E. Eriksen

4.2

Problem Session 2

6RA6035

a) $A = \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}$

$\lambda^2 - tr(A)\lambda + det(A) = 0$

Eigenvalues: $\begin{vmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{vmatrix} = 0$

$\lambda^2 + 6\lambda + 5 = 0$

$\underline{\underline{\lambda = -1, \lambda = -5}}$

$A\underline{x} = \lambda\underline{x}$

$(A - \lambda\underline{I})\underline{x} = \underline{0}$

Eigenvectors:

$\lambda = -1$: $\begin{pmatrix} 3 & -7 \\ 3 & -7 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\downarrow$

$\begin{pmatrix} 3 & -7 \\ 0 & 0 \end{pmatrix}$ $\quad 3x_1 - 7x_2 = 0$

$\begin{cases} x_1 = \frac{7}{3}x_2 \\ x_2 \text{ free} \end{cases}$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7/3\, x_2 \\ x_2 \end{pmatrix} = \underline{\underline{x_2 \cdot \begin{pmatrix} 7/3 \\ 1 \end{pmatrix}}}$

$\lambda = -5$: $\begin{pmatrix} 7 & -7 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad 7x_1 - 7x_2 = 0$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \underline{\underline{x_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$

4.3.

b) $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$

Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix} = (2-\lambda)\cdot(\lambda^2+\lambda-2) - 0 + 2\cdot(1+1-\lambda)$$

$$= (2-\lambda)(\lambda^2+\lambda-2) + 2\cdot(2-\lambda) = 0$$

$$= (2-\lambda)(\lambda^2+\lambda-2+2) = 0$$

$\lambda = 2$ or $\lambda^2+\lambda=0$

$\lambda=0,\ \lambda=-1$

Eigenvectors:

$\lambda=2$: $\begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 2 & 0 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x-2z=0$, $y-z=0$, z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ z \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

4.5

$$A = \begin{pmatrix} 1 & 18 & 30 \\ -2 & -11 & -10 \\ 2 & 6 & 5 \end{pmatrix} \qquad \underline{v_1} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \underline{v_2} = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}, \underline{v_3} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$A \cdot \underline{v_1} = \begin{pmatrix} +15 \\ -5 \\ 0 \end{pmatrix} = \underset{\substack{\| \\ (-5)}}{\lambda_1} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \quad \text{ok.}$$

$$A \cdot \underline{v_2} = \begin{pmatrix} 25 \\ 0 \\ -5 \end{pmatrix} = \underset{\substack{\| \\ (-5)}}{\lambda_2} \cdot \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \quad \text{ok.}$$

$\underline{v_1}, \underline{v_2}$ are linearly indep. since

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

i.e.

$$\operatorname{rk} \begin{pmatrix} -3 & -5 \\ +1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$$A \cdot \underline{v_3} = \begin{pmatrix} 15 \\ -5 \\ 5 \end{pmatrix} = \underset{\substack{\| \\ 5}}{\lambda_3} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \quad \underline{\text{ok.}}$$

$$D = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad P = \begin{pmatrix} -3 & -5 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow P^{-1}AP = D$$

$$A = PDP^{-1}$$

4.9

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 5 \end{pmatrix}$$ Is A diagonalizable?

Eigenvalues:

$$\begin{vmatrix} 4-\lambda & 1 & 2 \\ 0 & 3-\lambda & 0 \\ 1 & 1 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda) \begin{vmatrix} 4-\lambda & 2 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$\lambda_1 = 3$ or $\lambda^2 - 9\lambda + 18 = 0$

$\lambda_2 = 3, \; \lambda_3 = 6$

$\lambda = 3$ (mult. 2) $\quad \lambda_3 = 6$ (mult. 1)

→ Eigenvector $\lambda = 6$:

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -3 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x = z$

$y = 0$

z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Eigenvectors for $\lambda = 3$:

$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{matrix} \\ -1 \\ \end{matrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x = -y - 2z$

y free

z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-2z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

A is diagonalizable

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad P = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

5.4

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$$

$D_1 = \underline{1}$

$D_2 = \underline{0}$

$$D_3 = 0 - 5 \cdot \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} + 6 \cdot \overset{0}{\overset{\shortparallel}{D_2}}$$

$$= -5 \cdot 5 = \underline{-25}$$

$\Delta_1 = 1, 4, 6$

$\Delta_2 = 0, 6, -1$

$\Delta_3 = -25$

A is <u>indefinite</u>

pos. semidefn. $\Delta_1, \Delta_2, \Delta_3 \geq 0$ No.

neg semidefn. $\Delta_1 \leq 0, \Delta_2 \geq 0, \Delta_3 \leq 0$ No.

