Problem Session 6 — GRA 6035

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Problem Sheet 11:

3, 6a, 7, 8, 9

Problem Sheet 12:

6c, 8, load, 11, 12

Problem Sheet 11:

3) $\ddot{x} = \dot{x} + t, \quad x(0)=1, \quad x(1)=2$

$\ddot{x} - \dot{x} = t \Rightarrow x = x_h + x_p = \underline{C_1 + C_2 e^t - \frac{1}{2}t^2 - t}$

$\underline{x_h}$: $\ddot{x} - \dot{x} = 0$

$r^2 - r = 0$

$r = 0, 1 \Rightarrow x_h = C_1 e^{0t} + C_2 e^{1\cdot t} = \underline{C_1 + C_2 e^t}$

$\underline{x_p}$: $\left.\begin{array}{l} f(t) = t \\ f'(t) = 1 \\ f''(t) = 0 \end{array}\right\}$ $\begin{array}{l} x = At + B \\ \dot{x} = A \\ \ddot{x} = 0 \end{array} \Rightarrow \begin{array}{l} \ddot{x} - \dot{x} = t \\ 0 - A = t \\ -A = t \text{ impossible.} \end{array}$

$x = At^2 + Bt \Rightarrow 2A - (2At + B) = t$

$\dot{x} = 2A\cdot t + B \qquad \underline{2A} - 2A\cdot t - \underline{B} = t$

$\ddot{x} = 2A \qquad (2A - B) + (-2A)\cdot t = t$

$2A - B = 0$

$-2A = 1 \Rightarrow \boxed{\begin{array}{l} B = -1 \\ A = -1/2 \end{array}}$

$x_p = \underline{-\frac{1}{2}t^2 - t}$

$$x(t) = C_1 + C_2 e^t - \frac{1}{2}t^2 - t \quad , \quad \begin{cases} x(0)=1 \\ x(1)=2 \end{cases}$$

$$1 = C_1 + C_2 e^0 - 0 - 0 \Rightarrow C_1 + C_2 = 1$$

$$2 = C_1 + C_2 e^1 - \frac{1}{2}\cdot 1^2 - 1 \Rightarrow C_1 + C_2 e = 3/2$$

$$C_2 = 1 - C_1 \Rightarrow C_1 + (1-C_1)e = 3/2$$

$$C_1(1-e) = 3/2 - e$$

$$\underline{C_1 = \frac{3/2 - e}{1-e}} \quad , \quad C_2 = \underline{1 - C_1}$$

6 a) $\ddot{x} + 2\dot{x} + x = t^2 \quad , \; x(0)=0 \; , \; \dot{x}(0) = 1$

$$x = x^h + x^p = \underline{C_1 e^{-t} + C_2 t e^{-t} + t^2 - 4t + 6}$$

$\underline{x_h}$: $\ddot{x} + 2\dot{x} + x = 0$

$r^2 + 2r + 1 = 0$

$\underline{r = -1}$ (double root) $\Rightarrow x_h = C_1 e^{-1\cdot t} + C_2 \cdot t e^{-1\cdot t}$

$= \underline{C_1 e^{-t} + C_2 t e^{-t}}$

$\underline{x_p}$: $\left.\begin{array}{l} f(t) = t^2 \\ f' = 2t \\ f'' = 2 \end{array}\right\}$ $\begin{array}{l} x = \underline{At^2 + Bt + C} \\ \dot{x} = 2A\cdot t + B \\ \ddot{x} = 2A \end{array}$ $\Rightarrow \ddot{x} + 2\dot{x} + x = t^2$

$\left\{\begin{array}{l} (\underline{At^2} + \underline{Bt} + \underline{C}) \\ + 2(\underline{2At} + \underline{B}) + \underline{2A} \end{array}\right\} = t^2$

$\underset{\substack{\shortparallel \\ 1}}{(A)}t^2 + \underset{\substack{\shortparallel \\ 0}}{(B + 4A)}t + \underset{\substack{\shortparallel \\ 0}}{(C + 2B + 2A)} = t^2$

