Problem Sheet 9 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. Maximize the function

$$f(x_1,x_2) = x_1^2 + x_2^2 + x_2 - 1$$

subject to $g(x_1,x_2) = x_1^2 + x_2^2 \leq 1$.

2. Solve max $(1 - x^2 - y^2)$ subject to $x \geq 2$ and $y \geq 3$ by a direct argument and then see what the Kuhn-Tucker conditions have to say about the problem.

3. Solve the following problem (assuming it has a solution):

$$\min\ 4\ln(x^2+2) + y^2 \text{ subject to } x^2 + y \geq 2, x \geq 1$$

4. Mock Final Exam in GRA6035 12/2010, 4
We consider the following optimization problem: Maximize $f(x,y,z) = xy + yz - xz$ subject to the constraint $x^2 + y^2 + z^2 \leq 1$.

a) Write down the first order conditions for this problem, and solve the first order conditions for x,y,z using matrix methods.
b) Solve the optimization problem. Make sure that you check the non-degenerate constraint qualification, and also make sure that you show that the problem has a solution.

5. Final Exam in GRA6035 10/12/2010, 4
We consider the function $f(x,y,z) = xyz$.

a) The function g is defined on the set $D = \{(x,y,z) : x > 0, y > 0, z > 0\}$, and it is given by
$$g(x,y,z) = \frac{1}{f(x,y,z)} = \frac{1}{xyz}$$
Is g a convex or concave function on D?
b) Maximize $f(x,y,z)$ subject to $x^2 + y^2 + z^2 \leq 1$.

Solutions

1 The Lagrangian is

$$\mathscr{L}(\mathbf{x}) = x_1^2 + x_2^2 + x_2 - 1 - \lambda(x_1^2 + x_2^2 - 1),$$

and the Kuhn-Tucker conditions are

$$\mathscr{L}_1' = 2x_1 - 2\lambda x_1 = 2x_1(1-\lambda) = 0$$
$$\mathscr{L}_2' = 2x_2 + 1 - 2\lambda x_2 = 2x_2(1-\lambda) + 1 = 0$$

and

$$\lambda \geq 0 \text{ and } \lambda = 0 \text{ if } x_1^2 + x_2^2 < 1.$$

From $\mathscr{L}_1' = 0$ we obtain $2x_1(1-\lambda) = 0$. Thus $x_1 = 0$ or $\lambda = 1$. If $\lambda = 1$, then $\mathscr{L}_2' = 2x_2 + 1 - 2x_2 = 1 \neq 0$, so we conclude that $x_1 = 0$.

CASE $x_1^2 + x_2^2 = 1$:

From $x_1^2 + x_2^2 = 1$ and $x_1 = 0$ we obtain $x_2^2 = 1$ or $x_2 = \pm 1$. We have $\mathscr{L}_2' = 2x_2(1-\lambda) + 1 = \pm 2(1-\lambda) + 1 = 0 \implies (1-\lambda) = \frac{-1}{\pm 2} = \mp \frac{1}{2} \implies \lambda = 1 \pm \frac{1}{2}$. Thus

$$(0,-1) \text{ corresponding to } \lambda = \frac{1}{2} \text{ and}$$
$$(0,1) \text{ corresponding to } \lambda = \frac{3}{2}$$

are candidates for maximum.

CASE $x_1^2 + x_2^2 < 1$:

From $x_1 = 0$ we get $x_2^2 < 1$. This is the same as to say $-1 < x_2 < 1$. Since $\lambda = 0$, $\mathscr{L}_2' = 2x_2 + 1 = 0$ gives $x_2 = -\frac{1}{2}$. We conclude that

$$(0, -\frac{1}{2}) \text{ corresponding to } \lambda = 0$$

is a candidate for maximum. We compute

$$f(0,1) = 1$$
$$f(0,-1) = -1 \text{ and}$$
$$f(0,-\frac{1}{2}) = -\frac{5}{4}$$

and conclude that $f(0,1) = 1$ is the maximal value.

