GRA6035 Mathematics

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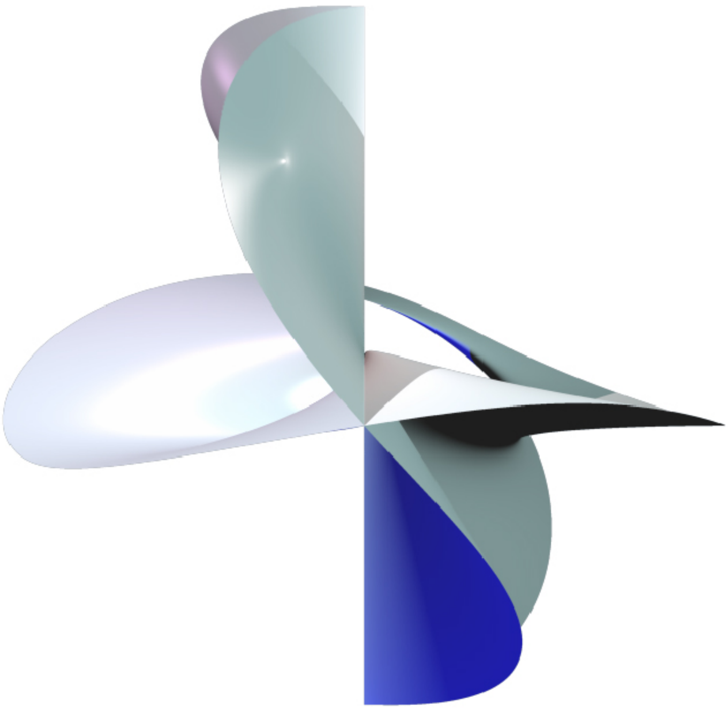

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Part 1

Lecture Notes

CHAPTER 1

Lecture Notes

The course GRA6035 Mathematics traditionally consists of 13 lectures, and this chapter contains the lecture notes from these lectures. The first lecture is labeled Lecture 0, since it (for the most part) covers the prerequisites for the course.

The lecture notes contain the most important notions, techniques and methods in GRA6035 Mathematics, and are written in a concise style. We refer to the corresponding material in the textbook FMEA [**2**] for further details. For each lecture, a number of exercise problems with solutions are given in Chapter 2 and 3.

0. Review of matrix algebra and determinants

Reading. This lecture covers topics from Section 1.1, pages 2-4 in FMEA [**2**]. For a more detailed treatment, see Sections 15.2-15.5 and 16.1-16.5 in EMEA [**3**]. See also 8.1-8.3 in [**1**]. Note that most of this material is part of the required background for the course.

0.1. Matrix algebra. In this section we review the notion of matrices, matrix summation and matrix multiplication.

Matrix addition and subtraction. We first look at the notion of a matrix.

Definition 0.1. A matrix is a rectangular array of numbers considered as an entity.

We write down two matrices of small size.

Example 0.2.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 6 & 5 \end{pmatrix}$$

Here A is a 2×2 matrix (two by two matrix) and B is a 3×2 matrix. We also say that A has size, order or dimension 2×2.

An $m \times n$ matrix is a matrix with m rows and n columns. If we denote the matrix by A, the number at the ith row and at the jth column is often denoted by a_{ij} :

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

We often write $A = (a_{ij})_{m \times n}$ for short. The *diagonal* entries of A are a_{11}, a_{22}, $a_{33}, \ldots$. In the case $m = n$, these elements form the *main diagonal.* A *zero matrix* is matrix consisting only of 0. A zero matrix is often written 0. The size of a zero matrix is usually clear from the context.

We say that two matrices are *equal* if the have they same size (order) and their corresponding entries are equal.

A word on brackets: A matrix is usually embraced by a pair of brackets, () or [] . The book FMEA [**2**] uses () , so I try to use these pairs of brackets in these notes. During the lectures I will however mostly write [] , since it is easier.

Definition 0.3. The sum of to matrices of same the order is computed by adding the corresponding entries.

It is easy to see how this works in an example.

Example 0.4.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 6 & 5 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix}$$

We can calculate the sum of A and C

$$A + C = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2+4 & 1+(-1) \\ 3+2 & 6+0 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 5 & 6 \end{pmatrix},$$

but the sum of A and B is not defined, since they do not have the same order.

A matrix can be multiplied by a number.

Definition 0.5. Let A be a matrix and let k be a real number. Then kA is calculated by multiplying each entry of A by k.

In an example this looks as follows:

Example 0.6. If

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix}$$

we get that

$$4A = \begin{pmatrix} 4 \cdot 2 & 4 \cdot 1 \\ 4 \cdot 3 & 4 \cdot 6 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 12 & 24 \end{pmatrix}.$$

If A and B are two matrices of the same size we define $A - B$ to be $A + (-1)B$. This means that matrices are subtracted by subtracting the corresponding entries. The following rules applies:

Proposition 0.7. Let A, B and C be matrices of the same size (order), and let r and s be scalars (numbers). Then:

(1) $A + B = B + A$
(2) $(A + B) + C = A + (B + C)$
(3) $A + 0 = A$
(4) $r(A + B) = rA + rB$
(5) $(r + s)A = rA + sA$
(6) $r(sA) = (rs)A$

Matrix multiplication. In this section we review matrix multiplication which is a bit more complicated than addition and subtraction of matrices.

Definition 0.8. Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. The product AB is an $m \times p$ matrix that is calculated according to the following scheme:

$$
\begin{array}{r|l}
 & \begin{pmatrix} b_{11} & \dots & \mathbf{b}_{1j} & \dots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \dots & \mathbf{b}_{nj} & \dots & b_{np} \end{pmatrix} \quad = B \\
\hline
A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ \mathbf{a}_{i1} & \dots & \mathbf{a}_{in} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} & \begin{pmatrix} c_{11} & \dots & c_{1j} & \dots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \dots & \mathbf{c}_{ij} & \dots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \dots & c_{mj} & \dots & c_{mp} \end{pmatrix} \quad = AB
\end{array}
$$

The element c_{ij} is computed as

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

You should think of this in the following way: The line through row i in A and the line through a column j in B meet in the entry c_{ij} in the product. The entry c_{ij} is obtained by multiplying the first number in row i in A by the first number in the column j in B, the second number in row i of A with the second number in column j of B and so on, and then adding all these numbers.

We see that the product AB have the same number of rows as A and the same number of columns as B.

Example 0.9. We take $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Then we get

$$
\begin{array}{cc}
 & \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix} \\
 & \\
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & \begin{pmatrix} 1 \cdot 4 + 2 \cdot 0 & 1 \cdot 3 + 2 \cdot 1 \\ 3 \cdot 4 + 4 \cdot 0 & 3 \cdot 3 + 4 \cdot 1 \end{pmatrix}
\end{array}
$$

Thus we get that

$$AB = \begin{pmatrix} 1 \cdot 4 + 2 \cdot 0 & 1 \cdot 3 + 2 \cdot 1 \\ 3 \cdot 4 + 4 \cdot 0 & 3 \cdot 3 + 4 \cdot 1 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 12 & 13 \end{pmatrix}$$

We need another example.

Example 0.10. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{pmatrix}$. Calculate BA.

Solution.

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 3 \\ 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Thus we get that

$$BA = \begin{pmatrix} 0 & 3 \\ 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Note that the product AB is not defined.

Matrix multiplication follows rules that we know from multiplication of numbers.

Proposition 0.11. We have the following rules for matrix multiplication:

(1) $(AB)C = A(BC)$ (associative law)
(2) $A(B + C) = AB + AC$ (left distributive law)
(3) $(A + B)C = AC + BC$ (right distributive law)
(4) $k(AB) = (kA)B = A(kB)$ where k is a scalar.

The following easy example shows how to calculate with the distributive laws.

Example 0.12. Assume that A and B are square matrices of the same order. Show that

$$(A + B)(A - B) = A^2 - B^2$$

if and only if $AB = BA$.

Solution.
We have that

$$(A + B)(A - B) = A(A - B) + B(A - B)$$

by the right distributive law. By applying the left distributive law two times, we get

$$A(A - B) + B(A - B) = A^2 - AB + BA - B^2.$$

This is equal to $A^2 - B^2$ if and only if $-AB + BA = 0$ which is the same as to say that $AB = BA$. (Note that we may move a summand from one side of the equation to the other side of the equation if we change the sign. Why?)

Problem 0.1.
Simplify the expression where A, B and C are matrices.

$$A(3B - C) + (A - 2B)C + 2B(C + 2A)$$

0.2. Determinants. In this section we review how to compute determinants and cofactors.

Definition 0.13. Let A be a general 2×2 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The determinant $|A|$ of A is defined by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

Let us compute the determinant in an example.

Example 0.14. Compute

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}.$$

Solution.

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 2 \cdot 4 - 3 \cdot 1 = 8 - 3 = 5.$$

$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 1 \cdot 2 - 2 \cdot 1 = 2 - 2 = 0.$$

Cofactors are very important. They will allow us to calculate the determinant of larger matrices and to find the inverse matrix.

Definition 0.15. Let A be an $n \times n$ matrix. The cofactor A_{ij} is $(-1)^{i+j}$ times the determinant obtained by deleting row i and column j in A.

Example 0.16. Let

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Compute the cofactor A_{23}.

Solution.

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = (-1)(1 \cdot 0 - 2 \cdot 3) = 6$$

Using cofactors, we can now calculate the determinant of larger matrices.

Definition 0.17. Assume that

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is an $n \times n$ matrix. Then the determinant $|A|$ of A is given by

$$|A| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}.$$

Moreover, this is called the *cofactor expansion of* A along the first row.

In the case A is a 3×3 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

this gives the following formula for the determinant:

$$\begin{aligned} |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \end{aligned}$$

We use this formula to calculate the determinant in an example.

Example 0.18. Compute

$$\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}.$$

Solution.

$$\begin{aligned} \begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} &= 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 3 \cdot \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 5 \cdot \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} \\ &= 1 \cdot (1 \cdot 1 - 0 \cdot 1) - 3 \cdot (0 \cdot 1 - 2 \cdot 1) + 5 \cdot (0 \cdot 0 - 2 \cdot 1) \\ &= 1 - 3(-2) + 5(-2) \\ &= -3. \end{aligned}$$

Actually, the concept of cofactor expansion is more general. The determinant may be computed by cofactor expansion along any row or column. We will not state this in a formal theorem, but rather we show how this works in an example.

Example 0.19. Let

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Compute $|A|$ by cofactor expansion along the second row.

Solution.
We get

$$\begin{aligned} |A| &= 0 \cdot A_{21} + 1 \cdot A_{22} + 1 \cdot A_{23} \\ &= 0 \cdot (-1)^{2+1} \cdot \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} + (-1)^{2+2} \cdot \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} + (-1)^{2+3} \cdot \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} \\ &= 0 + (1 \cdot 1 - 2 \cdot 5) - (1 \cdot 0 - 2 \cdot 3) \\ &= 0 - 9 + 6 \\ &= -3. \end{aligned}$$

An important notion is the transpose of a matrix A. This is written A^T. The books FMEA [**2**] and EMEA [**3**] use the notation A', but A^T is more common, so I will stick to that notation.

Definition 0.20. Let A be an $n \times m$ matrix. The transpose of A, denoted A^T, is the matrix obtained form A by interchanging the rows and columns in A.

For a 3×3 matrix, the transpose matrix is easy to write down.

Example 0.21. Let

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Write down A^T.

Solution.

$$A^T = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{pmatrix}.$$

We now return to calculation of determinants, and note the following properties. These are very useful, and can simplify calculations tremendously.

Proposition 0.22. Let A and B be $n \times n$ matrices.

(1) $|A^T| = |A|$
(2) $|AB| = |A||B|$
(3) If two rows in A are interchanged, the sign of the determinant changes.
(4) If a row in A is multiplied by a constant k, the determinant is multiplied by k.
(5) If a multiple of one row is added to an other row, the determinant is unchanged.

We demonstrate the utility of this proposition in some examples.

Example 0.23. Let

$$A = \begin{pmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 5 & 1 & 1 \end{pmatrix}.$$

Compute $|A|$.

Solution.
Since the determinant is unchanged if we take two times the first row and add to the second, we obtain:

$$|A| = \begin{vmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 5 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 5 & 1 & 1 \end{vmatrix}.$$

By taking cofactor expansion along the second row, we see that $|A| = 0$.

Example 0.24. We have that

$$\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -3.$$

Compute

$$\begin{vmatrix} -2 & -6 & -10 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{vmatrix}.$$

Solution.

Note that $\begin{vmatrix} -2 & -6 & -10 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}$ is obtained from $\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}$ by multiplying the first row by -2. Thus

$$\begin{vmatrix} -2 & -6 & -10 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = (-2)(-3) = 6.$$

Note further that $\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{vmatrix}$ is obtained from $\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}$ by interchanging two rows. Thus

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{vmatrix} = (-1)(-3) = 3.$$

Example 0.25. Let

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Compute $|AB|$.

Solution.

$$|A| = 0 \implies |AB| = |A||B| = 0 \cdot |B| = 0.$$

Problem 0.2.
Compute the determinants.

(a) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \\ 1 & 2 & 6 \end{vmatrix}$

1. The inverse matrix and linear dependence

Reading. This lecture covers topics from Sections 1.1, 1.2 and 1.9 in FMEA [**2**]. See also 8.3-8.5 in [**1**].

1.1. The inverse matrix. We start by recalling the notion of cofactors from Lecture 0.

Example 1.1. Let

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

Compute the cofactor A_{32}.

Solution.
To compute the cofactor A_{32} we cross out row 3 and column 2 from the matrix A. We then get

$$\begin{aligned} A_{32} &= (-1)^{3+2} \begin{vmatrix} 1 & 5 \\ 0 & 1 \end{vmatrix} = (-1)^5(1 \cdot 1 - 0 \cdot 5) \\ &= -1. \end{aligned}$$

The cofactors will enable us to compute the inverse matrix, but first we define the cofactor matrix and the (classical) adjoint matrix.

