BI NORWEGIAN BUSINESS SCHOOL

Solutions:	**GRA 60353** **Mathematics**	
Examination date:	11.06.2013 09:00 – 12:00	Total no. of pages: 4
Permitted examination support material:	A bilingual dictionary and BI-approved calculator TEXAS INSTRUMENTS BA II Plus	
Answer sheets:	Squares	
	Counts 80% of GRA 6035	The subquestions have equal weight
Re-sit exam		Responsible department: Economics

Question 1.

(a) We compute the determinant of A using cofactor expansion along the first column, and find that

$$\det(A) = \begin{vmatrix} 2 & 4 & s \\ -4 & -6 & -3 \\ s & s & 1 \end{vmatrix} = 2(-6+3s) + 4(4-s^2) + s(-12+6s) = \mathbf{2s^2 - 6s + 4}$$

Since $\det(A) \neq 0$ for $s \neq 1, 2$, and the minor $\begin{vmatrix} 2 & 4 \\ -4 & -6 \end{vmatrix} = 4$ of order two is non-zero, we have that

$$\operatorname{rk}(A) = \begin{cases} \mathbf{3}, & s \neq 1, 2 \\ \mathbf{2}, & s = 1, 2 \end{cases}$$

(b) When $s = 0$, the characteristic equation of A is given by

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 & 0 \\ -4 & -6-\lambda & -3 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Cofactor expansion along the third row gives

$$(1-\lambda)(\lambda^2 + 4\lambda + 4) = (1-\lambda)(\lambda+2)^2 = 0$$

The eigenvalues are therefore $\lambda = \mathbf{1}$ and $\lambda = \mathbf{-2}$, where the last eigenvalue has multiplicity two. When $\lambda = -2$, the eigenvectors are given by $(A + 2I)\mathbf{x} = \mathbf{0}$, and the matrix

$$A + 2I = \begin{pmatrix} 4 & 4 & 0 \\ -4 & -4 & -3 \\ 0 & 0 & 3 \end{pmatrix}$$

has rank two since $A + 2I$ has a non-zero minor $\begin{vmatrix} 4 & 0 \\ -4 & -3 \end{vmatrix} = -12$ of order two — it cannot have rank three since $\lambda = -2$ is an eigenvalue. Therefore, the linear system has just one free variable while $\lambda = -2$ is an eigenvalue of multiplicity two. So A is **not diagonalizable** when $s = 0$.

Question 2.

(a) We compute the partial derivatives and the Hessian matrix of f:

$$\begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \begin{pmatrix} y^2 + 15x^2y - a^2y \\ 2xy + 5x^3 - a^2x \end{pmatrix}, \quad f'' = \begin{pmatrix} 30xy & 2y + 15x^2 - a^2 \\ 2y + 15x^2 - a^2 & 2x \end{pmatrix}$$

(b) We compute the stationary points, which are given by the equations

$$y^2 + 15x^2y - a^2y = 0, \quad 2xy + 5x^3 - a^2x = 0$$

The first equation give $y = 0$ or $y + 15x^2 - a^2 = 0$. If $y = 0$, then the second equation gives $5x^3 - a^2x = 0$, which means that $x = 0$ or $x = \pm a/\sqrt{5}$. This gives stationary points

$$(0,0), (a/\sqrt{5}, 0), (-a/\sqrt{5}, 0)$$

If $y \neq 0$, then $y = a^2 - 15x^2$, and the second equation gives $x = 0$ or $2y + 5x^2 - a^2 = 0$. In the first case, $x = 0$ and $y = a^2$. In the second case, $2(a^2 - 15x^2) + 5x^2 - a^2 = a^2 - 25x^2 = 0$, or $x = \pm a/5$ and $y = 2a^2/5$. We get stationary points with $y \neq 0$ given by

$$(0, a^2), (\pm a/5, 2a^2/5)$$

To find the local maximum, we look at the leading principal minors of $f''(x^*, y^*)$ for each stationary point (x^*, y^*). We see that all the stationary points with $x = 0$ or $y = 0$ are saddle points, since $D_2 < 0$ when $a > 0$. For $(x^*(a), y^*(a)) = (\pm a/5, 2a^2/5)$, we have

$$D_2 = \frac{24}{25}a^4 - (\frac{2}{5}a^2)^2 = \frac{20}{25}a^4 > 0$$

and $D_1 = 30xy = \pm 12/5\ a^3$. This means that there is exactly one local maximum point for a given $a > 0$, given by $(x^*(a), y^*(a)) = (-a/5, 2a^2/5)$. The point $(a/5, 2a^2/5)$ is a local minimum point.