$Q(x,y,z) = x^2 + 4xy + 4y^2 + 10yz + 6z^2$ is <u>indefinite</u>

$(x,y,z) = (0,0,0)$ is a <u>saddle point</u>.

5.5

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

$D_1 = a$ $\quad\quad$ $\Delta_1 = a, c$

$D_2 = ac - b^2$ $\quad\quad$ $\Delta_2 = ac - b^2$

A pos. semidefinite: $\Delta_1 \geq 0, \Delta_2 \geq 0$

$$a \geq 0, \quad c \geq 0, \quad ac - b^2 \geq 0$$

A negative semidefinite: $\Delta_1 \leq 0, \Delta_2 \geq 0$

$$a \leq 0, \quad c \leq 0, \quad ac - b^2 \geq 0$$

A positive definite: $D_1 > 0, D_2 > 0$

$$a > 0, \quad ac - b^2 > 0$$

A negative definite: $D_1 < 0, D_2 > 0$

$$a < 0, \quad ac - b^2 > 0$$

5.10

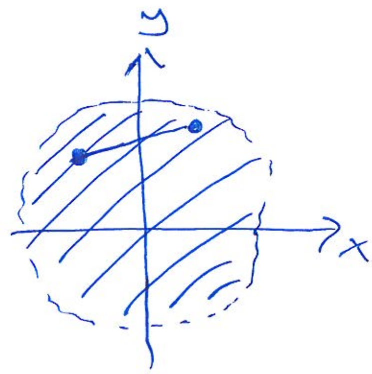
$$Q(x_1, x_2) = -2x_1^2 + 12x_1x_2 + 2x_2^2$$

i) $A = \begin{pmatrix} -2 & 6 \\ 6 & 2 \end{pmatrix}$ $\left.\begin{array}{l} D_1 = -2 \\ D_2 = -40 \end{array}\right\}$ indefinite

ii) $H(Q) = \begin{pmatrix} -4 & 12 \\ 12 & 4 \end{pmatrix}$ $\left.\begin{array}{l} D_1 = -4 \\ D_2 = -80 \end{array}\right\}$ indefn.

Problem 6.2

a) $\{ (x,y) : x^2 + y^2 \leq 2 \}$



Boundary: $x^2 + y^2 = 2$

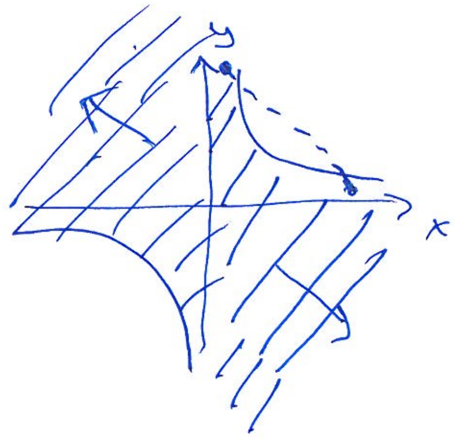
circle

open: boundary pts not in the set (not closed)

bounded: yes

convex: yes

c) $\{ (x,y) : xy \leq 1 \}$



Boundary: $xy = 1$

$y = 1/x$

$x > 0$: $xy \leq 1 \quad | \cdot \frac{1}{x}$

$y \leq 1/x$

$x < 0$: $xy \leq 1 \quad | \cdot \frac{1}{x}$

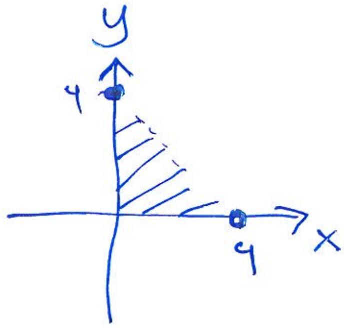
$y \geq 1/x$

closed: boundary pts are in the set (not open)