$\underline{A = 1}$ $\quad B = \underline{-4}$ $\quad C = -2(-4) - 2\cdot 1 = \underline{6}$

$$x_p = \underline{t^2 - 4t + 6}$$

$$x = c_1 e^{-t} + c_2 t e^{-t} + t^2 - 4t + 6, \quad \begin{cases} x(0) = 0 \\ \dot{x}(0) = 1 \end{cases}$$

$$\dot{x} = -c_1 e^{-t} + c_2 (1 \cdot e^{-t} + t \cdot (-1) e^{-t}) + 2t - 4$$

$$= -c_1 e^{-t} + c_2 \left(e^{-t} - t e^{-t} \right) + 2t - 4$$

$\underline{x(0) = 0:}$ $\quad 0 = c_1 + 6 \Rightarrow \underline{c_1 = -6}$

$\underline{\dot{x}(0) = 1:}$ $\quad 1 = -c_1 + c_2 - 4 \Rightarrow c_2 = 1 + 4 + (-6)$

$$= \underline{-1}$$

$$\underline{\underline{x = -6e^{-t} - te^{-t} + t^2 - 4t + 6}}$$

7.) $\ddot{x} + a\dot{x} + bx = 0$

$\underline{\text{Assumption:}}$ $\frac{1}{4}a^2 - b = 0$

$\Downarrow$

double root

$r = -\frac{a}{2}$

$$\left\{ \begin{aligned} & r^2 + ar + b = 0 \\ & r = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \\ & = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b} \end{aligned} \right.$$

Show that: $\left\{ \begin{aligned} & \text{Let } x(t) = u(t) \cdot e^{rt} \text{ is a solution} \\ & \Updownarrow \\ & \ddot{u} = 0 \end{aligned} \right.$

$$x = u \cdot e^{rt}$$

$$\dot{x} = \dot{u} \cdot e^{rt} + u \cdot e^{rt} \cdot r$$

$$\ddot{x} = \underbrace{\ddot{u} e^{rt} + \dot{u} e^{rt} \cdot r}_{} + \underbrace{\dot{u} e^{rt} r + u \cdot e^{rt} \cdot r \cdot r}_{}$$

$$\ddot{x} + a\dot{x} + bx = 0$$

$$(\ddot{u}e^{rt} + 2\dot{u}re^{rt} + ur^2e^{rt}) + (\dot{u}e^{rt} + ure^{rt}) \cdot a$$

$$+ (ue^{rt}) b = 0$$

$$e^{rt} \cdot (\ddot{u} + 2r\dot{u} + ur^2 + \dot{u}a + ura + ub) = 0$$

$$\ddot{u} + 2r\dot{u} + r^2u + \dot{u}a + ura + ub = 0$$

$r = -\frac{a}{2}$:
$$\ddot{u} + \cancel{(-a)\dot{u}} + \frac{a^2}{4}u + \cancel{\dot{u}a} + u \cdot \left(-\frac{a}{2}\right) \cdot a + ub = 0$$
$$\ddot{u} + \frac{a^2}{4}u - \frac{a^2}{2}u + bu = 0$$
$$\ddot{u} + -\frac{a^2}{4}u + bu = 0$$

$\frac{a^2}{4} - b = 0$:
$$\ddot{u} = 0 \Rightarrow \dot{u} = A \Rightarrow u = At + B$$

Solution: $u \cdot e^{rt} = (At + B)e^{rt}$

8. a) $t^2\ddot{x} + 5t\dot{x} + 3x = 0$ (Euler equation)

Check if $x = t^r$ is a solution:

$$\left.\begin{array}{l} x = t^r \\ \dot{x} = r \cdot t^{r-1} \\ \ddot{x} = r(r-1)t^{r-2} \end{array}\right\} \begin{array}{l} t^2\ddot{x} + 5t\dot{x} + 3x = 0 \\ t^2 \cdot r(r-1)t^{r-2} + 5t \cdot rt^{r-1} + 3 \cdot t^r = 0 \\ t^r \cdot (r(r-1) + 5r + 3) = 0 \\ t^r(r^2 + 4r + 3) = 0 \end{array}$$

t^r solution $\Longleftrightarrow r^2 + 4r + 3 = 0$
$r = -1, -3 \Rightarrow x = c_1 \cdot t^{-1} + c_2 \cdot t^{-3}$

9) $\ddot{x} + 2a\dot{x} - 3a^2 x = 100\, e^{bt}$ (a,b parameters)

$x = x_h + x_p$

$\underline{x_h}$: $r^2 + 2a\, r - 3a^2 = 0$

$$r = \frac{-2a \pm \sqrt{4a^2 - 4\cdot 1\cdot(-3a^2)}}{2\cdot 1}$$

$$= -a \pm \frac{\sqrt{16a^2}}{2} = -a \pm \frac{4|a|}{2}$$

$$= -a \pm 2|a|$$

$a>0$: $\begin{cases} r_1 = -a + 2a = +a \\ r_2 = -a - 2a = -3a \end{cases}$

$\underline{a=0}$: $r_1 = 0$ (double root)

$\underline{a<0}$: $r_1 = -a - 2a = -3a$
$r_2 = -a + 2a = +a$

$$x_h = \begin{cases} C_1 + C_2 t, & a = 0 \\ C_1 e^{at} + C_2 e^{-3at}, & a \neq 0 \end{cases}$$