Definition 1.2. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

be an $n \times n$ matrix. We define the *cofactor matrix* as

$$\operatorname{cof}(A) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}.$$

The *(classical) adjoint matrix* (also called the adjugate matrix) is defined as the transpose of the cofactor matrix,

$$\operatorname{adj}(A) = (\operatorname{cof}(A))^T.$$

We compute the cofactor matrix in an example.

Example 1.3. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Compute $\operatorname{adj}(A)$.

Solution.
We find

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = (-1)^{1+1} \cdot 2 = 2$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = (-1)^{1+2} \cdot (-2) = 2$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = (-1)^{1+3} \cdot (-4) = -4$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = (-1)^{2+2} \cdot 1 = 1$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} = 0$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (-1)^{3+2} \cdot 1 = -1$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = (-1)^{3+3} \cdot 2 = 2$$

From this we get

$$\operatorname{adj}(A) = \begin{pmatrix} 2 & 2 & -4 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}^T = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & -1 \\ -4 & 0 & 2 \end{pmatrix}.$$

We will soon see that the (classical) adjoint matrix has very useful properties. This is suggested in the following example.

Example 1.4. Let A be as in the example above. Compute

$$A \operatorname{adj}(A).$$

Solution.

$$A \operatorname{adj}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & -1 \\ -4 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

We compute the determinant $|A|$ of the matrix in the previous example as

$$|A| = 1 \cdot \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

and we see that

$$A \operatorname{adj}(A) = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Definition 1.5. The $n \times n$ matrix with 1's on the main diagonal and 0's elsewhere, is called the identity matrix and is denoted by I or I_n.

Example 1.6.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If A is an $m \times n$ matrix one verifies that

$$I_m A = A$$

and that

$$AI_n = A.$$

As the previous example suggested, we have that the adjoint matrix satisfy the following property:

Theorem 1.7. Let A be an $n \times n$ matrix. Then

$$A \operatorname{adj}(A) = \operatorname{adj}(A) A = |A| I_n.$$

Definition 1.8. Let A be an $n \times n$ matrix. If $|A| \neq 0$, then A is said to be invertible and the *inverse matrix* denoted A^{-1} is given by

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A).$$

When $|A| = 0$, we say that A is not invertible.

Example 1.9. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

as above. Compute A^{-1}.

Solution.
We have already found that

$$\operatorname{adj}(A) = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & -1 \\ -4 & 0 & 2 \end{pmatrix} \text{ and } |A| = 2.$$

Thus we get

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & -1 \\ -4 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ -2 & 0 & 1 \end{pmatrix}$$

Remark 1.10.

$$AA^{-1} = A^{-1}A = I_n$$

Example 1.11. Let
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
and assume that $|A| = ad - bc \neq 0$. Find a formula for A^{-1}.

Solution.
We need to find the adjoint matrix of A, so we calculate the cofactors. We find A_{11} by crossing out the first row and column of A. This leaves us with d, and we get
$$A_{11} = (-1)^{1+1} d = d.$$
Similarly, we get
$$\begin{aligned} A_{12} &= (-1)^{1+2} c = -c \\ A_{21} &= (-1)^{2+1} b = -b \\ A_{22} &= (-1)^{2+2} a = a \end{aligned}$$
Thus we get that
$$\operatorname{adj}(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
This gives
$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ (remember this formula).}$$

1.2. Partitioned matrices. We will not go deeply into the subject of partitioned matrices, but we show that this can be useful in an example. For more one partitioned matrices, see Section 1.2 in FMEA [**2**].

Example 1.12. Let
$$M = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Use partition to compute M^2.

Solution.
The matrix M, may be written as

$$M = \left(\begin{array}{cc|cc} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} P & Q \\ 0 & I \end{pmatrix}$$

where

$$P = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \; Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

and I is the identity matrix. Now we compute

$$M^2 = MM = \begin{pmatrix} P & Q \\ 0 & I \end{pmatrix}^2$$

pretending that P and Q are numbers. HOWEVER, we must be careful since PQ is not the same as QP. We get

$$\begin{aligned} M^2 &= \begin{pmatrix} P & Q \\ 0 & I \end{pmatrix}\begin{pmatrix} P & Q \\ 0 & I \end{pmatrix} = \begin{pmatrix} PP & PQ + QI \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} P^2 & PQ + Q \\ 0 & I \end{pmatrix} \end{aligned}$$

Thus we need to calculate P^2 and PQ and $PQ + Q$. We get

$$\begin{aligned} P^2 &= \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ PQ &= \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \\ PQ + Q &= \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix} \end{aligned}$$

Thus

$$M^2 = \begin{pmatrix} 2 & 0 & 1 & 4 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1.3. Linear independence. Before we define the notion of linear independence we recall the notion of a vector.

Definition 1.13. A matrix consisting of either a single row or a single column, is called a *vector*.

A vector consisting of m numbers is called an m-vector. We may speak about row vectors and column vectors.

Example 1.14. The following are column vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \; \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ -1 \\ 2 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & -2 & 5 \end{pmatrix}, \; \begin{pmatrix} -1 & 1 \end{pmatrix}$$

are row vectors. These row vectors could also have been written as

$$(1, -2, 5), \; (-1, 1).$$

We perform calculations on vectors in the same way as we calculate with matrices.

Example 1.15. We have for instance

$$3\begin{pmatrix}1\\2\\-1\end{pmatrix}+2\begin{pmatrix}-1\\0\\2\end{pmatrix}=\begin{pmatrix}1\\6\\1\end{pmatrix}.$$

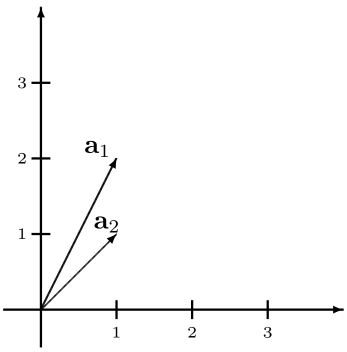
It is common to draw 2-vectors in the plane, as suggested in the following example:

Example 1.16. Draw the vectors

$$\mathbf{a}_1=\begin{pmatrix}1\\2\end{pmatrix} \text{ and } \mathbf{a}_2=\begin{pmatrix}1\\1\end{pmatrix}$$

in the plane.

Solution.



Definition 1.17. The set of all m-vectors is denoted by $\mathbb{R}^m$.

Definition 1.18. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be in $\mathbb{R}^m$. Let $\mathbf{a}$ be another m-vector. If we can find numbers $c_1, c_2, \ldots, c_n$ such that $\mathbf{a}$ can be expressed as

$$\mathbf{a}=c_1\mathbf{a}_1+c_2\mathbf{a}_2+\cdots+c_n\mathbf{a}_n,$$

we say that $\mathbf{a}$ is a *linear combination* of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.

Some examples will hopefully clarify this definition.

Example 1.19. We have that

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so that the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Since

$$\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} = 2\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

we have that $\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$. Finally,

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so that $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Example 1.20. Is it possible to express the vector $\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$ as a linear combination of

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Solution.
The answer is yes, since

$$\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We now come to the important concept of linear dependence.

Definition 1.21. The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^m$ are *linearly dependent* if there exists numbers $c_1, c_2, \ldots, c_n$ *not all zero* such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}.$$

If this equation holds only when $c_1 = 0$, $c_2 = 0, \ldots, c_n = 0$, we say that the vectors $\mathbf{a}_1$, $\mathbf{a}_2, \ldots, \mathbf{a}_n$ are *linearly independent.*

It is not always easy to determine if a set of vectors are linearly dependent, but the following examples should not be difficult.

Example 1.22. Show that $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{a}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ are linearly dependent.

Solution.
We see that
$$2\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix},$$
so $2\mathbf{a}_1=\mathbf{a}_2$. From this we have that $2\mathbf{a}_1+(-1)\mathbf{a}_2=\mathbf{0}$. Thus we may choose $c_1=2$ and $c_2=-1$ in the definition above, so we see that $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly dependent.

Example 1.23. Show that $\mathbf{e}_1=\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{e}_2=\begin{pmatrix}0\\1\end{pmatrix}$ are linearly independent.

Solution.
We must prove that the only possible values for c_1 and c_2 are 0, in the equation
$$c_1\mathbf{e}_1+c_2\mathbf{e}_2=\mathbf{0}.$$
But the left side of this equation is
$$c_1\mathbf{e}_1+c_2\mathbf{e}_2=c_1\begin{pmatrix}1\\0\end{pmatrix}+c_2\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}c_1\\c_2\end{pmatrix},$$
so the equation says that
$$\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
which means that booth c_1 and c_2 have to be 0.

Example 1.24. Determine if the vectors
$$\begin{pmatrix}1\\2\end{pmatrix}\text{ and }\begin{pmatrix}1\\1\end{pmatrix}$$
are linearly independent.

Solution.
We must look at the equation
$$c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$
This is equivalent to
$$\begin{pmatrix}c_1\\2c_1\end{pmatrix}+\begin{pmatrix}c_2\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
which again is the same as
$$\begin{pmatrix}c_1+c_2\\2c_1+c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$
We must find the solution of
$$\begin{aligned}c_1+c_2&=0\\2c_1+c_2&=0,\end{aligned}$$
and by subtraction the first equation from the second, we see that $c_1=0$. Substituting this into one of the equations, gives that $c_2=0$. We conclude that the only possible solution of
$$c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
is $c_1=0$ and $c_2=0$, and this proves that
$$\begin{pmatrix}1\\2\end{pmatrix}\text{ and }\begin{pmatrix}1\\1\end{pmatrix}$$
are linearly independent.

Finally we look at an example which indicate how to use the determinant to see linear independence.

Example 1.25. Show that

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{a}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

are linearly independent.

Solution.
We must show that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = 0$$

only has the solution $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$. In other words we must look at

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1}$$

The left hand side can be calculated to be

$$\begin{pmatrix} c_1 + c_2 + c_3 \\ c_2 + c_3 \\ c_2 - c_3 \end{pmatrix}.$$

Moreover, it is possible to write this as a matrix product

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

The matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

is obtained by taking the coefficients in front of the c_i's. If we put

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

the equation (1) may be written as

$$A\mathbf{c} = 0.$$

One calculates that $|A| = -2$. In particular $|A| \neq 0$, so that A is invertible. We may thus multiply the equation $A\mathbf{c} = \mathbf{0}$ by A^{-1} from the left, and we obtain

$$A^{-1}A\mathbf{c} = A^{-1}0.$$

But $A^{-1}A = I_3$ and $A^{-1}0 = 0$, so we have $I_3\mathbf{c} = 0$. But $I_3\mathbf{c} = \mathbf{c}$, so we conclude that $\mathbf{c} = 0$ or

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This shows that equation (1) only has the solution $c_1 = c_2 = c_3 = 0$, so the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are linearly independent.

2. The rank of a matrix and applications

Reading. This lecture covers topics from Sections 1.2, 1.3 and 1.4 in FMEA [**2**]. See also 8.4 and 8.5 in [**1**].

2.1. Linear dependence and systems of linear equations. A system of linear equations may be written as a vector equation.

Example 2.1. Consider the system

$$\begin{array}{rrrcl} 2x_1 & +2x_2 & -x_3 & = & 0 \\ 4x_1 & & +2x_3 & = & 0 \\ & 6x_2 & -3x_3 & = & 0 \end{array} \tag{2}$$

of linear equations. Show that it can be written as a vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}.$$

Solution.
Note that

$$\begin{aligned} x_1\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + x_3\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} &= \begin{pmatrix} 2x_1 \\ 4x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ 0 \\ 6x_2 \end{pmatrix} + \begin{pmatrix} -x_3 \\ 2x_3 \\ -3x_3 \end{pmatrix} \\ &= \begin{pmatrix} 2x_1 + 2x_2 - x_3 \\ 4x_1 + 2x_3 \\ 6x_2 - 3x_3 \end{pmatrix}. \end{aligned}$$

We see that (2) is equivalent to

$$x_1\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + x_3\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We now put

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

and thus we have

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}.$$

Note that the vector $\mathbf{a}_1$ consists of the numbers in (2) in front of x_1, the vector $\mathbf{a}_2$ consists of the numbers in (2) in front of x_2 etc.

Systems of equations like (2) always have the solution $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. This is called the *trivial solution.* If this is the only solution, the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are linearly independent. This leads to the following criterion for linear independence.

Theorem 2.2. The n column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ of the $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \text{ where } \mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

are linearly independent if and only if $|A| \neq 0$.

Example 2.3. Show that

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

are linearly independent.

Solution.

$$\begin{vmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -3 \end{vmatrix} = 4 \cdot (-1)^{2+1} \cdot \begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix} + 0 + 2 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 2 & 2 \\ 0 & 6 \end{vmatrix}$$
$$= (-4) \cdot 0 - 2 \cdot 12 = -24 \neq 0$$

By the theorem we conclude that $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are linearly independent.

2.2. The rank of a matrix. In this section we define the important concept of *rank* of a matrix. This will be used in the next section to describe the solutions of a systems of linear equations.

Definition 2.4. The *rank* of a matrix A, written $r(A)$, is the maximum number of linearly independent column vectors in A. If A is a zero matrix, we put $r(A) = 0$.

If A is and $n \times n$ matrix, i.e. a square matrix, we know from Theorem 2.2, that the n column vectors of A are linearly independent if and only if $|A| \neq 0$, so $r(A) = n$ if and only if $|A| \neq 0$. More generally, the rank can be found by looking at the so-called *minors* of the matrix.

Definition 2.5. Let A be an $n \times m$ matrix. A *minor of order* k is a determinant obtained from A by deleting $m - k$ columns and $n - k$ rows in A.

Before we state how to compute the rank using minors, we give an example.