bounded: no

convex: no

f) $\{(x,y): \sqrt{x} + \sqrt{y} \le 2\}$



<u>Note</u>: $x \geq 0, \; y \geq 0$

<u>Boundary</u>: $\sqrt{x} + \sqrt{y} = 2$

$$\sqrt{y} = 2 - \sqrt{x}$$

$$y = (2-\sqrt{x})^2, \qquad 0 \le x \le 4$$

$$\sqrt{x} + \sqrt{y} \le 2$$

$$\sqrt{y} \le 2 - \sqrt{x}$$

the set →

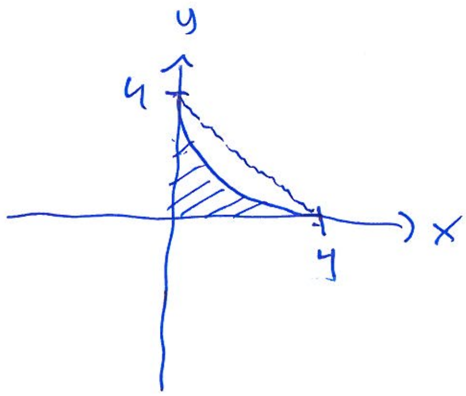
<u>Closed</u>: yes
(not open)
<u>bounded</u>: yes
<u>convex</u>: no

Convex funct.　　Concave funct.

Check if $y = (2-\sqrt{x})^2$ is a convex function

$$y' = \cancel{2} \cdot (2-\sqrt{x}) \cdot -\frac{1}{\cancel{2}\sqrt{x}}$$

$$= -\frac{2-\sqrt{x}}{\sqrt{x}} = -2x^{-1/2} + 1$$



$$y'' = (-2)\cdot(-1/2)x^{-3/2} = x^{-3/2}$$

$$= \frac{1}{x\sqrt{x}} > 0 \Rightarrow \text{convex fn.}$$

$$y = (2-\sqrt{x})^2$$

the function.

6.4 $f(x,y) = x - y - x^2$

Show that f is concave on $\mathbb{R}^2$.

$$H(f) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

Eigenvalues: $0, -2$ semi-

$\Rightarrow H(f)$ is neg. defn.

$\Rightarrow f$ is concave

$$g(x,y) = e^{x-y-\cancel{x}^2} = e^{f(x,y)} = e^u$$

$u = f(x,y)$:

$$g'_x = e^u \cdot u'_x = e^u \cdot (1-2x)$$

$$g'_y = e^u \cdot u'_y = e^u \cdot (-1)$$

$$g''_{xx} = e^u \cdot (1-2x) \cdot (1-2x) + e^u \cdot (-2)$$
$$= e^u \left((1-2x)^2 - 2 \right)$$

$$g''_{xy} = (1-2x) \cdot e^u \cdot (-1) = -(1-2x) e^u$$

$$g''_{yy} = -1 \cdot e^u \cdot (-1) = e^u$$

$$H(g) = \begin{pmatrix} e^u \cdot \left((1-2x)^2 - 2 \right) & -(1-2x) e^u \\ -(1-2x) e^u & e^u \end{pmatrix}$$

$$= e^u \cdot \begin{pmatrix} (1-2x)^2 - 2 & -(1-2x) \\ -(1-2x) & 1 \end{pmatrix}$$

$$D_2 = |H(g)| = (e^u)^2 \cdot \begin{vmatrix} (1-2x)^2-2 & -(1-2x) \\ -(1-2x) & 1 \end{vmatrix}$$

$$= (e^u)^2 \cdot \left(\cancel{(1-2x)^2} - 2 - \cancel{(1-2x)^2} \right)$$

$$= -2(e^u)^2 < 0 \Rightarrow H(g) \text{ is } \underline{\text{indefinite}}$$

$\Rightarrow$ g is neither convex nor concave