LECTURE 11

GRA 6035

MATHEMATICS

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PLAN:

Reading:

[FMEA] 5.1 – 5.7, 6.1 – 6.4

1. First order differential equation
   - Review of diff. eqn's solvable by direct integration and separable diff. eqn's.
   - Linear first order diff. eqn. (integrating factor)
   - Exact differential eqn.

2. Second order linear diff. eqn.
   - Homogeneous case

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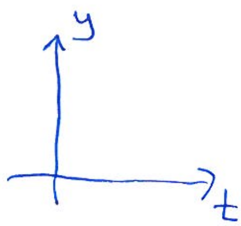
1. First order differential equations

equations that contain y', y, t



A. Solvable by direct integration

$$\boxed{y' = f(t)} \qquad \text{Solution: } y = \int f(t)\,dt$$

Ex: $ty' = 1$

$$\Updownarrow$$

$$y' = 1/t \qquad y = \int 1/t \, dt = \underline{\underline{\ln|t| + C}}$$

(B) Separable diff. eqn's

$$\boxed{y' = f(y) \cdot g(t)}$$

Solution: $\frac{1}{f(y)} \cdot y' = g(t)$

$\int \ldots dt$ $\quad \int \frac{1}{f(y)}\, dy = \int g(t)\, dt$

$y'\,dt = dy$

Ex: $y' = \underbrace{e^{y}}_{f(y)} \cdot \underbrace{t}_{g(t)}$

$\frac{1}{e^y} \cdot y' = e^{-y} y' = t \qquad \int e^{-y}\, dy = \int t\, dt$

$$-e^{-y} = \frac{1}{2}t^2 + C$$

solution in implicit form

$$e^{-y} = -\frac{1}{2}t^2 - C$$

$$-y = \ln\left|-\frac{1}{2}t^2 - C\right|$$

$$y = -\ln\left|-\frac{1}{2}t^2 - C\right| \quad \left(= -\ln\left|\frac{t^2}{2} + C\right|\right)$$

(explicit solution)

(C) Linear first order differential equations

$$\boxed{y' + a(t)y = b(t)}$$

where $a(t)$, $b(t)$ are functions in t

$$\left(y' = \underbrace{-a(t)y + b(t)}_{\text{linear in } y} \right)$$

Method: Integrating factor

Ex: $y' + ay = b$ where a, b are constants

$uy' + auy = bu$

$(uy)' = bu$

$uy = \int bu\,dt$

IC: integrating factor ($\overset{u(t)}{"u"}$)

$uy' + auy = (u \cdot y)'$

$\cancel{uy'} + auy = \cancel{u \cdot y'} + u'y$

$au \cdot y = u' \cdot y$

Choose u s.t. $u' = a \cdot u$

$\boxed{u = e^{at}}$ gives $u' = e^{at} \cdot a$
$= a \cdot u$

Let us start again:

$y' + ay = b \quad | \cdot e^{at}$

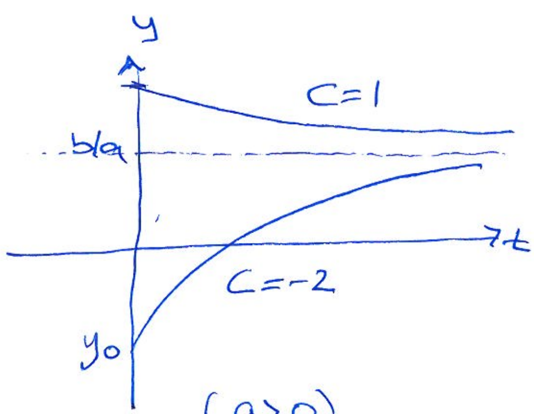
$\underbrace{e^{at} \cdot y' + a e^{at} y}$ $= b e^{at}$

$(e^{at} \cdot y)' = b e^{at}$

$e^{at} \cdot y = \int b e^{at}\,dt$

$e^{at} \cdot y = b \cdot e^{at} \cdot \frac{1}{a} + C$

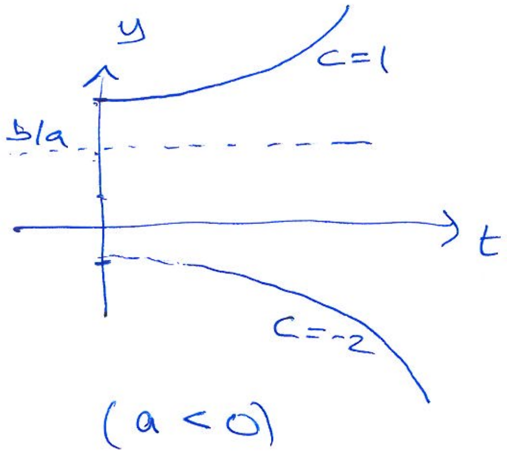
$$y = \frac{b}{a} + C e^{-at}$$



$(a > 0)$

$t=0$: $y_0 = \frac{b}{a} + C \cdot e^0 = \frac{b}{a} + C$

$$\boxed{y_0 = \frac{b}{a} + C}$$



$$y = \frac{b}{a} + C \cdot e^{-at}$$

Long term behaviour: $t \to \infty$

a>0: $\lim_{t \to \infty} y(t) = \frac{b}{a}$ so $y = \frac{b}{a}$ is equilibrium state

diff. eqn. stable

it is also globally asymptotically stable

$$\left(\lim_{t \to \infty} C e^{-at} = 0 \right)$$

a<0: $\lim_{t \to \infty} y(t) = \pm \infty$ diff. eqn. is unstable

(unless $C = 0 \iff y_0 = b/a$)