Example 2.6. If

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

we get for instance the minor

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

by deleting column number 3 in A. (Note that there are three other minors of order 3.)

Deleting the second row and the two last columns we obtain the minor

$$\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$$

of order 2. Note that there are several other minors of order 2.

The following theorem shows how to compute the rank of a matrix.

Theorem 2.7. The rank $r(A)$ of a matrix A, is equal to the order of the largest non-zero minor in A.

For large matrices one may have to calculate many minors, but for matrices of moderate size, it is often easy to find the rank using the theorem above.

Example 2.8. Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Solution.
We have that

$$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 \cdot 1 - 3 \cdot 2 = -5 \neq 0.$$

Since we cannot have minors of order greater than 2, we conclude that $r(A) = 2$.

Example 2.9. Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Solution.
Deleting one column in A gives a minor of order 3. There are four such minors. Each of these minors will however contain a column with only zeros. Thus all minors of order 3 are zero.

Deleting the two columns in A consisting entirely of zeros, and deleting for instance the last row, we get the following minor of order 2:

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

From this we conclude that $r(A) = 2$.

Example 2.10. Find the rank of

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution.
We see that all 3-minors will contain a row of zeros, so all 3-minors will be zero. This means that $r(A) < 3$. Picking a 2-minor,

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$$

we see that $r(A) = 2$.

Example 2.11. Show that

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

are linearly independent.

Solution.
We form the matrix
$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 3 \end{pmatrix}$$
with the two vectors as columns. We recall that the rank is the largest number of linearly independent columns. This means that we have to show that $r(A) = 2$. But, just by picking a 2-minors,
$$\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0$$
we see that $r(A) = 2$.

2.3. Rank and linear systems. In this section we will see how the notion of rank of a matrix is connected to the existence of solutions of systems of linear equations. The first observation is that any system of linear equations may be written as a matrix equation.

Example 2.12. Show that
$$\begin{array}{cccccc} 2x_1 & +2x_2 & -x_3 & = & -3 \\ 4x_1 & & +2x_3 & = & 8 \\ & 6x_2 & -3x_3 & = & -12 \end{array} \tag{3}$$
may be written as
$$A\mathbf{x} = \mathbf{b}$$
where
$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Solution.
The coefficients of the system, may be arranged in a matrix,
$$A = \begin{pmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -3 \end{pmatrix}$$
where we put a zero at a position if the corresponding term is missing in the system. By matrix multiplication, we find that
$$\begin{pmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 2x_2 - x_3 \\ 4x_1 + 2x_3 \\ 6x_2 - 3x_3 \end{pmatrix}$$
From this we see that the system (3) of linear equation is equivalent to the matrix equation
$$\begin{pmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ -12 \end{pmatrix}$$
or
$$A\mathbf{x} = \mathbf{b}$$
where
$$\mathbf{b} = \begin{pmatrix} -3 \\ 8 \\ -12 \end{pmatrix}.$$

The matrix formed in the example, is called the *coefficient matrix* of (3). The matrix obtained from the coefficient matrix by augmenting $\mathbf{b}$ as a last column, is called the *augmented matrix* of (3). In the example above, this would be denoted by $A_{\mathbf{b}}$ and

$$A_{\mathbf{b}} = \begin{pmatrix} 2 & 2 & -1 & -3 \\ 4 & 0 & 2 & 8 \\ 0 & 6 & -3 & -12 \end{pmatrix}.$$

The notion of coefficient matrix and augmented matrix generalizes to any system of linear equations.

Theorem 2.13. Consider a system of linear equations, written as $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $r(A) = r(A_{\mathbf{b}})$.

We will use the theorem in an example.

Example 2.14. Show that

$$\begin{aligned} 2x_1 - x_2 &= 3 \\ 4x_1 - 2x_2 &= 5 \end{aligned}$$

is inconsistent (i.e. has no solutions.)

Solution.
Writing the system on matrix form $A\mathbf{x} = \mathbf{b}$, we have

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \text{ and } A_{\mathbf{b}} = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -2 & 5 \end{pmatrix}.$$

First we find the rank of A. We have that

$$\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} = 2 \cdot (-2) - 4 \cdot (-1) = -4 + 4 = 0.$$

We must then look at minors of order 1, but these are only the entries of A. Since not all of these are zero (in fact none of them are), we conclude that the there are minors of order 1 that are not zero. From this we know that $r(A) = 1$.

We now find the rank of $A_{\mathbf{b}}$. Deleting the middle row, we get the minor

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2 \cdot 5 - 4 \cdot 3 = -2 \neq 0$$

of order 2. Since $r(A) = 1$ and $r(A_{\mathbf{b}}) = 2$, we have $r(A) \neq r(A_{\mathbf{b}})$ and we conclude by the theorem that the system has no solutions.

Theorem 2.15. Consider a linear system of equations in n variables, $x_1, x_2, \ldots, x_n$, written as $A\mathbf{x} = \mathbf{b}$. Assume that $r(A) = r(A_{\mathbf{b}}) = k$. Then the following holds.

(1) Choose any set of k equations corresponding to k linearly independent rows. Then any solution of these equations will also be a solution of the remaining equations.
(2) If $k < n$ then there exists $n - k$ variables that can be chosen freely. (We say that there are $n - k$ degrees of freedom.)

We modify the previous example slightly.

Example 2.16. Consider the system

$$\begin{aligned} 2x_1 - x_2 &= 3 \\ 4x_1 - 2x_2 &= 6. \end{aligned}$$

Show that $r(A) = r(A_{\mathbf{b}}) = 1$.

Solution.
We have
$$A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \text{ and } A_{\mathbf{b}} = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \end{pmatrix}$$
As in the example preceding the theorem, $r(A) = 1$. However for $A_{\mathbf{b}}$ we have that all minors of order 2 are zero:
$$\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ -2 & 6 \end{vmatrix} = 0.$$
Thus we also have $r(A_{\mathbf{b}}) = 1$.

In the example, the theorem tells us that we may choose $2x_1 - x_2 = 3$ and that any solution of this equation will also be a solution of $4x_1 - 2x_2 = 6$. (In this simple example you should also be able convince yourself about this with out referring to the theorem.)

The equation $2x_1 - x_2 = 3$ may be rewritten as $2x_1 = 3 + x_2$ or $x_1 = \frac{3}{2} + \frac{1}{2}x_2$. We may choose any value for x_2 , and find a corresponding value for x_1 from this equation. This will be a solution of the system

$$\begin{aligned} 2x_1 - x_2 &= 3 \\ 4x_1 - 2x_2 &= 6. \end{aligned}$$

We say that the system has 1 degree of freedom.

Example 2.17. What is the number of free variables in the following system?
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + 2x_2 + 2x_3 + 2x_4 &= 0. \end{aligned}$$

Solution.
We have that there are $n = 4$ variables. The coefficient matrix is
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$
Since all 2-minors look like
$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0$$
We see that $r(A) = 1$. From the theorem above, we have $n - k = 4 - 1 = 3$ free variables. Another way to see this is to observe that the two equations are equivalent, so in reality there is only one equation. This means that we may freely choose a value for three of the variables, and then the equation gives the value of the last variable.

Example 2.18. Solve the following system of linear equations
$$\begin{array}{cccccc} x_1 & -x_2 & +2x_3 & +3x_4 & = & 0 \\ & x_2 & +x_3 & & = & 0 \\ x_1 & & +3x_3 & +3x_4 & = & 0. \end{array}$$

Solution.
The system may be written as

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One calculates that the three 3-minors are zero, so the rank of the coefficient matrix is less than 3. Looking at the 2-minor

$$\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

we see that the rank must be 2. From this we see that we should look for two independent rows. In fact, we see that the third row in the coefficient matrix is the sum of the two first rows. We also see that the two first rows are linearly independent. From the theorem, we know that any solution of the first two equations will automatically be a solution of the last equation. This means that the last equation is *superfluous.* We may thus write the system as

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To write up a solution of the system, we simplify it by multiplying by

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

is just the matrix consisting of the first two columns of the coefficient matrix. We calculate

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

The system

$$\begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is equivalent to our original system. Written in the usual way, this is

$$\begin{array}{ccccccc} x_1 & & +3x_3 & +3x_4 & = & 0 \\ & x_2 & +x_3 & & = & 0 \end{array}$$

We may regard x_1 and x_2 as the *base* variables and isolate these on the left side of the equations. This gives

$$\begin{aligned} x_1 &= -3x_3 - 3x_4 \\ x_2 &= -x_3 \\ x_3 &= \text{free} \\ x_4 &= \text{free} \end{aligned}$$

It is to be understood that for any choice of x_3 and x_4 we get a solution.

3. Eigenvalues and diagonalization

Reading. This lecture covers topics from Sections 1.5 and 1.6 in FMEA [**2**]. See also [**1**] pages 516-517.

3.1. Eigenvalues. Assume that we are facing the following problem. Given a very large $n \times n$ matrix A and an n-vector $\mathbf{x}$, compute, say $A^{1000}\mathbf{x}$. This would involve a very large number of computations. We ask if it is possible to simplify this by reducing the computational load, which in practical examples may be high even for computers. For this and many other reasons, we will examine the notion of eigenvectors and eigenvalues.

Definition 3.1. An eigenvector for an $n \times n$ matrix A, is an n-vector $\mathbf{x} \neq \mathbf{0}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

where λ (lambda, greek l) is a number called an eigenvalue of A.

Given a matrix and a vector it is easy to determine if the given vector is an eigenvector for the matrix.

Example 3.2. Let

$$A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}, \text{ and } \mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors for A?

Solution.
To determine if $\mathbf{u}$ is an eigenvector for A, we calculate $A\mathbf{u}$:

$$A\mathbf{u} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -24 \\ 20 \end{pmatrix}.$$

Taking a closer look we see that $-24 = (-4) \cdot 6$ and that $20 = (-4) \cdot (-5)$. In other words

$$\begin{pmatrix} -24 \\ 20 \end{pmatrix} = (-4) \begin{pmatrix} 6 \\ -5 \end{pmatrix}$$

so that

$$A\mathbf{u} = (-\mathbf{4})\mathbf{u}.$$

This means that $\mathbf{u}$ is an eigenvector for A with eigenvalue $\lambda = -4$.
Doing the same calculation for $\mathbf{v}$, we have

$$A\mathbf{v} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ 11 \end{pmatrix}$$

we see that $-9 = (-3) \cdot 3$ but $(-3) \cdot (-2) = 6 \neq 11$. This means that $\mathbf{v}$ is *not* an eigenvector for A.

We want to proceed to find the eigenvectors for a given matrix A. It turns out, however, that it is advantageous to find the eigenvalues first.

Example 3.3. Find the possible eigenvalues for the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}.$$

Solution.
We must find the possible values for λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for at least one vector $\mathbf{x} \neq \mathbf{0}$. This is the same as to say that the matrix equation $A\mathbf{x} = \lambda\mathbf{x}$ should have at least one non-trivial solution. We may rearrange this equation slightly

$$A\mathbf{x} = \lambda\mathbf{x} \Longleftrightarrow A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0} \Longleftrightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}.$$

Note here that $I\mathbf{x} = \mathbf{x}$. If A is an $n \times n$ matrix, we know that we must have $r(A - \lambda I) < \mathbf{0}$ for the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ to have at least one degree of freedom (see Lecture 3.) This is the same as to say that $|A - \lambda I| = 0$. On the other hand if $|A - \lambda I| \neq 0$, the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$. We conclude that for λ to be an eigenvalue we must have that $|A - \lambda I| = 0$:

$$A - \lambda I = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix}$$

so that

$$\begin{aligned} |A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} &= (2-\lambda)(-6-\lambda) - 3 \cdot 3 \\ &= \lambda^2 + 4\lambda - 21 = 0. \end{aligned}$$

The equation $\lambda^2 + 4\lambda - 21 = 0$ is called the characteristic equation, and we want to find its solutions. Using the formula for the quadratic equation, we get

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 21}}{2} = \begin{cases} -7 \\ 3 \end{cases}$$

and these are the eigenvalues of A.

The procedure suggested in the example, generalizes to an arbitrary square matrix A.

Definition 3.4. Assume that A is an $n \times n$ matrix. The equation

$$|A - \lambda I| = 0$$

is called the *characteristic equation* of A.

The characteristic equation is important not only because its solution gives the eigenvalues, but this remains the main interest in the present context.

Proposition 3.5. The eigenvalues of a square matrix is given as the solutions of the characteristic equation.

3.2. Diagonalization. The diagonal matrices are particular easy to compute with.

Example 3.6. Let

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}.$$

Compute D^2, $D^3, \ldots, D^k$.

Solution.

$$
\begin{aligned}
D^2 = DD &= \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5^2 & 0 \\ 0 & 3^2 \end{pmatrix} \\
D^3 = D^2 D &= \begin{pmatrix} 5^2 & 0 \\ 0 & 3^2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5^3 & 0 \\ 0 & 3^3 \end{pmatrix} \\
&\vdots \\
D^k = D^{k-1} D &= \begin{pmatrix} 5^{k-1} & 0 \\ 0 & 3^{k-1} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix}
\end{aligned}
$$

Although it is easier to multiply diagonal matrices, most matrices are not diagonal. Some matrices are however what we call *diagonalizable.*

Definition 3.7. An $n \times n$ matrix A is said to be *diagonalizable* if we can find an invertible matrix P so that

$$P^{-1}AP = D$$

is diagonal.

Before we explore some of the advantages of diagonalizable matrices, we give a procedure for how to *diagonalize* a matrix (if it is possible.)

Procedure 3.8. How to diagonalize an $n \times n$ matrix A.

Step 1 Find the eigenvalues of A.

Step 2 Find the eigenvectors of A, by solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for each eigenvalue λ.

Step 3 Pick n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ if possible. If this is not possible, then A is not diagonalizable.