2 See answers in FMEA (Exercise 3.5.1)

3 We will use the following general method of solving

$$\max f(x_1,\dots,x_n) \text{ subject to } \begin{cases} g_1(x_1,\dots,x_n) \le b_1 \\ \vdots \\ g_m(x_1,\dots,x_n) \le b_m \end{cases}$$

by applying the following steps:

a) $\mathscr{L} = f - \lambda_1 g_1 - \cdots - \lambda_m g_m$
b) $\mathscr{L}'_1 = 0, \mathscr{L}'_2 = 0, \dots, \mathscr{L}'_n = 0$ (FOC's)
c) $\lambda_j \ge 0$ and $\lambda_j = 0$ if $g_j(x_1,\dots,x_n) < b_j$
d) Require $g_j(x_1,\dots,x_n) \le b_j$

To transform the problem into this setting, we define

$$f(x,y) = -(4\ln(x^2+2)+y^2)$$

since minimizing $4\ln(x^2+2)+y^2$ is the same as maximizing $-(4\ln(x^2+2)+y^2)$. We also rewrite the constraints as

$$\begin{aligned} g_1(x,y) &= -x^2 - y \le -2 \\ g_2(x,y) &= -x \le -1 \end{aligned}$$

We define the Lagrange function:

$$\begin{aligned} \mathscr{L} &= -(4\ln(x^2+2)+y^2) - \lambda_1(-x^2-y) - \lambda_2(-x) \\ &= -4\ln(x^2+2) - y^2 + \lambda_1(x^2+y) + \lambda_2 x \end{aligned}$$

The first order conditions are the

$$\begin{aligned} &\mathscr{L}'_1 = -4\frac{1}{x^2+2}\cdot 2x + 2\lambda_1 x + \lambda_2 = \frac{-8x}{x^2+2} + 2\lambda_1 x + \lambda_2 = 0 \\ &\mathscr{L}'_2 = -2y + \lambda_1 = 0 \end{aligned}$$

Since there are two constraints, there are four cases to consider:

The case $-x^2 - y = -2$ and $-x = -1$:

Since $x = 1$, we deduce from $\mathscr{L}'_1 = 0$ that

$$\frac{-8\cdot 1}{1^2+2} + 2\lambda_1 \cdot 1 + \lambda_2 = 0 \iff 2\lambda_1 + \lambda_2 - \frac{8}{3} = 0$$

From $x^2 + y = 2$ and $x = 1$ we obtain that $y = 1$. From $\mathscr{L}'_2 = -2y + \lambda_1 = 0$ we the obtain that $\lambda_1 = 2$. Substituting this into $2\lambda_1 + \lambda_2 - \frac{8}{3} = 0$ we get

$$2\cdot 2 + \lambda_2 - \frac{8}{3} = 0 \iff \lambda_2 = -\frac{4}{3} < 0$$

This violates the complementary slackness conditions that says that $\lambda_2 \geq 0$ since the second constraint is active. We conclude that the case case $-x^2 - y = -2$ and $-x = -1$ does not lead to a solution.

The case $-x^2 - y = -2$ and $-x < -1$:

Since the second constraint is inactive, we get $\lambda_2 = 0$. Substituting this into $\frac{-8x}{x^2+2} + 2\lambda_1 x + \lambda_2 = 0$ we get

$$\frac{-8x}{x^2+2} + 2\lambda_1 x = 0 \Longleftrightarrow 2x(\lambda_1 - \frac{4}{x^2+2}) = 0$$

Since $x > 1$ this gives

$$\lambda_1 = \frac{4}{x^2+2}$$

From $-x^2 - y = -2$ we have that $y = 2 - x^2$ and substituting this and $\lambda_1 = \frac{4}{x^2+2}$ into $-2y + \lambda_1 = 0$ gives

$$-2(2-x^2) + \frac{4}{x^2+2} = 0 \Longleftrightarrow (x^2+2)(x^2-2) + 2 = 0 \Longleftrightarrow x^4 = 2$$

From this we obtain that

$$x = \pm\sqrt[4]{2} \cong \pm 1.1892$$

Since $x > 1$ we get that

$$x = \sqrt[4]{2}$$

From $y = 2 - x^2$ we obtain

$$y = 2 - \sqrt{2}$$

and from $\lambda_1 = 2y$ we get

$$\lambda_1 = 2(2 - \sqrt{2})$$

Thus we have the following candidate for optimum

$$\boxed{(\sqrt[4]{2}, 2 - \sqrt{2}) \longleftrightarrow \lambda_1 = 2(2-\sqrt{2}), \lambda_2 = 0}$$

The case $-x^2 - y < -2$ and $-x = -1$:

Since the first constraint is inactive, we get $\lambda_1 = 1$. Substituting this into $-2y + \lambda_1 = 0$ we get

$$y = 0.$$

Since $x = 1$ by assumption, we see that $-x^2 - y = -1$ which is not less that -2 so the first constraint is not satisfied. Thus the case $-x^2 - y < -2$ and $-x = -1$ does not give any solution

The case $-x^2 - y < -2$ and $-x < -1$:

Since both constraints are inactive, we get $\lambda_1 = 0$ and $\lambda_2 = 0$. Thus we get from $-2y + \lambda_1 = 0$ that

$$y = 0$$

and from $\frac{-8x}{x^2+2} + 2\lambda_1 x + \lambda_2 = 0$ that

$$x = 0$$

But $-x = 0$ is not less that -1, so this gives no solutions.

