EXAMPLE 2 Let $U(\mathbf{x}) = U(x_1, \ldots, x_n)$ denote a consumer's utility function. If $U(\mathbf{x}^0) = a$, then the **upper level set or upper contour set** $\Gamma_a = \{\mathbf{x} : U(\mathbf{x}) \geq a\}$ consists of all commodity vectors $\mathbf{x}$ that the consumer weakly prefers to $\mathbf{x}^0$. In consumer theory, Γ_a is often assumed to be a convex set for every a. (The function U is then called *quasiconcave*.) Figure 5 shows a typical upper level set for the case of two goods.

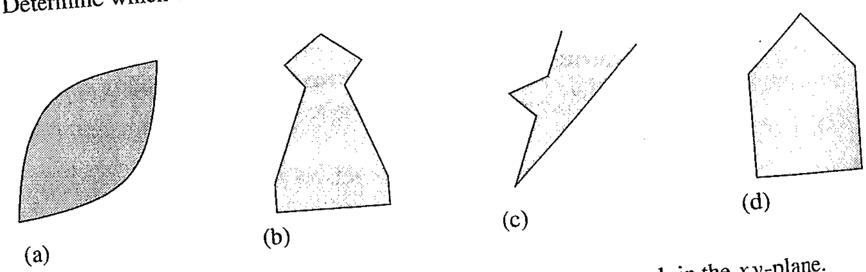
Let $\mathbf{x} = (x_1, \ldots, x_n)$ represent a commodity vector and $\mathbf{p} = (p_1, \ldots, p_n)$ the corresponding price vector. Then $\mathbf{p}\cdot\mathbf{x} = p_1x_1 + \cdots + p_nx_n$ is the cost of buying $\mathbf{x}$. A consumer with m dollars to spend on the commodities has a *budget set* $\mathcal{B}(\mathbf{p}, m)$ defined by the inequalities

$$\mathbf{p}\cdot\mathbf{x} = p_1x_1 + \cdots + p_nx_n \leq m \quad \text{and} \quad x_1 \geq 0, \ldots, x_n \geq 0 \tag{4}$$

The budget set $\mathcal{B}(\mathbf{p}, m)$ consists of all commodity vectors that the consumer can afford. Let $\mathbb{R}^n_+$ denote the set of all $\mathbf{x}$ for which $x_1 \geq 0, \ldots, x_n \geq 0$. Then $\mathcal{B}(\mathbf{p}, m) = H_- \cap \mathbb{R}^n_+$, where H_- is the convex half space introduced in Example 1. It is easy to see that $\mathbb{R}^n_+$ is a convex set. (If $\mathbf{x} \geqq \mathbf{0}$ and $\mathbf{y} \geqq \mathbf{0}$ and $\lambda \in [0, 1]$, then evidently $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \geqq \mathbf{0}$.) Hence $\mathcal{B}(\mathbf{p}, m)$ is convex according to (3). Note that this means that if the consumer can afford either of the commodity vectors $\mathbf{x}$ and $\mathbf{y}$, she can also afford any convex combination of these two vectors.

PROBLEMS FOR SECTION 2.2

1. Determine which of the following four sets are convex:



2. Determine which of the following sets are convex by drawing each in the xy-plane.

   (a) $\{(x, y) : x^2 + y^2 < 2\}$ (b) $\{(x, y) : x \geq 0,\ y \geq 0\}$

   (c) $\{(x, y) : x^2 + y^2 > 8\}$ (d) $\{(x, y) : x \geq 0,\ y \geq 0,\ xy \geq 1\}$

   (e) $\{(x, y) : xy \leq 1\}$ (f) $\{(x, y) : \sqrt{x} + \sqrt{y} \leq 2\}$

3. Let S be the set of all points $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ that satisfy all the m inequalities

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}$$

and moreover are such that $x_1 \geq 0, \ldots, x_n \geq 0$. Show that S is a convex set.

We end this section by proving Theorems 2.3.3 and 2.3.2.