Step 3 Form the matrix $P = (\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_n)$ with $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ as column vectors.

Step 5 We get $D = P^{-1}AP$ where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

and where λ_i is the eigenvalue corresponding to $\mathbf{v}_i$.

Example 3.9. Diagonalize

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$$

if possible.

Solution.

Step 1: The characteristic equation is:

$$\begin{vmatrix} 1-\lambda & -1 \\ -2 & 0-\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - (-2)(-1)$$
$$= \lambda^2 - \lambda - 2 = 0.$$

The solutions are $\lambda = -1$ and $\lambda = 2$.

Step 2: $\lambda = -1$**:**

$$A - \lambda I = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$$

Thus $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is equivalent to

$$\begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or equivalently

$$\begin{array}{rrl} 2x_1 & -x_2 & = 0 \\ -2x_1 & +x_2 & = 0 \end{array}$$

We see that the two equations are equivalent as they must be, since the rank of the coefficient matrix is 1. So we need to find the solutions of the single equation $2x_1 - x_2 = 0$. There are of course infinitely many solutions, and since $2x_1 - x_2 = 0 \iff x_1 = \frac{1}{2}x_2$ we see that for each choice of x_2 we get a value for x_1. We write this as

$$x_1 = \frac{1}{2}x_2$$
$$x_2 = t \text{ is free}$$

Using vector notation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t \\ t \end{pmatrix} = t\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}.$$

We choose $\mathbf{v}_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$ as the eigenvector corresponding to the eigenvalue $\lambda = -1$.

$\lambda = 2$**:** We have that

$$A - \lambda I = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix}$$

and thus $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is equivalent to

$$\begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or equivalently

$$\begin{array}{rrl} -x_1 & -x_2 & = 0 \\ -2x_1 & -2x_2 & = 0 \end{array}$$

This is equivalent to $x_1 + x_2 = 0$ or $x_1 = -x_2$. We say that x_2 is free and write

$$x_1 = -x_2$$
$$x_2 = t \text{ is free}$$

Using vector notation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

We choose $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ as the eigenvector corresponding to the eigenvalue $\lambda = 2$.

Solution.
(continues)

Step 3: We have fond $\mathbf{v}_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$ corresponding to $\lambda = -1$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ corresponding to $\lambda = 2$.

Step 4: We can now form the matrix P with $\mathbf{v}_1$ and $\mathbf{v}_2$ as columns:
$$P = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix}$$

Step 5: We get the corresponding diagonal matrix
$$D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

One can check by matrix multiplication that $D = P^{-1}AP$. We have that
$$P^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$
and
$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} \\ -\frac{4}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = D. \end{aligned}$$

Example 3.10. Compute A^{100}.

Solution.
We have that $D = P^{-1}AP$. Multiplying this equality with P from the left gives us $PD = PP^{-1}AP = IAP = AP$. Likewise, multiplying with P^{-1} from the right, we get $PDP^{-1} = A$. Using this we have:
$$\begin{aligned} A^{100} &= \underbrace{A \cdots A}_{\text{100 factors}} = PDP^{-1}PDP^{-1} \cdots PDP^{-1} \\ &= PDIDI \cdots IDP^{-1} \\ &= PD^{100}P^{-1} \end{aligned}$$
From this we get that
$$\begin{aligned} A^{100} &= \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}^{100} \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(-1)^{100} & -2^{100} \\ (-1)^{100} & 2^{100} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3}(-1)^{100} + \frac{2}{3}2^{100} & \frac{1}{3}(-1)^{100} - \frac{1}{3}2^{100} \\ \frac{2}{3}(-1)^{100} - \frac{2}{3}2^{100} & \frac{2}{3}(-1)^{100} + \frac{1}{3}2^{100} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} + \frac{2}{3}2^{100} & \frac{1}{3} - \frac{1}{3}2^{100} \\ \frac{2}{3} - \frac{2}{3}2^{100} & \frac{2}{3} + \frac{1}{3}2^{100} \end{pmatrix}. \end{aligned}$$

The following theorem tells us when it is possible to diagonalize a square matrix.

Theorem 3.11. Assume that A is an $n \times n$ matrix,

(1) A is diagonalizable if and only if A has n linearly independent eigenvectors.
(2) If A has n distinct eigenvalues then A is diagonalizable.

(3) If A is symmetric (i.e. $A = A^T$) then A is diagonalizable.

We explore how to use this theorem in some examples.

Example 3.12. Determine if

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

is diagonalizable.

Solution.
The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$$

This equation has $\lambda = 1$ as its only solution. Having found the eigenvalues, we must find the corresponding eigenvectors by solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$. We have

$$\begin{aligned} A - \lambda I &= \begin{pmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{pmatrix} = \begin{pmatrix} 1-1 & 2 \\ 0 & 1-1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

so $(A - \lambda I)\mathbf{x} = \mathbf{0}$ gives only the equation $2x_2 = 0$. This means that $x_2 = 0$ and that x_1 is free. In other words

$$\begin{aligned} x_1 &= t \text{ is free} \\ x_2 &= 0 \end{aligned}$$

Using vector notation, we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From this it is clear that we can not find two linearly independent eigenvectors, and this means that A is *not* diagonalizable.

In the next example the situation is different.

Example 3.13. Determine if

$$A = \begin{pmatrix} 4 & 2 \\ -3 & 11 \end{pmatrix}$$

is diagonalizable.

Solution.
The characteristic equation

$$\begin{aligned} \begin{vmatrix} 4-\lambda & 2 \\ -3 & 11-\lambda \end{vmatrix} &= (4-\lambda)(11-\lambda) - (-3)\cdot 2 \\ &= \lambda^2 - 15\lambda + 50 = 0 \end{aligned}$$

This equation has the solutions $\lambda = 5$ and $\lambda = 10$. We thus see that there are two distinct eigenvalues. From the theorem above, the matrix A is diagonalizable.

Example 3.14. Determine if

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 11 \end{pmatrix}$$

is diagonalizable.

Solution.
Since A is symmetric, we conclude that it is diagonalizable.

We finally look at the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 3 = 0.$$

This equation has no real solutions, so A has no real eigenvalues. In particular there are no real eigenvectors, so the matrix is not diagonalizable.

4. Quadratic forms and concave/convex functions

Reading. This lecture covers topics from Sections 1.7, 2.2 and 2.3 in FMEA [**2**]. For background material, see for instance Sections 4.1-4.4, 4.6-4.10, 6.1-6.11 and 11.1-11.3 in EMEA [**3**].

4.1. Quadratic forms. We start with an example of a quadratic form.

Definition 4.1. Define a function f in three variables x_1, x_2 and x_3 by $f(x_1, x_2, x_3) = \mathbf{x}^T A\mathbf{x}$, where

$$A = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Find an expression for f.

Solution.

$$\begin{aligned} f(x_1, x_2, x_3) &= \mathbf{x}^T A\mathbf{x} \\ &= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 3x_1 + 3x_3 \\ x_2 - 2x_3 \\ 3x_1 - 2x_2 + 8x_3 \end{pmatrix} \\ &= 6x_1x_3 - 4x_2x_3 + 3x_1^2 + x_2^2 + 8x_3^2. \end{aligned}$$

A function as in the example is called an *quadratic form.* Note that each term in the expression for the function is either a square or a product of two variables with constant coefficients.

Definition 4.2. A function in n variables, is said to be a quadratic form if it can be expressed as

$$f(x_1, x_2, \ldots, x_n) = \mathbf{x}^T A\mathbf{x}$$

where A is a symmetric $n \times n$ matrix and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Let us look at a quadratic form in two variables.

Example 4.3. Write the quadratic form

$$Q(x_1, x_2) = x_1^2 + 4x_1x_2 + 3x_2^2$$

as $Q(x_1, x_2) = \mathbf{x}^T A\mathbf{x}$ for a symmetric A.

Solution.

$$\begin{aligned} Q(x_1, x_2) &= x_1^2 + 4x_1x_2 + 3x_2^2 \\ &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

The method used in the example is the following: First write

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The coefficient in front of $x_1^2 = x_1x_1$ is 1, so we put $a_{11} = 1$. The coefficient in front of $x_2^2 = x_2x_2$ is 3, so we put $a_{22} = 3$. Thus the coefficients in front of the squares goes to the diagonal. The coefficient in front of x_1x_2 is 4. To create a symmetric matrix we divide this number by 2, and we put $a_{14} = a_{41} = 4/2 = 2$.

Definition 4.4. A quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ and its associated symmetric matrix A is said to be

(1) positive definite if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$.
(2) positive semidefinite if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$.
(3) negative definite if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$.
(4) negative semidefinite if $Q(\mathbf{x}) \leq$ for all $\mathbf{x} \neq \mathbf{0}$.
(5) indefinite if $Q(\mathbf{x})$ assumes both positive and negative values.

It turns out that it is possible to use the eigenvalues of the matrix A to determine the definiteness of the quadratic form.

Theorem 4.5. Let be a quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ where A is symmetric, and denote by $\lambda_1, \dots, \lambda_n$ the eigenvalues of A. Then

(1) $Q(\mathbf{x})$ is positive definite if and only if $\lambda_1 > 0, \dots, \lambda_n > 0$.
(2) $Q(\mathbf{x})$ is positive semidefinite if and only if $\lambda_1 \geq 0, \dots, \lambda_n \geq 0$.
(3) $Q(\mathbf{x})$ is negative definite if and only if $\lambda_1 < 0, \dots, \lambda_n < 0$.
(4) $Q(\mathbf{x})$ is negative semidefinite if and only if $\lambda_1 \leq 0, \dots, \lambda_n \leq 0$.
(5) $Q(\mathbf{x})$ is indefinite if it has both positive and negative eigenvalues.

We show in an example how to use this theorem.

Example 4.6. Determine the definiteness of $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ where

$$A = \begin{pmatrix} -1 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Solution.
The characteristic equation is

$$\begin{aligned}|A-\lambda I| &= \begin{vmatrix} -1-\lambda & 3 & 0 \\ 3 & -9-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} \\ &= (-2-\lambda)\begin{vmatrix} -1-\lambda & 3 \\ 3 & -9-\lambda \end{vmatrix} \\ &= -(\lambda+10)\,\lambda\,(\lambda+2) = 0.\end{aligned}$$

From this we get the eigenvalues $\lambda = -10, \lambda = -2$ and $\lambda = 0$. Consequently, $Q(\mathbf{x})$ is negative semidefinite.

In the next few lectures we will be concerned with optimization, i.e. the problem of finding maximum and minimum. For quadratic forms this is closely connected to definiteness. In fact, it turns out that our observations regarding quadratic forms will be a fundamental tool in the general optimization theory.

Example 4.7. Let

$$Q(x_1, x_2, x_3) = -9x_1^2 - 6x_2^2 - x_3^2 + 4x_1x_2.$$

Show that $Q(0,0,0)$ is the maximum value for Q.

Solution.
We have that $Q = \mathbf{x}^T A\mathbf{x}$ where

$$A = \begin{pmatrix} -9 & 2 & 0 \\ 2 & -6 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Solving the characteristic equation we find that the eigenvalues are $\lambda = -1$, $\lambda = -5$ and $\lambda = -10$. This means that Q is negative definite, and by the definition this means that $Q(x_1, x_2, x_3) < 0$ for all $(x_1, x_2, x_3) \neq (0,0,0)$. In other word 0 is the largest value attained by Q.

4.2. Convex sets. We will need to consider point sets in Euclidean space $\mathbb{R}^n$. Earlier we defined $\mathbb{R}^n$ to be the set of n-vectors. We will identify this with the set of points in n-space and interchangeably speak of points and vectors in $\mathbb{R}^n$. For instance $(1,2,3)$ is both a point in $\mathbb{R}^3$ and a vector in $\mathbb{R}^3$. If $x_1, x_2, \ldots, x_n$ are n variables, we write $\mathbf{x}$ for the vector $(x_1, x_2, \ldots, x_n)$ but we may also consider this as a variable point in $\mathbb{R}^n$. We start with subsets of the plane $\mathbb{R}^{2.}$

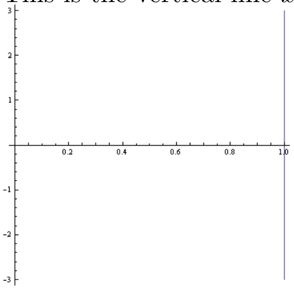
I apologize for the following trivial example, but it is important that we can relate to the notion of point sets.

Example 4.8. Draw the set $\{(1,2),(0,1),(3,1)\} \subset \mathbb{R}^2$.

In the example we considered a subset of the plane consisting of a finite set of points. We will mostly be interested in other subsets of an Euclidean space.

Example 4.9. Draw the set $\{(x,y) : x = 1\} \subseteq \mathbb{R}^2$.

Solution.
This is the vertical line $x = 1$.



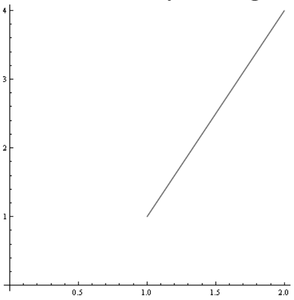
For the definition of convex and concave, the following particular class of sets is important.

Example 4.10. Draw the set $S = \{(x, y) : (x, y) = s(1, 1) + (1 - s)(2, 4) \text{ for } s \in [0, 1]\}$.

Solution.
We try different values for $s \in [0, 1]$.

$$\begin{aligned} s = 0\textbf{:} &\implies (x, y) = 0 \cdot (1, 1) + (1 - 0) \cdot (2, 4) = (2, 4) \\ s = \tfrac{1}{2}\textbf{:} &\implies (x, y) = \tfrac{1}{2}(1, 1) + (1 - \tfrac{1}{2}) \cdot (2, 4) = (\tfrac{1}{2}, \tfrac{1}{2}) + \tfrac{1}{2}(2, 4) = (\tfrac{3}{2}, \tfrac{5}{2}) \\ s = 1\textbf{:} &\implies (x, y) = 1 \cdot (1, 1) + (1 - 1) \cdot (2, 4) = (1, 1) \end{aligned}$$

For instance by taking more points, one sees that S is the line segment from $(1, 1)$ to $(2, 4)$.