Conclusion:

The minimum value (subject to the constraints) is given by

$$(x,y) = (\sqrt[4]{2}, 2-\sqrt{2}) \implies 4\ln(x^2+2) + y^2 = 4\ln(\sqrt{2}+2) + (2-\sqrt{2})^2 \cong 5.2549$$

4 Mock Final Exam in GRA6035 12/2010, Problem 4

See handwritten solution on the coarse page for GRA 6035 Mathematics 2010/11.

5 Final Exam in GRA6035 10/12/2010, Problem 4

a) The Hessian of f is indefinite for all $(x,y,z) \neq (0,0,0)$ since it is given by

$$f'' = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}$$

and has principal minors $-z^2, -y^2, -x^2$ of order two. Hence f is not convex or concave. We compute the Hessian of g, and find

$$g'' = \frac{1}{xyz} \begin{pmatrix} \frac{2}{x^2} & \frac{1}{xy} & \frac{1}{xz} \\ \frac{1}{xy} & \frac{2}{y^2} & \frac{1}{yz} \\ \frac{1}{xz} & \frac{1}{yz} & \frac{2}{z^2} \end{pmatrix}$$

Hence the leading principal minors are

$$D_1 = \frac{1}{xyz}\frac{2}{x^2} > 0, \quad D_2 = \frac{1}{(xyz)^2}\frac{3}{(xy)^2} > 0, \quad D_3 = \frac{1}{(xyz)^3}\frac{4}{(xyz)^2} > 0$$

This means that g is convex.

b) The set $\{(x,y,z) : x^2 + y^2 + z^2 \leq 1\}$ is closed and bounded, so the problem has solutions by the extreme value theorem. The NDCQ is satisfied, since the rank of $\begin{pmatrix} 2x & 2y & 2z \end{pmatrix} = 1$ when $x^2 + y^2 + z^2 = 1$. We form the Lagrangian

$$\mathscr{L} = xyz - \lambda(x^2 + y^2 + z^2 - 1)$$

and solve the Kuhn-Tucker conditions, consisting of the first order conditions

$$\begin{aligned} \mathscr{L}'_x &= yz - \lambda \cdot 2x = 0 \\ \mathscr{L}'_y &= xz - \lambda \cdot 2y = 0 \\ \mathscr{L}'_z &= xy - \lambda \cdot 2z = 0 \end{aligned}$$

together with one of the following conditions: i) $x^2 + y^2 + z^2 = 1$ and $\lambda \geq 0$ or ii) $x^2 + y^2 + z^2 < 1$ and $\lambda = 0$. We first solve the equations/inequalities in case i): If $x = 0$, then we see that $y = 0$ or $z = 0$ from the first equation, and we get the solutions $(x, y, z; \lambda) = (0, 0, \pm 1; 0), (0, \pm 1, 0; 0)$. If $x \neq 0$, we get $2\lambda = yz/x$ and the remaining first order conditions give $(x^2 - y^2)z = 0$ and $(x^2 - z^2)y = 0$. If $y = 0$, we get the solution $(\pm 1, 0, 0; 0)$. Otherwise, we get $x^2 = y^2 = z^2$, hence

$$(x, y, z; \lambda) = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}; \pm\frac{1}{2\sqrt{3}}\right)$$

The condition that $\lambda \geq 0$ give that either all three coordinates (x, y, z) are positive, or that one is positive and two are negative. In total, we obtain four different solutions. We note that $f(x, y, z) = \frac{1}{3\sqrt{3}}$ for each of these four solutions, while $f(x, y, z) = 0$ for either of the first three solutions. Finally, we consider case ii), where $\lambda = 0$. This gives $xy = xz = yz = 0$, and we obtain

$$(x, y, z; \lambda) = (a, 0, 0; 0), (0, a, 0; 0), (0, 0, a; 0)$$

The condition that $x^2 + y^2 + z^2 < 1$ give $a^2 \leq 1$ or $a \in (-1, 1)$. For all these solutions, we get $f(x, y, z) = 0$. We can therefore conclude that the solution to the optimization problem is a maximum value of

$$\mathbf{\frac{1}{3\sqrt{3}}}$$