(c) Let $a > 0$. By the Envelope Theorem, we have that

$$\frac{d}{da} f^*(a) = \frac{\partial f}{\partial a}\Big|_{(x,y)=(x^*(a),y^*(a))} = (-2axy)\Big|_{(x,y)=(x^*(a),y^*(a))} = \frac{4}{25}\ a^4 > 0$$

Since the derivative is positive, the local maximal value will **increase** when a increases. We could also compute $f^*(a) = f(x^*(a), y^*(a)) = 4/125\ a^5$ explicitly for $a > 0$, and use this to see that $f^*(a)$ increases when a increases.

Question 3.

(a) The equation $y'' = -15$ gives $y' = -15t + C_1$ and $y = -7.5t^2 + C_1t + C_2$. Alternatively, we can solve the differential equation as a second order linear equation. The conditions $y(0) = 695$ and $y'(0) = 55.5$ give $C_2 = 695$ and $C_1 = 55.5$, and the solution is

$$y = -7.5t^2 + 55.5t + 695$$

(b) The equation $y' = (1 - 3t^2)y^2$ is separable, and can be written as

$$1/y^2 \cdot y' = 1 - 3t^2 \quad \Rightarrow \quad \int y^{-2}\,\mathrm{d}y = \int 1 - 3t^2\,\mathrm{d}t$$

This gives $-1/y = t - t^3 + \mathcal{C}$. The initial condition $y(0) = -1$ gives $1 = \mathcal{C}$, and therefore $1/y = t^3 - t - 1$. The solution is

$$y = \frac{1}{t^3 - t - 1}$$

(c) The differential equation $(2y - t)e^{y^2 - yt}y' - ye^{y^2 - yt} = 0$ is exact if and only if there is a function $h(t, y)$ such that

$$\frac{\partial h}{\partial t} = -ye^{y^2 - yt}, \quad \frac{\partial h}{\partial y} = (2y - t)e^{y^2 - yt}$$

We see that $h(t,y) = e^{y^2-yt}$ is a solution to these equations. Therefore the solution of the exact differential equation is given by

$$h(t,y) = e^{y^2-yt} = C$$

The initial condition $y(0) = 1$ gives $e^1 = C$. This means that the solution is given implicitly by

$$e^{y^2-yt} = e \quad \Rightarrow \quad y^2 - yt = 1 \quad \Rightarrow \quad y^2 - yt - 1 = 0$$

This has explicit solution

$$y = \frac{t \pm \sqrt{t^2+4}}{2} = \frac{t}{2} + \frac{\sqrt{t^2+4}}{2}$$

where the sign is + since $y(0) = 1$.

Question 4.

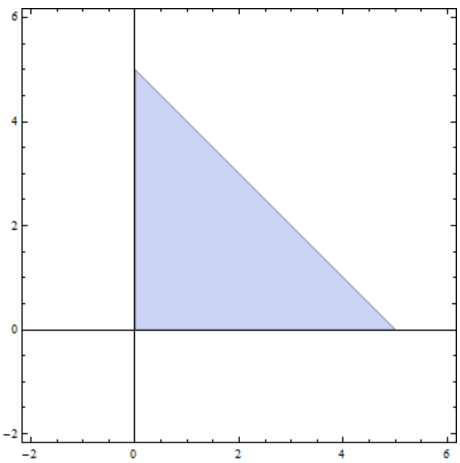
(a) For the sketch, see the figure below. We see from the sketch that the region defined by the constraints is closed and bounded. Alternatively, it is closed since it is defined by closed inequalities $\leq$ and $\geq$ and bounded since any point in the region must satisfy $0 \leq x \leq 5$ and $0 \leq y \leq 5$. Since f is continuous, there is a solution to the optimization problem, and we know that a solution must either satisfy the Kuhn-Tucker conditions, or else be a point where NDCQ fails. We therefore check NDCQ: Each admissible point is either i) an interior point (with $x > 0$, $y > 0$ and $x + y < 5$), ii) a point on one of the sides in the triangle but not a corner (where one of constraints is binding) or iii) one of the three corners (where two of the constraints are binding). In case i) the NDCQ condition is empty and therefore satisfied. In case ii) we must chech that the rank of the matrix of partial derivatives of g_i has rank one, where $g_1 = -x$, $g_2 = -y$ or $g_3 = x + y$ correspond to the three sides. The corresponding matrices

