solution:

$$\underline{\underline{y = c_1 \cdot t^2 + c_2 t^6}}$$

2 Difference equations

A difference equation is an equation that relates y_t to $y_{t-1}, y_{t-2}, \ldots$ That is, a relation between a number in a sequence and one or more of the preceeding numbers in the sequence.

Ex:

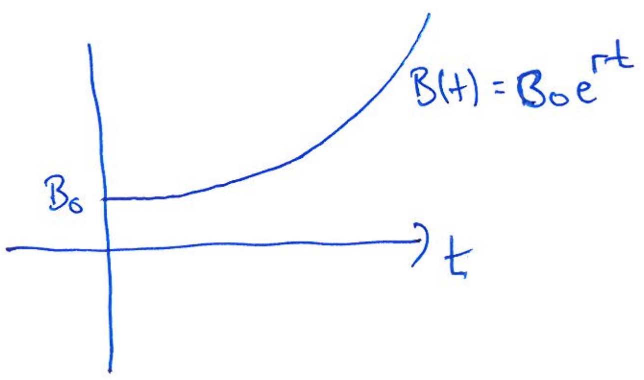
$$\underbrace{y_{t+1} = 2y_t}_{\text{difference equation}}, \quad \underbrace{y_0 = 1}_{\text{initial condition}}$$

Solution:

$$y_0 = 1$$
$$y_1 = 2y_0 = 2$$
$$y_2 = 4$$
$$y_3 = 8$$
$$y_4 = 16$$
$$\vdots$$

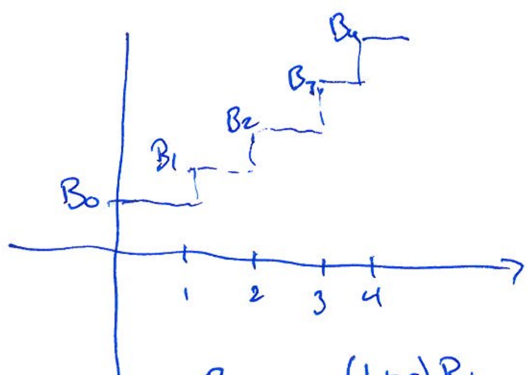
Closed form:

$$y_t = 2^t$$



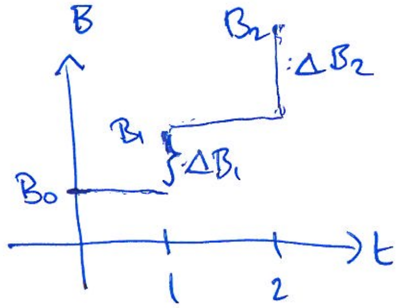
$$B' = r \cdot B$$

(differential equation)



$$B_{t+1} = (1+r)B_t$$

(difference equation) $B_{t+1} - B_t = rB_t$



$$\Delta B_t = B_t - B_{t-1} \quad (\Delta = \text{change})$$

$$\Delta^2 B_t = (B_t - B_{t-1}) - (B_{t-1} - B_{t-2})$$
$$= B_t - 2B_{t-1} + B_{t-2}$$

(change in the change)

$$\Delta^2 B_2 = \Delta B_2 - \Delta B_1$$

First order linear difference equations:

$$\boxed{y_{t+1} + a y_t = f(t)}$$

a: constant

- $f(t) = 0$: homogeneous
- $f(t)$ is a sequence : inhomogeneous

i) Homogeneous case:

$$y_{t+1} + a y_t = 0$$
$$y_{t+1} = (-a) \cdot y_t$$

$$y_1 = (-a) \cdot y_0$$
$$y_2 = (-a)\, y_1 = (-a)^2 y_0$$
$$y_3 = (-a)\, y_2 = (-a)^3 y_0$$
$$\vdots$$
$$\boxed{y_t = (-a)^t \cdot y_0}$$

general solution of difference eqn.

ii) Inhomogeneous case

$$y_{t+1} + a y_t = f_t$$

Ex: $y_{t+1} - 1.05 y_t = -100$, $y_0 = 5000$

$$\boxed{y_{t+1} - y_t = 0.05 y_t - 100}$$

In general: $y_{t+1} + a y_t = f_t$

General solution: $y_t = y_t^h + y_t^p$ (superposition principle)

y_t^h: $y_{t+1} + a y_t = 0 \implies y_t = (-a)^t \cdot y_0$

$$y_t^h = C \cdot (-a)^t$$

y_t^p: A particular solution of $\boxed{y_{t+1} + a y_t = f_t}$

Guess y_t^p, and verify the guess.

Ex: $y_{t+1} - 1.05 y_t = -100$

$$y_t = y_t^h + y_t^p$$

$$= \underbrace{C \cdot 1.05^t}_{y_t^h} + y_t^p$$

$$= C \cdot 1.05^t + 2000$$

y_t^p: $\begin{cases} y_t = A \\ y_{t+1} = A \end{cases}$

$$A - 1.05 \cdot A = -100$$

$$-0.05 A = -100$$

$$A = \frac{-100}{-0.05} = 2000$$

$$y_t^p = 2000$$

General solution: $y_t = C \cdot 1.05^t + 2000$
Initial condition: $y_0 = 5000$

$$y_0 = C \cdot 1.05^0 + 2000 = C + 2000 = 5000$$

$$C = 5000 - 2000 = \underline{3000}$$

$$y_t = \underline{\underline{3000 \cdot 1.05^t + 2000}}$$

Second order linear difference equations

$$\boxed{y_{t+2} + a y_{t+1} + b y_t = f_t}$$

a, b constants
f_t : sequence in t

i) Homogeneous case: $f_t = 0$ $\quad \boxed{y_{t+2} + a y_{t+1} + b y_t = 0}$

Ex: $y_{t+2} - 5y_{t+1} + 6y_t = 0$

Guess: $y_t = r^t$ gives $r^{t+2} - 5 \cdot r^{t+1} + 6r^t = 0$

$$r^t(r^2 - 5r + 6) = 0$$

Char. eqn: $\longrightarrow \boxed{r^2 - 5r + 6 = 0}$

$$r = 2, \ r = 3$$

General solution:

$$\underline{\underline{y_t = C_1 \cdot 2^t + C_2 \cdot 3^t}}$$