$\underline{x_p}$: $f(t) = 100\, e^{bt}$, $f' = 100 b e^{bt}$, $f'' = 100 b^2 e^{bt}$

$x = A\cdot e^{bt}$
$\dot{x} = A\cdot b\, e^{bt}$
$\ddot{x} = A\cdot b^2 e^{bt}$

$\ddot{x} + 2a\dot{x} - 3a^2 x = 100\, e^{bt}$

$A b^2 \cancel{e^{bt}} + 2a A b \cancel{e^{bt}} - 3a^2 A \cancel{e^{bt}} = 100\cdot \cancel{e^{bt}}$

$A\cdot(b^2 + 2ab - 3a^2) = 100$

$A = \dfrac{100}{b^2 + 2ab - 3a^2}$ if $b^2 + 2ab - 3a^2 \neq 0$

$b^2 + 2ab + a^2 - 4a^2$

$(b+a)^2 - 4a^2 \neq 0$

$b + a \neq \pm 2a$

$$x_p = \frac{100}{(b+a)^2 - 4a^2}\, e^{bt}$$

if $b \neq a, -3a$

$x_p = ?$

if $b = a, -3a$

Problem Sheet 12:

10.) Exam Dec '07, no. 3

a) $\dot{x} = (t-2)\,x^2$, $x(0)=1$

* not linear
* separable? Yes

$$\frac{1}{x^2}\cdot\dot{x} = t-2 \quad \leftarrow \text{separated}$$

$$\int \frac{1}{x^2}\dot{x}\cdot dt = \int t-2\,dt$$

$$\int \frac{1}{x^2}\,dx = \int t-2\,dt$$

$$\frac{1}{-1}\cdot x^{-1} + c_1 = \frac{1}{2}t^2 - 2t + c_2$$

$$-\frac{1}{x} = \frac{1}{2}t^2 - 2t + c \qquad (c = c_2 - c_1)$$

$$\frac{1}{x} = -\frac{1}{2}t^2 + 2t - c$$

$$x = \frac{1}{-\frac{1}{2}t^2 + 2t - c} \qquad 1 = \frac{1}{0+0-c}$$

$$c = -1$$

$$x = \frac{1}{-\frac{1}{2}t^2 + 2t + 1}$$

d) $\left(3x^2 e^{x^3+3t}\right)\cdot\dot{x} + \left(3e^{x^3+3t} - 2e^{2t}\right) = 0$, $x(1) = -1$

- not linear - not seperable - <u>Exact?</u>

$$p\cdot\dot{x} + q = 0 \qquad \boxed{\begin{aligned} p &= 3x^2 e^{x^3+3t} \\ q &= 3e^{x^3+3t} - 2e^{2t} \end{aligned}}$$

<u>Defn:</u> $p\cdot\dot{x} + q = 0$ is <u>exact</u> if there is a function $u(x,t)$ such that

$$p = \frac{\partial u}{\partial x} \quad \text{and} \quad q = \frac{\partial u}{\partial t}$$

If it is exact, then the solution is $\boxed{u(x,t) = C}$

<u>Method I:</u> $p\dot{x} + q = 0$ is exact $\Longleftrightarrow \frac{\partial p}{\partial t} = \frac{\partial q}{\partial x}$

$$\frac{\partial p}{\partial t} = \left(3x^2 e^{x^3+3t}\right)'_t = 3x^2\cdot e^{x^3+3t}\cdot 3$$

$$\frac{\partial q}{\partial x} = \left(3e^{x^3+3t} - 2e^{2t}\right)'_x = 3e^{x^3+3t}\cdot 3x^2$$

ok Exact

<u>Method 2:</u> find $u(x,t)$

① $\boxed{\frac{\partial u}{\partial x} = p = 3x^2 e^{x^3+3t}}$ ② $\boxed{\frac{\partial u}{\partial t} = q = 3e^{x^3+3t} - 2e^{2t}}$

$$\frac{\partial u}{\partial x} = 3x^2 e^{x^3+3t} \Rightarrow u = \int 3x^2 e^{x^3+3t}\, dx = e^{x^3+3t} + C(t)$$

consider t as a constant

$$\Rightarrow \boxed{u = e^{x^3+3t} + C(t)}$$

$\underline{u = e^{x^3+3t} + C(t)}$: Use ② $\quad \underline{\frac{\partial u}{\partial t}} = \underline{3e^{x^3+3t} - 2e^{2t}}$

$$\frac{\partial u}{\partial t} = \left(e^{x^3+3t} + C(t)\right)'_t = \underline{e^{x^3+3t}} \cdot 3 + C'(t).$$