Proof: Let us first show the implication $\Leftarrow$ in part (a). Take two points $\mathbf{x}, \mathbf{y}$ in S and let $t \in [0, 1]$. Define $g(t) = f(\mathbf{y} + t(\mathbf{x} - \mathbf{y})) = f(t\mathbf{x} + (1-t)\mathbf{y})$. Then by using formula (2.1.7), $g'(t) = \sum_{i=1}^{n} f_i'(\mathbf{y} + t(\mathbf{x} - \mathbf{y}))(x_i - y_i)$. Using the chain rule again, we get (for more details see (2.6.6)):

$$g''(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}''(\mathbf{y} + t(\mathbf{x} - \mathbf{y}))(x_i - y_i)(x_j - y_j) \tag{i}$$

By the assumption in (a) that $\Delta_r(\mathbf{y}) \geq 0$ for all $\mathbf{y}$ in S and all $r = 1, \ldots, n$, Theorem 1.7.1(b) implies that the quadratic form in (i) is ≥ 0 for t in $[0, 1]$. This shows that g is convex. In particular,

$$g(t) = g(t \cdot 1 + (1-t) \cdot 0) \leq tg(1) + (1-t)g(0) = tf(\mathbf{x}) + (1-t)f(\mathbf{y}) \tag{ii}$$

But this shows that f is convex, since the inequality in (1) is satisfied with $\leq$.

To prove that $\Rightarrow$ is valid in case (a), suppose f is convex in S. According to Theorem 1.7.1(b), it suffices to show that for all $\mathbf{x}$ in S and all $h_1, \ldots, h_n$ we have

$$Q = \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}''(\mathbf{x})h_i h_j \geq 0 \tag{iii}$$

Now S is an open set, so if $\mathbf{x} \in S$ and $\mathbf{h} = (h_1, \ldots, h_n)$ is an arbitrary vector, there exists a positive number a such that $\mathbf{x} + t\mathbf{h} \in S$ for all t with $|t| < a$. Let $I = (-a, a)$. Define the function p on I by $p(t) = f(\mathbf{x} + t\mathbf{h})$. According to (8), p is convex in I. Hence $p''(t) \geq 0$ for all t in I. But

$$p''(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}''(\mathbf{x} + t\mathbf{h})h_i h_j \tag{iv}$$

Putting $t = 0$, we get inequality (iii).

This proves the equivalence in part (a) of the theorem. The equivalence in (b) follows from (a) if we simply replace f with $-f$. ■

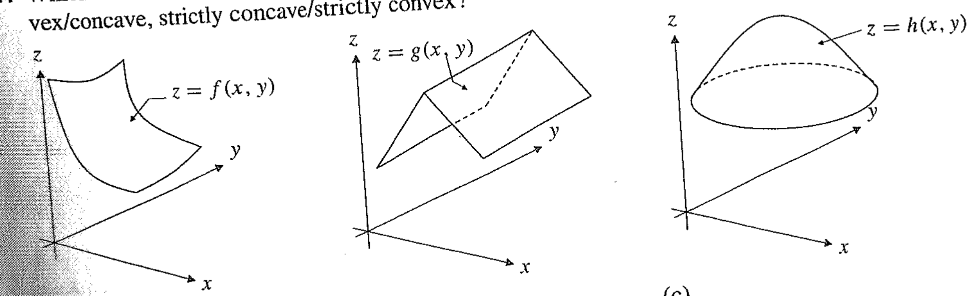
Proof of Theorem 2.3.2: Define g as in the proof of Theorem 2.3.3.

(a) If the specified conditions are satisfied, the Hessian matrix $\mathbf{f}''(\mathbf{x})$ is positive definite according to Theorem 1.7.1(a). So for $\mathbf{x} \neq \mathbf{y}$ the sum in (i) is > 0 for all t in $[0, 1]$. It follows that g is strictly convex. The inequality in (ii) of the proof above is then strict for t in $[0, 1]$, so f is strictly convex. ■

(b) Follows from (a) by replacing f with $-f$.

PROBLEMS FOR SECTION 2.3

1. Which of the functions whose graphs are shown in the figure below are (presumably) convex/concave, strictly concave/strictly convex?



(a) (b) (c)