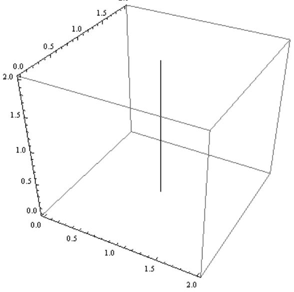


The set in the example was a line segment. We may consider line segments in $\mathbb{R}^n$ for any n. If $\mathbf{x}$ and $\mathbf{y}$ are two points in $\mathbb{R}^n$, we denote by $[\mathbf{x}, \mathbf{y}]$ line segment between these points.

Example 4.11. Let $\mathbf{x} = (1, 1, 0)$ and $\mathbf{y} = (1, 1, 2)$. Draw the *line segment*

$$[\mathbf{x}, \mathbf{y}] = \{\mathbf{z} : \mathbf{z} = s\mathbf{x} + (1 - s)\mathbf{y},\ s \in [0, 1]\}$$

Solution.
This is a vertical line segment parallel to the z-axis starting in the point $(1, 1, 0)$ and ending in the point $(1, 1, 2)$.



We may now define the important notion of a convex set.

Definition 4.12. A subset S of $\mathbb{R}^n$ is called convex if for all $\mathbf{x}$ and $\mathbf{y}$ in S, the line segment of $[\mathbf{x}, \mathbf{y}]$ from $\mathbf{x}$ to $\mathbf{y}$ is also contained in S.

You should draw examples of convex and non-convex sets in the plane.

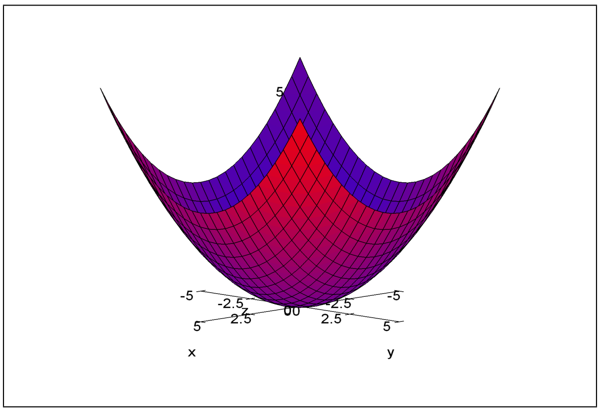
4.3. Concave/convex functions. Recall that a function $f : [a, b] \to \mathbb{R}$ in two variables is said to be concave if the second derivative $f''(x) \leq 0$. This corresponds to the graph having the form $\cap$. Similarly, f is said to be convex if $f''(x) \geq 0$. This corresponds to the graph having the form $\cup$. We will now extend this notions to functions of several variables. To do this, we will start with functions in two variables and state the definition informally.

Recall that we can draw the graph of a function in two variables, in 3-space $\mathbb{R}^3$. The graph of a function f in two variables x and y defined on a set $S \subseteq \mathbb{R}^2$ consists of the points (x, y, z) where (x, y) are in S and $z = f(x, y)$.

Definition 4.13. Let f be a function in two variables defined on a convex set S.

(1) The function f is called *concave* if a line segment joining two points on the graph is never above the graph.
(2) The function f is called *convex* if a line segment joining two points on the graph is never below the graph.

You should draw a picture to clarify this definition. Here is an example of a the convex function $f(x, y) = x^2 + y^2$



The formal definition of concave and convex functions is as follows. Be careful to note the difference between the notion of a convex set and the notion of a convex function. These notions should not be confused.

Definition 4.14. A function f in n variables $x_1, x_2, \ldots, x_n$ defined on a convex set $S \subseteq \mathbb{R}^n$ is said to be *convex* if

$$f(s\mathbf{x} + (1-s)\mathbf{y}) \leq sf(\mathbf{x}) + (1-s)f(\mathbf{y}) \tag{4}$$

for all $\mathbf{x}$ and $\mathbf{y}$ in S and $s \in [0, 1]$. The function f is said to be *concave* if

$$f(s\mathbf{x} + (1-s)\mathbf{y}) \geq sf(\mathbf{x}) + (1-s)f(\mathbf{y}) \tag{5}$$

for all $\mathbf{x}$ and $\mathbf{y}$ in S and $s \in [0, 1]$. If $\leq$ is interchanged with $<$ the function is called *strictly convex*, and if $\geq$ is interchanged with $>$ the function is called *strictly concave*.

In the next lecture we will develop useful criteria for a function to be convex or concave, but in some cases it is possible to use the definition directly.

Example 4.15. Use the last definition to show that f given by

$$f(x_1, x_2) = x_1 + x_2$$

is both convex and concave.

Solution.
Note that if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ we have that

$$\begin{aligned} s\mathbf{x} + (1-s)\mathbf{y} &= s(x_1, x_2) + (1-s)(y_1, y_2) \\ &= (sx_1 + (1-s)y_1, sx_2 + (1-s)y_2). \end{aligned}$$

From this we see that

$$\begin{aligned} f(s\mathbf{x} + (1-s)\mathbf{y}) &= f(sx_1 + (1-s)y_1, sx_2 + (1-s)y_2) \\ &= sx_1 + (1-s)y_1 + sx_2 + (1-s)y_2 \\ &= s(x_1 + x_2) + (1-s)(y_1 + y_2). \end{aligned}$$

On the other hand,

$$f(\mathbf{x}) + (1-s)f(\mathbf{y}) = s(x_1 + x_2) + (1-s)(x_1 + x_2)$$

so we see that

$$f(s\mathbf{x} + (1-s)\mathbf{y}) = sf(\mathbf{x}) + (1-s)f(\mathbf{y})$$

and it follows that f is both convex and concave.

If we know some convex functions, the next theorem shows that we may construct other convex functions from these.

Theorem 4.16. Let f and g be functions defined on a convex set S in $\mathbb{R}^n$. We have that, if f and g are convex (concave) and $a \geq 0$, $b \geq 0$, then the function $af + bg$ is convex (concave).

From the theory of functions in one variable, we know that the function $f(x) = x^2$ is convex. (Draw its graph or note that $f''(x) = 2 > 0$). We may use this to find many examples of convex functions in several variables.

Example 4.17. Show that the function f defined by

$$f(x_1, x_2) = 2x_1^2 + 3x_2^2$$

is convex.

Solution.
The function $g(x_1, x_2) = x_1^2$ is easily seen to by convex as in the case of functions of one variable. Similarly, $h(x_1, x_2) = x_2^2$ is convex. From the theorem we conclude that $f = 2g + 3h$ is convex.

More generally, we have the following:

Proposition 4.18. Assume that $Q = \mathbf{x}^T A\mathbf{x}$ is a quadratic form in n variables.

(1) If Q is positive definite or positive semidefinite, then Q is convex.
(2) If Q is negative definite or negative semidefinite, then Q is concave.

5. The Hessian matrix

Reading. This lecture covers topics from 1.7, 2.3 and 3.1 in FMEA [**2**].

5.1. Principal minors. Principal minors are used to determine the definiteness of a quadratic form or a symmetric matrix, and they are defined as follows.

Definition 5.1. Let A be an $n \times n$ matrix. A principal minor of order r is the determinant obtained by deleting all but r rows and r columns with the same numbers.

The following example will clarify this notion.

Example 5.2. Let

$$A = \begin{pmatrix} 1 & 4 & 6 \\ 3 & 2 & 1 \\ 2 & 4 & 6 \end{pmatrix}.$$

The determinant

$$\begin{vmatrix} 1 & 4 & 6 \\ 3 & 2 & 1 \\ 2 & 4 & 6 \end{vmatrix}.$$

is a principal minor, since it is obtained by deleting *zero* columns and rows.
The minor

$$\begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix}$$

is however not a principal minor since we are deleting row number 1 but column number 3.
The minor

$$\begin{vmatrix} 1 & 6 \\ 2 & 6 \end{vmatrix}$$

on the other hand, is a principal minor since it is obtained by deleting row 2 and column 2.

The so-called *leading principal minors* will be useful for determining if a quadratic form is positive or negative definite or semidefinite.

Definition 5.3. A principal minor is called a *leading principal minor* if it is obtained by deleting the last rows and columns.

It is easy to spot the principal minors for the matrix in the previous example.

Example 5.4. Let

$$A = \begin{pmatrix} 1 & 4 & 6 \\ 3 & 2 & 1 \\ 2 & 4 & 6 \end{pmatrix}.$$

The leading principal minors are

$$D_1 = 1, \; D_2 = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} \text{ and } D_3 = \begin{vmatrix} 1 & 4 & 6 \\ 3 & 2 & 1 \\ 2 & 4 & 6 \end{vmatrix}.$$

We can use the leading principle minors to investigate the definiteness of a quadratic form and later to determine if a function is convex or concave.

Theorem 5.5. Let

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

be quadratic form given by a symmetric $n \times n$ matrix A. Denote by D_k the leading principal minor of order k, and let Δ_k denote any principal minor of order k. Then the quadratic form Q and the symmetric matrix A are

(1) positive definite $\Longleftrightarrow D_k > 0$ for $k = 1, \ldots, n$.
(2) positive semidefinite $\Longleftrightarrow \Delta_k \geq 0$ for all principal minors.
(3) negative definite $\Longleftrightarrow (-1)^k D_k > 0$ for $k = 1, \ldots, n$.
(4) negative semidefinite $\Longleftrightarrow (-1)^k \Delta_k \geq 0$ for all principal minors.

Note that the condition $(-1)^k D_k > 0$ for $k = 1, \ldots, n$, is the same as $D_1 < 0, D_2 > 0, D_3 < 0$, etc. We apply the theorem in an example.

Example 5.6. Show that the quadratic form defined by

$$Q(x_1, x_2, x_3) = 3x_1^2 + 6x_1x_3 + x_2^2 - 4x_2x_3 + 8x_3^2$$

is positive definite.

Solution.
We may write Q as

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and

$$A = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{pmatrix}.$$

The leading principal minors

$$3, \quad \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3, \text{ and } \begin{vmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{vmatrix} = 3$$

are all positive, and this means that Q is positive definite.

5.2. The Hessian matrix. Several of the result that we will state in this and the next few lectures are concerned with functions that belong to the class denoted C^2. That a function is C^2 means that the second order partial derivatives exists and are continuos. I will not state this condition, but rather assume that all functions in this course are C^2.

We start by recalling the notion of partial derivatives.

Example 5.7. Assume

$$f(x_1, x_2) = x_1^2 - x_2^2 - x_1x_2.$$

Find all the first order partial derivatives of f.

Solution.
We denote by f_1' the partial derivative of f with respect to the first variable x_1:

$$f_1' = \frac{\partial f}{\partial x_1} = 2x_1 - x_2.$$

We denote by f_2' the partial derivative of f with respect to the second variable x_2:

$$f_2' = \frac{\partial f}{\partial x_2} = -2x_2 - x_1.$$

To define the Hessian matrix we need to compute the second order partial derivatives. We denote by f_{12}'' the second order partial derivative, were we first differentiate with respect to the first variable x_1 and then differentiate with respect to the second variable x_2. We denote by f_{22}'' the second order partial derivative, were we differentiate twice with respect to x_2. Note also that it is common to write

$$f''_{12} = \frac{\partial^2 f}{\partial x_2 \partial x_1} \text{ and } f''_{21} = \frac{\partial^2 f}{\partial x_1 \partial x_2}.$$

At this point we should also make another remark on notation. We have already used $\mathbf{x}$ in two slightly different meanings. We have used $\mathbf{x}$ to denote the column vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

in connection with matrices. In connection with functions we have used $\mathbf{x} = (x_1, \dots, x_n)$ for instance when writing $f(\mathbf{x}) = f(x_1, \dots, x_n)$. We will continue this notational ambiguity, but the two meanings of $\mathbf{x}$ can be distinguished from the context.

Example 5.8. Find the second order partial derivatives f''_{12} and f''_{22} of the function $f(x_1, x_2) = x_1^2 - x_2^2 - x_1 x_2$.

Solution.
We get

$$\begin{aligned} f''_{12} &= \frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}(2x_1 - x_2) = -1 \\ f''_{22} &= \frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}(-2x_2 - x_1) = -2. \end{aligned}$$

Definition 5.9. Let f be a function in n variables $x_1, x_2, \dots, x_n$. The matrix

$$\mathbf{f}''(\mathbf{x}) = \left(f''_{ij}(\mathbf{x})\right)_{n \times n}$$

is called the Hessian (matrix) of f at $\mathbf{x}$.

Note that for all functions that we will consider in this course, the Hessian matrix will be a symmetric matrix. This follows from *Young's theorem.*

Example 5.10. Find the Hessian matrix of the function

$$f(x_1, x_2) = x_1^2 - x_2^2 - x_1 x_2.$$

Solution.
We get that

$$\begin{aligned} f''_{11} &= \frac{\partial}{\partial x_1}(2x_1 - x_2) = 2 \\ f''_{12} &= -1 \\ f''_{21} &= \frac{\partial}{\partial x_1}(-2x_2 - x_1) = -1 \\ f''_{22} &= -2 \end{aligned}$$

From this we get

$$\begin{aligned} \mathbf{f}''(x_1, x_2) &= \begin{pmatrix} f''_{11}(\mathbf{x}) & f''_{12}(\mathbf{x}) \\ f''_{21}(\mathbf{x}) & f''_{22}(\mathbf{x}) \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}. \end{aligned}$$

Some theorems can be used only for functions defined on open subsets of $\mathbb{R}^n$. We will not give precise definition of a open set, but refer to FMEA [**2**] page 460. For our purposes it will be enough to know that the Euclidian spaces $\mathbb{R}^n$ are open and also that subsets of $\mathbb{R}^n$ defined by strict inequalities ($>$ and $<$) are usually open.