Ex:

$$\left.\begin{aligned} D &= a - bP \\ S &= \alpha + \beta P \\ p' &= \lambda \cdot (D - S) \end{aligned}\right\} \quad \begin{array}{l} a, b, \\ \alpha, \beta, \\ \lambda \end{array} \text{ are positive constants}$$

$$\Downarrow$$

$$p' = \lambda \cdot \left((a - bP) - (\alpha + \beta P) \right)$$

$$p' = \lambda (a - \alpha) - \lambda (b + \beta) p$$

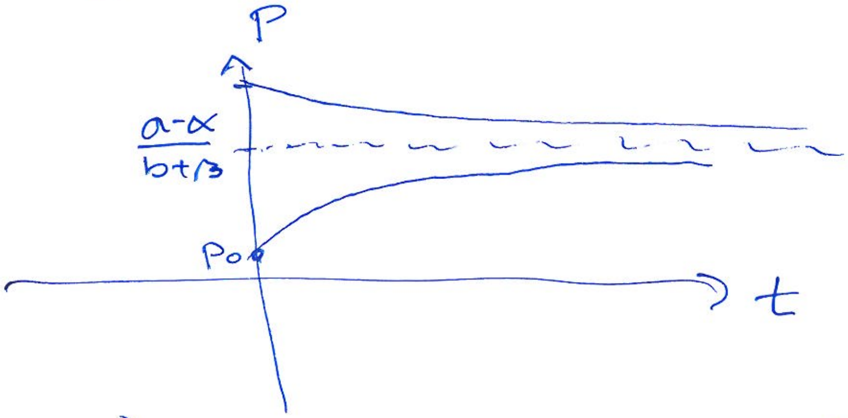
$$\boxed{p' + \lambda (b + \beta) p = \lambda (a - \alpha)}$$

linear first order differential equation

General solution:

$$p = \frac{\cancel{\lambda}(a - \alpha)}{\cancel{\lambda}(b + \beta)} + C e^{-\lambda (b + \beta) t} \qquad \lambda (b + \beta) > 0$$

$$p = \frac{a - \alpha}{b + \beta} + C e^{-\lambda (b + \beta) t}$$



Ex:

$$\left.\begin{aligned} D &= 5000 - 4P \\ S &= 1000 + 6P \\ p' &= 0.5 (D - S) \end{aligned}\right\} \qquad p(t) = \frac{4000}{10} + C \cdot e^{-0.5(10) t}$$

$$p = 400 + C e^{-5t}$$

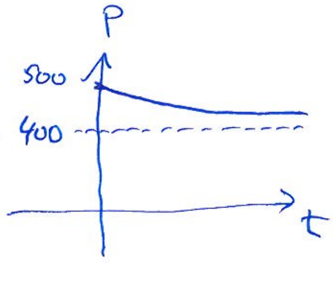
$$p_0 = 500$$

$$\Downarrow$$

$$p_0 = 400 + C$$

$$C = 100$$

$$p = \underline{400 + 100 e^{-5t}}$$



General case: Linear diff. eqn:

$$y' + a(t)y = b(t)$$

$$\underbrace{uy' + a(t)uy}_{} = b(t)u$$

$$(uy)' = b(t)u$$

$$u \cdot y = \int b(t) \cdot u \, dt$$

$$\boxed{y = \frac{1}{u} \cdot \int b(t) u \, dt}$$, where $u = e^{\int a(t)dt}$

Integrating factor: u

$$u' = a(t) \cdot u$$

$$u = e^{\int a(t)dt}$$

Ex: $y' - 2t \cdot y = t$

$$\left(e^{-t^2} \cdot y\right)' = te^{-t^2}$$

$$e^{-t^2} y = \int t \cdot e^{-t^2} dt$$

$$= \int \cancel{t} e^{u} \cdot \frac{du}{-2\cancel{t}}$$

IF: $\int a(t)dt = \int -2t \, dt$
$= -t^2 + C$
$\Downarrow$
$u = e^{-t^2}$ (C=0)

$$\boxed{\begin{aligned} u &= -t^2 \\ du &= -2t \cdot dt \end{aligned}}$$

$du = u' \cdot dt$

$$e^{-t^2} \cdot y = \frac{1}{-2} \int e^{u} du = -\frac{1}{2} e^{u} + C = -\frac{1}{2} e^{-t^2} + C$$

$$y = -\frac{1}{2} + Ce^{t^2}$$

(unstable)

$\lim_{t \to \infty} y(t) = \pm \infty$
(unless C=0)

D Exact differential equations

Ex: $1+ty^2+t^2y\cdot y'=0$

$y' = -\frac{1+ty^2}{t^2y}$

$(h(t,y))'_t = 0$ ← $y=y(t)$

$h(t,y) = C$

Question:

Is there a $h(t,y)$ s.t. $h(t,y)'_t$ is the left hand side of the diff. eqn.