$$\begin{pmatrix} -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \end{pmatrix}$$

all have rank one, so NDCQ is satisfied in case ii). In case iii) we must check that the rank of the matrix of partial derivatives of two of the g_i's is two. We see that the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

all have rank two, and NDCQ is also satisfied in case iii). We conclude that there are no admissible points where NDCQ fails, and therefore there must a solution to the optimization problem where the Kuhn-Tucker conditions are satisfied.



(b) We rewrite the constraints as $g_1 = -x \leq 0$, $g_2 = -y \leq 0$ and $g_3 = x + y \leq 5$, and obtain the Lagrangian

$$\begin{aligned}\mathcal{L} &= xy^2 + 5x^3y - xy - \lambda(x+y) - \nu_1(-x) - \nu_2(-y) \\ &= xy^2 + 5x^3y - xy - \lambda(x+y) + \nu_1 x + \nu_2 y\end{aligned}$$

The Kuhn-Tucker conditions for this problem are the first order conditions

$$\begin{aligned}\mathcal{L}'_x &= y^2 + 15x^2y - y - \lambda + \nu_1 = 0 \\ \mathcal{L}'_y &= 2xy + 5x^3 - x - \lambda + \nu_2 = 0\end{aligned}$$

the constraints $x+y \leq 5$ and $x, y \geq 0$, and the complementary slackness conditions $\lambda, \nu_1, \nu_2 \geq 0$ and

$$\lambda(x+y) = 0, \quad \nu_1 x = 0, \quad \nu_2 y = 0$$

There is one obvious solution of the Kuhn-Tucker conditions, given by $(x, y) = (0, 0)$, with $\lambda = \nu_1 = \nu_2 \geq 0$, where the value of f is given by $f(0,0) = 0$. Let us check if there are other solutions of the Kuhn-Tucker conditions: If $\nu_1 > 0$, then $x = 0$ and $\lambda = \nu_2 > 0$ by the second FOC, and this implies that $x + y = 0$ and therefore $y = 0$. Similarly, if $\nu_2 > 0$, then $y = 0$ and $\lambda = \nu_1 > 0$ by the first FOC, and this implies that $x + y = 0$ and therefore $x = 0$. We may therefore assume that $\nu_1 = \nu_2 = 0$ when we look for solutions $(x, y) \neq (0, 0)$. If $\lambda > 0$, then $x + y = 0$ and this implies $x = y = 0$ since $x, y \geq 0$. We may therefore assume that $\lambda = 0$ when we look for solutions $(x, y) \neq (0, 0)$. Then we have the FOC

$$y^2 + 15x^2y - y = 0, \quad 2xy + 5x^3 - x = 0$$

We see that this is exactly the same condition as in Question 2 with $a = 1$. Therefore the solution is $(x, y) = (0, 0), (\pm 1/\sqrt{5}, 0), (0, 1), (\pm 1/5, 2/5)$. Of these solutions to the FOC's, only $(0, 0)$, $(1/\sqrt{5}, 0)$, $(0, 1)$ and $(1/5, 2/5)$ satisfy the constraints $x, y \geq 0$ and $x + y \leq 5$. Since $f(0, 0) = f(1/\sqrt{5}, 0) = f(0, 1) = 0$ and $f(1/5, 2/5) = -4/125$, it follows that the solution to the optimization problem is

$$(x, y) = (1/5, 2/5) \text{ with } \lambda = \nu_1 = \nu_2 = 0$$

and with minimum value $f(1/5, 2/5) = -4/125$.