Example 5.11. The subset $S = \{(x_1, x_2) | x_1 > 0 \text{ and } x_2 > 0\}$ of $\mathbb{R}^2$ consists of points where both coordinates are positive. This set is open. The subset $\{(x_1, x_2) | x_1 \geq 0 \text{ and } x_2 > 0\}$ of $\mathbb{R}^2$ is however not open.

The following important theorem tells us that we can use the Hessian matrix to determine if a function is concave or convex.

Theorem 5.12. Suppose f is a function in n variables defined on an open convex set S in $\mathbb{R}^n$. Let $\mathbf{f}''(\mathbf{x})$ denote the Hessian matrix of f at $\mathbf{x}$.

(1) $\mathbf{f}''(\mathbf{x})$ is positive definite in $S \Longrightarrow f$ is strictly convex in S.
(2) $\mathbf{f}''(\mathbf{x})$ is negative definite in $S \Longrightarrow f$ is strictly concave in S.
(3) $\mathbf{f}''(\mathbf{x})$ is positive semidefinite in $S \Longleftrightarrow f$ is convex in S.
(4) $\mathbf{f}''(\mathbf{x})$ is negative semidefinite in $S \Longleftrightarrow f$ is concave in S.

The theorem is a powerful tool for determining if a function is convex or concave. Let us look at the function from the example above.

Example 5.13. Is the function

$$f(x_1, x_2) = x_1^2 - x_2^2 - x_1 x_2$$

convex or concave?

Solution.
We have already found that

$$\mathbf{f}''(x_1, x_2) = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 - 5 = 0$$

and has the solutions $\lambda = -\sqrt{5}$ and $\lambda = \sqrt{5}$. Since there are both positive and negative eigenvalues, $\mathbf{f}''(x_1, x_2)$ is indefinite, so f is neither convex nor concave.

We may also use principal minors to determine the definiteness of the Hessian matrix.

Example 5.14. Consider the function f defined by

$$f(x, y) = 2x - y - x^2 + xy - y^2$$

for all (x, y), i.e. f is defined on $S = \mathbb{R}^2$.
Is f concave or convex?

Solution.
We get

$$f_1' = 2 - 2x + y$$
$$f_2' = -1 + x - 2y$$

Differentiating again, we get

$$f_{11}' = -2$$
$$f_{12}' = 1$$
$$f_{21}' = 1$$
$$f_{22}' = -2.$$

Thus the Hessian matrix is

$$\mathbf{f}''(x, y) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

The leading principal minors are

$$D_1 = -2$$
$$D_2 = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3.$$

We get

$$(-1)^1 D_1 = (-1)(-2) = 2 > 0$$
$$(-1)^2 D_2 = 1 \cdot 3 = 3 > 0.$$

This means that the Hessian matrix is negative definite, and hence f is strictly concave.

Example 5.15. Show that the quadratic form

$$Q(x_1, x_2) = -x_1^2 + 2x_1x_2 - x_2^2$$

is negative semidefinite but not negative definite

Solution.
The quadratic form may be written as

$$Q = \mathbf{x}^T A \mathbf{x}$$

where

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The principal minors of order 1 are -1 and -1. Thus we have $\Delta_1 \leq 0$ for all principal minors Δ_1 of order 1. There is only one principal minor of order two

$$\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0.$$

Thus we have $\Delta_2 \geq 0$ for all principal minors Δ_2 of order 2. Since the last minor is also a leading principal minor, and since it is zero, we can conclude that the quadratic form is not negative definite.

5.3. Extreme points. In this section we come to the main concern of finding maxima and minima of a function of several variables.

Definition 5.16. Let f be a function in n variables $x_1, \ldots, x_n$. A *stationary point* of f is a point where all first order partial derivatives are 0.

To find the stationary points we must solve the system of equations obtained by setting the partial derivatives equal to zero.

Example 5.17. Find the stationary points of f defined by

$$f(x, y) = 2x - y - x^2 + xy - y^2.$$

Solution.
We get

$$\begin{aligned} f_1' &= 2 - 2x + y = 0 \\ f_2' &= -1 + x - 2y = 0. \end{aligned}$$

Rewriting we have

$$\begin{aligned} -2x + y &= -2 \\ x - 2y &= 1. \end{aligned}$$

This is a system of linear equation and may be written as

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

using matrix notation. The solution is given by

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

We conclude that $(x, y) = (1, 0)$ is the only stationary point of f.

Definition 5.18. Let f be a function in n variables $x_1, \ldots, x_n,$ defined on a set $S \subseteq \mathbb{R}^n$. A point $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ in S is called a global maximum for f if $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all other points $\mathbf{x}$ in S. We say that $\mathbf{x}^*$ is the maximum point and that $f(\mathbf{x}^*)$ is the maximum value. Minimum points and minimum values are defined similarly.

A maximum point and a minimum point will be stationary points.

Theorem 5.19. If $\mathbf{x}^*$ is an interior maximum point or a minimum point, then $\mathbf{x}^*$ is a stationary point.

On the other hand, if we know the stationary points we can often determine the maximum points and the minimum points.

Theorem 5.20. Suppose that f is defined on S and that $\mathbf{x}^*$ is an interior point. Then

(1) If f is concave, then $\mathbf{x}^*$ stationary $\Longleftrightarrow$ $\mathbf{x}^*$ is a global maximum.
(2) If f is convex, then $\mathbf{x}^*$ is stationary $\Longleftrightarrow$ $\mathbf{x}^*$ is a global minimum.

Consider the following example.

Example 5.21. Explain that

$$f(x, y) = 2x - y - x^2 + xy - y^2$$

has a global maximum.

Solution.
We have found that f is concave and that $(x, y) = (1, 0)$ is the only stationary point. By the theorem above, we conclude that $(1, 0)$ is a global maximum point.

We conclude with the following examples.

Example 5.22. Find all extreme points of f given by

$$f(x, y, z) = x^2 + 2y^2 + 3z^2 + 2xy + 2xz + 3.$$

Solution.
We get

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x + 2y + 2z = 0\\ \frac{\partial f}{\partial y} &= 4y + 2x = 0\\ \frac{\partial f}{\partial z} &= 6z + 2x = 0.\end{aligned}$$

This gives a system of linear equations that can be written as

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since

$$\begin{vmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{vmatrix} = 8 \neq 0$$

we have only the solution $(x, y, z) = (0, 0, 0)$. So $(0, 0, 0)$ is the only stationary point. The Hessian matrix becomes

$$\mathbf{f}'' = \begin{pmatrix} f''_{11} & f''_{12} & f''_{13} \\ f''_{21} & f''_{22} & f''_{23} \\ f''_{31} & f''_{32} & f''_{33} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{pmatrix}$$

and the leading principal minors are

$$\begin{aligned} D_1 &= 2, \\ D_2 &= \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} = 4, \\ D_3 &= \begin{vmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{vmatrix} = 8. \end{aligned}$$

Since the leading principal minors are all positive, the Hessian matrix $\mathbf{f}''$ is positive definite. This means that $(x, y, z) = (0, 0, 0)$ is a global minimum.

Example 5.23. Consider the function

$$f(x_1, x_2, x_3) = x_1^4 + x_2^4 + x_3^4 + x_1^2 + x_2^2 + x_3^2.$$

Find its extreme points.

Solution.
The first order conditions yields

$$\begin{aligned} f_1' &= 2x_1 + 4x_1^3 = 2x_1\left(1 + 2x_1^2\right) = 0 \\ f_2' &= 2x_2 + 4x_2^3 = 2x_2(1 + 2x_2^2) = 0 \\ f_3' &= 2x_3 + 4x_3^3 = 2x_3(1 + 2x_3^2) = 0 \end{aligned}$$

From this we see that $(x_1^*, x_2^*, x_3^*) = (0, 0, 0)$ is the only stationary point. The Hessian matrix is

$$\mathbf{f}'' = \begin{pmatrix} f_{11}'' & f_{12}'' & f_{13}'' \\ f_{21}'' & f_{22}'' & f_{23}'' \\ f_{31}'' & f_{32}'' & f_{33}'' \end{pmatrix} = \begin{pmatrix} 12x_1^2 + 2 & 0 & 0 \\ 0 & 12x_2^2 + 2 & 0 \\ 0 & 0 & 12x_3^2 + 2 \end{pmatrix}$$

For any choice of (x_1, x_2, x_3) this matrix will be positive definite, so the function f is convex. Hence $(x_1^*, x_2^*, x_3^*) = (0, 0, 0)$ is a minimum.

Example 5.24. Consider the function

$$f(x_1, x_2) = -x_1^3 - x_2^2$$

defined on

$$S = \{(x_1, x_2) | x_1 > 0 \text{ and } x_2 > 0\}.$$

Show that S is a convex set and that f is a concave function.

Solution.
Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ be two points in S. We must show that the line segment $[\mathbf{x}, \mathbf{y}]$ is also contained in S. To do this let $\mathbf{z} = (z_1, z_2)$ be any point on the line segment. Then

$$(z_1, z_2) = s(x_1, x_2) + (1 - s)(y_1, y_2)$$

for some $s \in [0, 1]$. Since $\mathbf{x}$ and $\mathbf{y}$ are in S, we know that $x_1 > 0$, $x_2 > 0$, $y_1 > 0$ and $y_2 > 0$. From this we see

$$\begin{aligned} z_1 &= sx_1 + (1 - s)y_1 > 0 \\ z_2 &= sx_2 + (1 - s)y_2 > 0 \end{aligned}$$

and we conclude that $\mathbf{z} = (z_1, z_2)$ is in S.
To show that the function f is concave, we find the Hessian matrix. We have

$$\begin{aligned} f_1' &= -3x_1^2 \\ f_2' &= -2x_2 \end{aligned}$$

and

$$\begin{aligned} \mathbf{f}''(x_1, x_2) &= \begin{pmatrix} f_{11}''(\mathbf{x}) & f_{12}''(\mathbf{x}) \\ f_{21}''(\mathbf{x}) & f_{22}''(\mathbf{x}) \end{pmatrix} \\ &= \begin{pmatrix} -6x_1 & 0 \\ 0 & -2 \end{pmatrix}. \end{aligned}$$

Since $x_1 > 0$ when (x_1, x_2) is in S, we see that the numbers on the diagonal will always be negative. For a diagonal matrix, the eigenvalues are on the diagonal, so we conclude that the Hessian is negative definite for all (x_1, x_2) in S. This means that the function is strictly concave.

6. Local extreme points and the Lagrange problem

Reading. This lecture covers topics from Sections 3.2 and 3.3 in FMEA [**2**].

6.1. Local extreme points. In the previous lecture we defined global maximum and minimum for a function in several variables. We will also consider the notion of a local maximum or minimum.

Definition 6.1. Let f be a function in $\mathbf{x} = (x_1, \dots, x_n)$. A point $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ is a local maximum point for f if $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x}$ in a small neighborhood of $\mathbf{x}^*$, and $\mathbf{x}^*$ is called a local minimum point for f if $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x}$ in a small neighborhood of $\mathbf{x}^*$.

If we compare this definition with the definition of a *global maximum* from the previous lecture, we note that we for a local maximum only need $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for $\mathbf{x}$ *close* to $\mathbf{x}^*$.

Example 6.2. We will show later that the function

$$f(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + 3x_1x_3 + x_2^3 + 3x_2x_3 + x_3^3$$

has $\mathbf{x}^* = (-2, -2, -2)$ as a local maximum. We have

$$f(-2, -2, -2) = 12$$

for the minimum value and

$$f(-1.8, -1.8, -1.8) = 11.664$$

which is less as we would expect since 12 is a local maximum. We have however

$$f(1, 1, 2) = 25$$

so $(-2, -2, -2)$ is not a global maximum.

As for global maxima and minima, if $\mathbf{x}^*$ is an interior local maxima or minima, then $\mathbf{x}^*$ is a stationary point. Conversely, if we know that $\mathbf{x}^*$ is a stationary point, we would like to know if it is a local maximum or a local minimum. The following theorem is known as the second derivative test.

Theorem 6.3. Suppose $f(\mathbf{x}) = f(x_1, \dots, x_n)$ is defined on a subset S of $\mathbb{R}^n$ and that $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ is an interior stationary point.

(1) If the Hessian matrix $\mathbf{f}''(\mathbf{x}^*)$ at $\mathbf{x}^*$ is positive definite, then $\mathbf{x}^*$ is a local minimum point.
(2) If the Hessian matrix $\mathbf{f}''(\mathbf{x}^*)$ at $\mathbf{x}^*$ is negative definite, then $\mathbf{x}^*$ is a local maximum point.
(3) If the Hessian matrix $\mathbf{f}''(\mathbf{x}^*)$ at $\mathbf{x}^*$ is indefinite, then $\mathbf{x}^*$ is neither a local maximum nor a local minimum.

We consider an example.