$$= \underline{3e^{x^3+3t}} - 2e^{2t}.$$

ok if and only if $C'(t) = -2e^{2t}$

$$C(t) = \int -2e^{2t}\,dt$$

$$= -e^{2t} + C_1$$

Conclusion: $u(x,t) = e^{x^3+3t} - e^{2t} + C_1$

Solution: $e^{x^3+3t} - e^{2t} + C_1 = C_2$

$$\boxed{e^{x^3+3t} - e^{2t} = C} \qquad (C = C_2 - C_1)$$

general solution in implicit form

$\cancel{e^{1+3} - e^{2} = C_1}$

$x(1) = -1$: $\quad e^{-1+3} - e^{2} = C$

$(t=1,\ x=-1)$ $\quad e^{2} - e^{2} = C \Rightarrow \underline{C = 0}$

$$e^{x^3+3t} - e^{2t} = 0$$

$$e^{x^3+3t} = e^{2t}$$

$$x^3 + 3t = 2t \Rightarrow x^3 = -t \Rightarrow x = \underline{\underline{\sqrt[3]{-t}}}$$

(explicit sol.)

11.) Borrow: K

Interest rate: r

Repayment: 500 first period

Increases with 10 for each period

b_t: outstanding balance after t periods

$$\boxed{b_{t+1} = b_t \cdot (1+r) - (500+10t)}$$

$$b_{t+1} - (1+r)b_t = -500-10t$$

$$b_t = b_t^h + b_t^p = \underline{\underline{C\cdot(1+r)^t + \frac{10}{r}\cdot t + \frac{500}{r} + \frac{10}{r^2}}}$$

$\underline{b_t^h}$: Char. eqn: $s-(1+r)=0$

$s = 1+r$ char. root

$$b_t^h = \underline{C\cdot(1+r)^t}$$

$\underline{b_t^p}$:

$$\left.\begin{aligned} f_t &= -500-10t \\ f_{t+1} &= -500-10(t+1) \\ &= -510-10t \end{aligned}\right\} \quad \begin{aligned} b_t &= \underline{At+B} \\ b_{t+1} &= A(t+1)+B \\ &= \underline{At+A+B} \end{aligned}$$

$$b_{t+1} - (1+r)b_t = -500-10t$$

$$(\underline{At}+\underline{A+B}) - (1+r)(\underline{At}+\underline{B}) = -500-10t$$

$$(\underbrace{-rA}_{=-10})t + (\underbrace{A+\cancel{B}-(\cancel{1}+r)B}_{=-500}) = -500-10t$$

$A = 10/r$, $A-rB = -500 \Rightarrow B = \dfrac{-500-10/r}{-r} = +\dfrac{500}{r} + \dfrac{10}{r^2}$

12)

a) $y' = y(1-y)$, $y(0) = 1/2$

$$\frac{1}{y(1-y)} y' = 1$$

$$\int \frac{1}{y(1-y)} dy = \int 1 \, dt$$

$$\int \frac{1}{y} + \frac{1}{1-y} dy = t + C_2$$

$$\ln |y| - \ln|1-y| = t + C$$

$$\ln \left|\frac{y}{1-y}\right| = t + C$$

$$\left|\frac{y}{1-y}\right| = e^{t+C}$$

$$\frac{y}{1-y} = \underbrace{\pm e^{c}}_{=K} e^{t} = K e^{t}$$

$$y = (1-y) \cdot K e^{t} = K e^{t} - y K e^{t}$$

$$y + y K e^{t} = K e^{t}$$

$$y = \frac{K e^{t}}{1 + K e^{t}}$$

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$

$$1 = \frac{A \cdot \cancel{y}(1-y)}{\cancel{y}} + \frac{B y \cancel{(1-y)}}{\cancel{1-y}}$$

$$1 = A \cdot (1-y) + B y$$

$A = B = 1$

c) $P_{t+2} = \frac{2}{3} P_{t+1} + \frac{1}{3} P_t$, $p_0 = 100$, $p_1 = 102$

$P_{t+2} - \frac{2}{3} P_{t+1} - \frac{1}{3} P_t = 0$ linear, homogeneous second order

$$r^2 - \frac{2}{3} r - \frac{1}{3} = 0$$

$$r = \frac{2/3 \pm \sqrt{4/9 - 4 \cdot 1 \cdot (-1/3)}}{2}$$

$$= 1/3 \pm \frac{\sqrt{4/9 + 12/9}}{2}$$

$$= 1/3 \pm \frac{\sqrt{16/9}}{2} = \frac{1}{3} \pm \frac{4/3}{2}$$

$$= \frac{1}{3} \pm \frac{2}{3} = \underline{1, -1/3}$$

$$P_t = c_1 \cdot t^1 + c_2 \cdot t^{-1/3}$$

$$= \underline{\underline{c_1 t + c_2 t^{-1/3}}}$$