If so, the diff. eqn. is called exact, and we can solve it

Try: $h = t^2 y^2/2 + t$

h'_t means $\frac{d}{dt}h$, the total derivative of h (not partial derivative)

$h'_t = \cancel{2}t\cdot y^2_{/\cancel{2}} + t^2\cdot\cancel{2}y\cdot y'_{/\cancel{2}}$

$= ty^2 + t^2y\cdot y' + 1$

Conclusion:

The eqn. is exact, with $h=\frac{t^2y^2}{2}+t$

$$\left(\frac{t^2y^2}{2}+t\right)'_t = 0$$

$$\frac{t^2y^2}{2}+t = C$$

implicit solution

$$t^2y^2/2 = C - t$$

$$y^2 = \frac{2(C-t)}{t^2}$$

$$y = \pm\sqrt{\frac{2(C-t)}{t^2}}$$

In general: Exact differential equations

$$\boxed{f(t,y) + g(t,y)\cdot y' = 0}$$

This equation is exact if there is a function $h(t,y)$ s.t.

$$\frac{d}{dt}h(t,y) = f(t,y) + g(t,y)\cdot y'$$

$$\Updownarrow$$

$$\boxed{\frac{\partial h}{\partial t} = f(t,y) \text{ and } \frac{\partial h}{\partial y} = g(t,y)}$$

$$\Updownarrow$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial t}$$

this can be used to check if the eqn. is exact, but not to find h

$$\frac{d}{dt}h(t,y) = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y}\cdot \underbrace{y'}_{=\frac{dy}{dt}}$$

Ex: $\underbrace{1 + ty^2}_{f} + \underbrace{t^2y}_{g}\cdot y' = 0$

$$\boxed{\begin{aligned}\frac{\partial h}{\partial t} &= 1+ty^2\\ \frac{\partial h}{\partial y} &= t^2y\end{aligned}}$$

① $\Rightarrow$ $h = \int 1+ty^2\,dt = t + y^2\cdot\frac{1}{2}t^2 + C$

$= t + \frac{1}{2}t^2y^2 + C(y)$

② $\Rightarrow$ $\frac{\partial h}{\partial y} = 0 + \frac{1}{\cancel{2}}t^2\cdot\cancel{2}y + C'(y)$

$= t^2y + C'(y) = t^2y$

ok if $C'(y) = 0$ $\quad C(y) = 0$

$$\Downarrow$$

$$\underline{h = t + \frac{1}{2}t^2y^2}$$

Solution: $h = C$

$t + \frac{1}{2}t^2y^2 = C$

$$\underline{y = \pm\sqrt{\frac{2(C-t)}{t^2}}}$$

(2) Second order linear diff. eqn.

In general:

$$y'' + ay' + by = f(t)$$

a, b constants

$f(t)$: fn. of t

Ex: $y'' - 5y' + 6y = 0$

Try:
$$\begin{cases} y = e^{rt} \\ y' = e^{rt} \cdot r \\ y'' = e^{rt} \cdot r^2 \end{cases}$$

$$\Downarrow$$

$$r^2 e^{rt} - 5re^{rt} + 6e^{rt} = 0$$

$$e^{rt} \cdot (r^2 - 5r + 6) = 0$$

$$\boxed{r^2 - 5r + 6 = 0}$$

Char. eqn:

Solution: $r=2$, $r=3$

Solution to diff. eqn:

e^{2t}, e^{3t}

General solution:

$$y = C_1 e^{2t} + C_2 \cdot e^{3t}$$

Comment: $y'' + ay' + by = 0$ is called homogeneous

$y'' + ay' + by = f(t)$ —"— inhomogeneous if $f(t) \not\equiv 0$

In both cases, it is called second order diff. eqn. that is linear with constant coefficients

(a, b are constants)

Homogeneous case:

$$\boxed{y'' + ay' + by = 0} \quad (a, b \text{ constants})$$

Characteristic equation:

$$r^2 + ar + b = 0 \implies r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Cases:

i) $a^2 - 4b > 0$: (two distinct characteristic roots r_1 and r_2)

General solution: $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

ii) $a^2 - 4b = 0$: (one double root $r = -\frac{a}{2}$)

General solution: $y = C_1 e^{rt} + C_2 t e^{rt} = (C_1 + C_2 t) e^{rt}$

iii) $a^2 - 4b < 0$: (no real roots) Put: $\begin{cases} \alpha = -a/2 \\ \beta = \sqrt{4b - a^2}/2 \end{cases}$

General solution: $y = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$