Example 6.4. Show that $(x_1^*, x_2^*, x_3^*) = (-2, -2, -2)$ is a local maximum for f given by

$$f(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + 3x_1x_3 + x_2^3 + 3x_2x_3 + x_3^3.$$

Solution.
First we show that $(x_1^*, x_2^*, x_3^*) = (-2, -2, -2)$ is a stationary point. The first order conditions are:

$$\begin{aligned} f_1' &= 3x_1^2 + 3x_2 + 3x_3 = 0 \\ f_2' &= 3x_1 + 3x_2^2 + 3x_3 = 0 \\ f_3' &= 3x_1 + 3x_2 + 3x_3^2 = 0. \end{aligned}$$

By substitution, we see that $(x_1^*, x_2^*, x_3^*) = (-2, -2, -2)$ satisfies these equations. Thus $(-2, -2, -2)$ is a stationary point. To show that it is a local maximum, we calculate the Hessian matrix

$$\mathbf{f}'' = \begin{pmatrix} f_{11}'' & f_{12}'' & f_{13}'' \\ f_{21}'' & f_{22}'' & f_{23}'' \\ f_{31}'' & f_{32}'' & f_{33}'' \end{pmatrix} = \begin{pmatrix} 6x_1 & 3 & 3 \\ 3 & 6x_2 & 3 \\ 3 & 3 & 6x_3 \end{pmatrix}.$$

In the point $(-2, -2, -2)$, we have that

$$\mathbf{f}''(-2, -2, -2) = \begin{pmatrix} -12 & 3 & 3 \\ 3 & -12 & 3 \\ 3 & 3 & -12 \end{pmatrix}.$$

We must show that this symmetric matrix is negative definite. To do this we calculate the leading principal minors:

$$D_1 = -12,\ D_2 = \begin{vmatrix} -12 & 3 \\ 3 & -12 \end{vmatrix} = 135 \text{ and } D_3 = \begin{vmatrix} -12 & 3 & 3 \\ 3 & -12 & 3 \\ 3 & 3 & -12 \end{vmatrix} = -1350$$

We thus have

$$(-1)^1 D_1 = 12,\ (-1)^2 D_2 = 135 \text{ and } (-1)^3 D_3 = 1350.$$

Since these are all positive, we conclude that the Hessian matrix is negative definite, and by the theorem we conclude that $(-2, -2, -2)$ is a local maximum.

Definition 6.5. A stationary point that is neither a local maximum nor a local minimum is called a *saddle point.*

From the theorem above, we know that if the Hessian matrix is indefinite in a stationary point, then the we have a saddle point.

Example 6.6. Show that $\mathbf{x}^* = (0, 0, 0)$ is a saddle point for

$$f(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + 3x_1x_3 + x_2^3 + 3x_2x_3 + x_3^3.$$

Solution.
From the first order conditions in the previous example, we see that $\mathbf{x}^* = (0,0,0)$ is a stationary point. We have already found the Hessian matrix

$$\mathbf{f}'' = \begin{pmatrix} f''_{11} & f''_{12} & f''_{13} \\ f''_{21} & f''_{22} & f''_{23} \\ f''_{31} & f''_{32} & f''_{33} \end{pmatrix} = \begin{pmatrix} 6x_1 & 3 & 3 \\ 3 & 6x_2 & 3 \\ 3 & 3 & 6x_3 \end{pmatrix}.$$

In the point $(0,0,0)$ we have

$$\mathbf{f}''(0,0,0) = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}.$$

We will show that this symmetric matrix is indefinite. One method is to find the eigenvalues:

$$\begin{vmatrix} 0-\lambda & 3 & 3 \\ 3 & 0-\lambda & 3 \\ 3 & 3 & 0-\lambda \end{vmatrix} = 0$$

To solve the characteristic equation we add the two first rows to the third row. Since the determinant is unchanged under these operations, we can solve

$$\begin{vmatrix} -\lambda & 3 & 3 \\ 3 & -\lambda & 3 \\ 6-\lambda & 6-\lambda & 6-\lambda \end{vmatrix} = 0$$

If we use cofactor expansion along the third row, we obtain

$$(6-\lambda)((-1)^{3+1}\begin{vmatrix} 3 & 3 \\ -\lambda & 3 \end{vmatrix} + (-1)^{3+2}\begin{vmatrix} -\lambda & 3 \\ 3 & 3 \end{vmatrix} + (-1)^{3+3}\begin{vmatrix} -\lambda & 3 \\ 3 & -\lambda \end{vmatrix}) = 0$$

or

$$(6-\lambda)(\lambda^2 + 6\lambda + 9) = 0.$$

From this we obtain the eigenvalues $\lambda = 6$ and -3. Since there are both positive and negative eigenvalues, we conclude that Hessian matrix is indefinite. Thus we have proven that $(0,0,0)$ is a saddle point.

The version of the second derivative test that we have considered is valid for any numbers of variables, and in the case of two variables it specializes to the more known version.

Example 6.7. Let f be a function in two variables. Then the Hessian matrix is

$$\mathbf{f}'' = \begin{pmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{pmatrix}.$$

The principal minors are

$$D_1 = f''_{11} \text{ and } D_2 = \begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} = f''_{11}f''_{22} - f''_{21}f''_{12}.$$

Since we only consider functions that are C^2, we have that $f''_{21} = f''_{12}$. Thus $D_2 = f''_{11}f''_{22} - (f''_{12})^2$.
If $D_1 > 0$ and $D_2 > 0$, we have a minimum. If $(-1)^1 D_1 > 0$ and $(-1)^2 D_2 > 0$ we have a maximum. We have seen that D_2 is equal to the product of the eigenvalues. So if $D_2 < 0$ we must have both a positive and a negative eigenvalue. Thus if $D_2 < 0$, the Hessian is indefinite. In conclusion we have

$f''_{11} > 0$ and $f''_{11}f''_{22} - (f''_{12})^2 > 0$	local minimum
$f''_{11} < 0$ and $f''_{11}f''_{22} - (f''_{12})^2 > 0$	local maximum
$f''_{11}f''_{22} - (f''_{12})^2 < 0$	saddle point

As we know, if $f''_{11}f''_{22} - (f''_{12})^2 = 0$, we cannot conclude on basis of the second derivative test. We know that one of the eigenvalues has to be zero, but we can still have a local minimum or a local maximum or a saddle point in this case.

If the Hessian matrix is positive or negative semidefinite, we cannot classify the point without further investigations. However on the other hand if we know that a point is a local maximum or a local minimum, it forces the Hessian matrix to be negative or positive semidefinite in these points.

Theorem 6.8. We have that:

(1) If $\mathbf{x}^*$ is a local minimum, then the Hessian $\mathbf{f}''(\mathbf{x}^*)$ at $\mathbf{x}^*$ is positive semidefinite.
(2) If $\mathbf{x}^*$ is a local maximum, then the Hessian $\mathbf{f}''(\mathbf{x}^*)$ at $\mathbf{x}^*$ is negative semidefinite.

Example 6.9. Classify the stationary points of

$$f(x_1, x_2, x_3) = -2x_1^4 + 2x_2x_3 - x_2^2 + 8x_1$$

Solution.
The first order conditions are

$$\begin{aligned} f_1' &= -8x_1^3 + 8 = 0 \\ f_2' &= 2x_3 - 2x_2 = 0 \\ f_3' &= 2x_2 = 0. \end{aligned}$$

From these we obtain

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= 0. \end{aligned}$$

Thus $\mathbf{x}^* = (x_1, x_2, x_3) = (1, 0, 0)$ is the only stationary point. The Hessian matrix is

$$\mathbf{f}'' = \begin{pmatrix} f_{11}'' & f_{12}'' & f_{13}'' \\ f_{21}'' & f_{22}'' & f_{23}'' \\ f_{31}'' & f_{32}'' & f_{33}'' \end{pmatrix} = \begin{pmatrix} -24x_1^2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

In the point $\mathbf{x}^* = (1, 0, 0)$, this becomes

$$\mathbf{f}''(1, 0, 0) = \begin{pmatrix} -24 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

We solve for the eigenvalues

$$\begin{vmatrix} -24 - \lambda & 0 & 0 \\ 0 & -2 - \lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} = (-24 - \lambda)(\lambda^2 + 2\lambda - 4) = 0.$$

From this we obtain that $\lambda = -24$ or $\lambda = -1 \pm \sqrt{5}$. Since $-1 + \sqrt{5} > 0$, we have both positive and negative eigenvalues. This means that $\mathbf{f}''(1, 0, 0)$ is indefinite, and thus that $(1, 0, 0)$ is a saddle point.
We may also conclude by looking at the leading principal minors. We have

$$D_1 = -24, \; D_2 = 48 \text{ and } D_3 = 96$$

Since none of them are zero and since they are not all positive and do not have alternating signs, we can conclude that the Hessian is indefinite.

6.2. The Lagrange problem. The Lagrange Problem is the problem of finding the maximum or minimum of a function $f(x_1, \ldots, x_n)$ subject to condition constraints $g_1(x_1, \ldots, x_n) = b_1, \ldots, g_m(x_1, \ldots, x_n) = b_m$. Here f is called the objective function and $g_1, g_2 \ldots, g_n$ are called the constraint functions.

Definition 6.10. The *Lagrange function* or the *Lagrangian* is defined as

$$\mathcal{L}(x_1, \dots, x_n) = f(x_1, \dots, x_n) - \lambda_1 g_1(x_1, \dots, x_n) - \cdots - \lambda_m g_m(x_1, \dots, x_n).$$

The equations

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \text{ for } i = 1, \dots, n$$

are called the first order conditions.

The variables λ_i are called the *Lagrange multipliers.* In some texts the Lagrangian is defined as

$$\mathcal{L}(x_1, \dots, x_n) = f(x_1, \dots, x_n) - \lambda_1(g_1(x_1, \dots, x_n) - b_1) - \cdots - \lambda_m(g_m(x_1, \dots, x_n) - b_m).$$

The advantage of this definition is that on obtain the constraints as $\frac{\partial \mathcal{L}}{\partial \lambda_i} = 0$.

Theorem 6.11. Suppose f, $g_1, \dots, g_m$ are defined on a subset S of $\mathbb{R}^n$, and that $\mathbf{x}^* = (x_1, \dots, x_n)$ is an interior point of S that solves the Lagrange problem. Suppose furthermore that the matrix

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial g_1}{\partial x_n}(\mathbf{x}^*) \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial g_m}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

has rank m. (This is called the NDCQ (nondegenerate constraint qualification) condition.) Then there exist unique numbers $\lambda_1, \dots, \lambda_m$ such that the first order conditions are satisfied.

The theorem ensures that (under the given condition) that one can find all maxima and minima by solving the first order conditions together with the constraints. We consider an example.

Example 6.12. Consider the problem of optimizing

$$f(x_1, x_2) = x_1 + 3x_2 \text{ subject to } g_1(x_1, x_2) = x_1^2 + x_2^2 = 10$$

Discuss the first order conditions.

Solution.
We first consider the NDCQ condition of the theorem above. Since there is only one constraint, we get the matrix

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 \end{pmatrix}.$$

This matrix has rank $m = 1$ except when $x_1 = x_2 = 0$. But if $x_1 = x_2 = 0$, then $x_1^2 + x_2^2 = 0^2 + 0^2 = 0 \neq 10$ violating the constraint, so the matrix has rank $m = 1$ at every point that satisfies the constraint. The Lagrangian is

$$\begin{aligned}\mathcal{L}(x_1, x_2) &= f(x_1, x_2) - \lambda_1 g_1(x_1, x_2) \\ &= x_1 + 3x_2 - \lambda_1(x_1^2 + x_2^2).\end{aligned}$$

For convenience we put $\lambda = \lambda_1$. We obtain

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= 1 - 2\lambda x_1 = 0 \implies x_1 = \frac{1}{2\lambda} \\ \frac{\partial \mathcal{L}}{\partial x_2} &= 3 - 2\lambda x_2 = 0 \implies x_2 = \frac{3}{2\lambda}\end{aligned}$$

We substitute these expressions in the constraint and obtain

$$g_1\left(\frac{1}{2\lambda}, \frac{3}{2\lambda}\right) = \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = \frac{10}{4\lambda^2} = 10.$$

From this we obtain that $\lambda^2 = 1/4$ or $\lambda = \pm 1/2$. This gives

$$\begin{aligned}x_1 &= \frac{1}{2\lambda} = \frac{1}{2(\pm 1/2)} = \pm 1 \\ x_2 &= \frac{3}{2\lambda} = \frac{3}{2(\pm 1/2)} = \pm 3.\end{aligned}$$

Thus we have

$$(-1, -3) \text{ corresponding to } \lambda = -\frac{1}{2}$$
$$(1, 3) \text{ corresponding to } \lambda = \frac{1}{2}$$

These two points are the candidates for optimum. By the theorem, the solution of the Lagrange maximization problem is given by one of these points.

If we are able to infer that the Lagrangian is concave or convex, we may use this to conclude that a point satisfying the first order conditions and the constraints is maximum or a minimum.

Theorem 6.13. Assume that there exists numbers $\lambda_1, \ldots, \lambda_m$ together with $\mathbf{x}^* = (x_1, \ldots, x_n)$ such that the first order conditions are satisfied. Then:

(1) If $\mathcal{L}$ is concave as a function in $\mathbf{x}^* = (x_1, \ldots, x_n)$ (and with $\lambda_1, \ldots, \lambda_m$ as the fixed numbers), then $\mathbf{x}^*$ solves the maximization Lagrange Problem.
(2) If $\mathcal{L}$ is convex as a function in $\mathbf{x}^* = (x_1, \ldots, x_n)$, then $\mathbf{x}^*$ solves the minimization Lagrange Problem.

Let us revisit our previous example.

Example 6.14. Consider the problem of optimizing

$$f(x_1, x_2) = x_1 + 3x_2 \text{ subject to } g_1(x_1, x_2) = x_1^2 + x_2^2 = 10.$$

Show that $(1, 3)$ is a maximum under the given constraint.

Solution.
The point $\mathbf{x}^* = (x_1, x_2) = (1, 3)$ corresponded to the value $\lambda = \frac{1}{2}$. Thus the Lagrangian is

$$\begin{aligned}\mathcal{L}(x_1, x_2) &= f(x_1, x_2) - \frac{1}{2} g_1(x_1, x_2) \\ &= x_1 + 3x_2 - \frac{1}{2}(x_1^2 + x_2^2).\end{aligned}$$

In Lecture 5 we showed that that $h(x_1, x_2) = x_1 + x_2$ is both concave and convex. This may easily be adapted to show that $h(x_1, x_2) = x_1 + 3x_2$ is both concave and convex. We have also seen that the function $h(x_1, x_2) = x_1^2 + x_2^2$ is convex. From this it follows that $-h(x_1, x_2)$ is concave. From the theorem at the end of Lecture 5 it follows that $\mathcal{L}(x_1, x_2)$ is concave. From the theorem above, this means that $(1, 3)$ is a maximum.

We end by finding the first order conditions in a somewhat more complicated example.

Example 6.15. Solve the first order conditions for max of

$$f(x_1, x_2, x_3) = x_1 x_2 x_3$$

subject to

$$g_1(x_1, x_2, x_3) = x_1^2 + x_2^2 = 1 \text{ and } g_2(x_1, x_2, x_3) = x_1 + x_3 = 1.$$

Solution.
We have that

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

We see that this matrix has rank equal to $m = 2$ unless both $x_1 = 0$ and $x_2 = 0$. But $x_1 = x_2 = 0$ violates the constraints. The Lagrangian is

$$\begin{aligned}\mathcal{L}(x_1, x_2, x_3) &= f(x_1, x_2, x_3) - \lambda_1 g_1(x_1, x_2, x_3) - \lambda_2 g_2(x_1, x_2, x_3) \\ &= x_1 x_2 x_3 - \lambda_1(x_1^2 + x_2^2) - \lambda_2(x_1 + x_3).\end{aligned}$$

The first order conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= x_2 x_3 - 2\lambda_1 x_1 - \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= x_1 x_3 - 2\lambda_1 x_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_3} &= x_1 x_2 - \lambda_2 = 0.\end{aligned}$$

We obtain $\lambda_1 = \frac{x_1 x_3}{2x_2}$ and $\lambda_2 = x_1 x_2$. Substituting this into $x_2 x_3 - 2\lambda_1 x_1 - \lambda_2 = 0$ gives

$$x_2^2 x_3 - x_1^2 x_3 - x_1 x_2^2 = 0.$$

From the constraints we obtain $x_2^2 = 1 - x_1^2$ and $x_3 = 1 - x_1$. Substituting this into the last equation gives

$$(1 - x_1^2)(1 - x_1) - x_1^2(1 - x_1) - x_1(1 - x_1^2) = 0.$$

This factors as

$$(1 - x_1)\left(-3x_1^2 - x_1 + 1\right) = 0$$

From this we get $x_1 = 1$ or $x_1 = (-1 \pm \sqrt{13})/6$, and from the equations $x_2^2 = 1 - x_1^2$, $x_3 = 1 - x_1$, $\lambda_1 = \frac{x_1 x_3}{2x_2}$ and $\lambda_2 = x_1 x_2$, we find

$x_1 = 1$	$x_2 = 0$	$x_3 = 0$	$\lambda_1 = 0$	$\lambda_2 = 0$
$x_1 = (-1 - \sqrt{13})/6$	$x_2 = \pm\sqrt{-\frac{1}{18}\sqrt{13} + \frac{11}{18}}$	$x_3 = \frac{1}{6}\sqrt{13} + \frac{7}{6}$	$\lambda_1 = \frac{x_1 x_3}{2x_2}$	$\lambda_2 = x_1 x_2$
$x_1 = (-1 + \sqrt{13})/6$	$x_2 = \pm\sqrt{\frac{1}{18}\sqrt{13} + \frac{11}{18}}$	$x_3 = \frac{7}{6} - \frac{1}{6}\sqrt{13}$	$\lambda_1 = \frac{x_1 x_3}{2x_2}$	$\lambda_2 = x_1 x_2$

7. Envelope theorems and the bordered Hessian

Reading. This lecture covers topics from Sections 3.1, 3.3, and 3.4 in FMEA [**2**].

7.1. Envelope theorems. In economic optimization problems the objective function, that is the function that we are maximizing, usually involves parameters like prices in addition to variables like quantities. In such cases it is of interest to know how the maximum is affected by a change in these parameters. This explains the interest in the so-called envelope theorems.

Theorem 7.1. Assume that $f(\mathbf{x}; a)$ is a function of $\mathbf{x} = (x_1, \ldots, x_n)$ that depends on a parameter a. For each choice of the parameter a, consider the maximization problem

$$\max\ f(\mathbf{x}; a) \text{ with respect to } \mathbf{x}.$$

Let $\mathbf{x}^*(a)$ be a solution to this problem. Then

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \left(\frac{\partial f}{\partial a}\right)_{\mathbf{x}=\mathbf{x}^*(a)}$$

To illustrate this theorem, we consider a simple example in one variable.

Example 7.2. Consider the function

$$f(x; a) = -x^2 + 2ax + 4a^2$$

in the one variable x. The function depends on the parameter a. The derivative of f with respect to x is

$$f'(x; a) = -2x + 2a$$

and we see that the function has a maximum for

$$x = a.$$

The *value function* f^* is given by $f^*(a) := f(x^*(a); a) = -a^2 + 2a \cdot a + 4a^2 = 5a^2$.
We get

$$\frac{d}{da}(f^*(a)) = \frac{d}{da}(5a^2) = 10a$$

and

$$\frac{\partial f}{\partial a} = 2x + 8a \implies \left(\frac{\partial f}{\partial a}\right)_{\mathbf{x}=\mathbf{x}^*} = 10a,$$

so $\frac{d}{da} f(x^*(a); a) = \left(\frac{\partial f}{\partial a}\right)_{x=x^*}$ as predicted by the theorem.

The following is a slight modification of one of the problems on last weeks problem sheet.

Example 7.3. A firm produces goods A and B. The price of A is 13 and the price of B is p. The cost function is

$$C(x, y) = 0.04x^2 - 0.01xy + 0.01y^2 + 4x + 2y + 500.$$

The profit function is

$$\pi(x, y) = 13x + py - C(x, y).$$

Determine the optimal value function $\pi^*(p)$ and verify the envelope theorem.

Solution.
We get

$$\pi(x, y) = 9x + (p-2)y - 0.04x^2 + 0.01xy - 0.01y^2 - 500.$$

The first order conditions yields:

$$\begin{aligned}\pi'_x &= 9 - 0.08x + 0.01y = 0 \\ \pi'_y &= (p-2) + 0.01x - 0.02y = 0.\end{aligned}$$

This gives the following system of linear equations

$$\begin{aligned}-0.08x + 0.01y &= -9 \\ 0.01x - 0.02y &= 2 - p.\end{aligned}$$

The solution is given by

$$\begin{aligned}\begin{pmatrix} x^* \\ y^* \end{pmatrix} &= \begin{pmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{pmatrix}^{-1} \begin{pmatrix} -9 \\ 2-p \end{pmatrix} \\ &= \begin{pmatrix} 6.666\,7p + 106.67 \\ 53.333p - 46.667 \end{pmatrix}.\end{aligned}$$

The Hessian is

$$\pi'' = \begin{pmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{pmatrix}$$

which has

$$D_1 = -0.08 < 0$$

and

$$\begin{aligned}D_2 &= (-0.08)(-0.02) - (0.01)^2 \\ &= 0.0015 > 0\end{aligned}$$

as leading principal minors. We conclude that the Hessian is negative definite, so π is concave. This means that (x^*, y^*) is a maximum.
The optimal value function is then

$$\begin{aligned}\pi^*(p) = \pi(x^*, y^*) &= \pi(6.666\,7p + 106.67, 53.333p - 46.667) \\ &= 26.667p^2 - 46.667p + 26.667.\end{aligned}$$

To verify the envelope theorem we compute

$$\frac{d}{dp}(\pi^*(p)) = 53.333p - 46.667.$$

On the other hand we have

$$\frac{\partial \pi}{\partial p} = y$$

so that

$$\left(\frac{\partial \pi}{\partial p}\right)_{x=x^*, y=y^*} = y^* = 53.333p - 46.667$$

in agreement with the envelope theorem.

There is also an envelope theorem for the Lagrange problem.

Theorem 7.4. Assume that $f(\mathbf{x}; a)$ and $g_1(\mathbf{x}; a), \ldots, g_m(\mathbf{x}; a)$ are functions of $\mathbf{x} = (x_1, \ldots, x_n)$ that depend on a parameter a. Suppose that the NDCQ condition holds. For each choice of the parameter a, consider

$$\max f(\mathbf{x}; a) \text{ subject to } g_1(\mathbf{x}; a) = 0, \ldots, g_m(\mathbf{x}; a) = 0.$$

Let $\mathbf{x}^*(a) = (x_1^*(a), \ldots, x_n^*(a))$ be a solution, and let $\lambda(a) = (\lambda_1(a), \ldots, \lambda_m(a))$ be the corresponding Lagrange multipliers, and let $f^*(a) = f(\mathbf{x}^*(a); a)$. Then

$$\frac{d}{da}(f^*(a)) = \left(\frac{\partial \mathcal{L}}{\partial a}\right)_{\mathbf{x}=\mathbf{x}^*(a), \lambda=\lambda(a)}$$

where $\mathcal{L}$ is the Lagrangian.

We show how the theorem may be used in an example.

Example 7.5. In Lecture 7 we considered

$$\max f(x_1, x_2) = x_1 + 3x_2 \text{ subject to } g(x_1, x_2) = x_1^2 + ax_2^2 = 10$$

when $a = 1$. We found that $\mathbf{x}^* = \mathbf{x}^*(1) = (1, 3)$ with Lagrange multiplier $\lambda = \frac{1}{2}$ solved the problem with $f(1, 3) = 10$. Let $\mathbf{x}^*(a)$ be the solution for each choice of the parameter a and let $f^*(a) = f(\mathbf{x}^*(a); a)$ be the optimal value function. Use the theorem above to estimate $f^*(1.01)$. Check the estimate by computing the optimal value function $f^*(a)$.

Solution.
The NDCQ condition is satisfied when $a \neq 0$. The Lagrangian is

$$\mathcal{L} = x_1 + 3x_2 - \lambda(x_1^2 + ax_2^2).$$

We get

$$\frac{\partial \mathcal{L}}{\partial a} = -\lambda x_2^2 \implies \left(\frac{\partial \mathcal{L}}{\partial a}\right)_{\mathbf{x}=(1,3),\lambda=1/2} = -\frac{1}{2}3^2 = -\frac{9}{2}.$$

By the theorem we have

$$\frac{d}{da}(f^*(a)) = -\frac{9}{2}.$$

By linear approximation we have

$$\begin{aligned} f^*(1.01) &\cong f^*(1) + \left(\frac{d}{da}(f^*(a))\right)_{a=1} \cdot 0.01 \\ &\cong 10 + (-\frac{9}{2}) \cdot 0.01 = 9.955 \end{aligned}$$

We now solve the maximization problem directly: We get

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= 1 - 2x_1\lambda = 0 \implies x_1 = \frac{1}{2\lambda} \\ \frac{\partial \mathcal{L}}{\partial x_2} &= 3 - 2a\lambda x_2 = 0 \implies x_2 = \frac{3}{2\lambda a} \end{aligned}$$

Substituting these expressions into the constraint $g(x_1, x_2) = x_1^2 + ax_2^2 = 10$, we obtain

$$\left(\frac{1}{2\lambda}\right)^2 + a\left(\frac{3}{2\lambda a}\right)^2 = \frac{a+9}{4a\lambda^2} = 10$$

For this we obtain

$$\lambda^2 = \frac{a+9}{40a} \implies \lambda = \pm\sqrt{\frac{a+9}{40a}}$$

We get

$$x_1 = \pm\frac{1}{2\sqrt{\frac{a+9}{40a}}} \text{ and } x_2 = \pm\frac{3}{2a\sqrt{\frac{a+9}{40a}}}$$

Thus

$$f^* = \pm\left(\frac{1}{2\sqrt{\frac{a+9}{40a}}} + 3\frac{3}{2a\sqrt{\frac{a+9}{40a}}}\right).$$

For $a = 1.01$ and positive sign we obtain

$$f^* = 9.9553.$$

7.2. The Bordered Hessian. In this section we consider briefly second order conditions for the Lagrange problem. Using the bordered Hessian one can find the *local* maxima and minima for a function subject to constraints. We restrict ourselves to the easiest case.

Theorem 7.6. Consider the problem of finding the local maxima of a function $f(x_1, x_2)$ in two variables, subject to one constraint $g(x_1, x_2) = b$. Let

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2)$$

be the corresponding Lagrangian. Suppose that for some λ, the first order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \text{ and } \frac{\partial \mathcal{L}}{\partial x_2} = 0$$

are satisfied at $\mathbf{x}^* = (x_1^*, x_2^*)$, that $g(x_1^*, x_2^*) = b$ and that the *bordered Hessian*

$$\begin{vmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \mathcal{L}''_{11} & \mathcal{L}''_{12} \\ \frac{\partial g}{\partial x_2} & \mathcal{L}''_{21} & \mathcal{L}''_{22} \end{vmatrix} > 0 \text{ at } \mathbf{x}^* = (x_1^*, x_2^*).$$

Then $\mathbf{x}^* = (x_1^*, x_2^*)$ is a local maximum for $f(x_1, x_2)$ subject to $g(x_1, x_2) = b$.

We will use this theorem on the function and constraint in a previous example.

Example 7.7. Consider

$$f(x_1, x_2) = x_1 + 3x_2 \text{ subject to } g(x_1, x_2) = x_1^2 + x_2^2 = 10.$$

Show that $(1, 3)$ is a local maximum.

Solution.
With

$$\mathcal{L} = x_1 + 3x_2 - \lambda(x_1^2 + x_2^2)$$

we found that $(1, 3)$ and $\lambda = \frac{1}{2}$ satisfied the first order conditions:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 1 - 2x_1\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 3 - 2\lambda x_2 = 0.$$

We have that

$$\frac{\partial g}{\partial x_1} = 2x_1 = 2 \cdot 1 = 2 \text{ and } \frac{\partial g}{\partial x_2} = 2x_2 = 2 \cdot 3 = 6.$$

The second order partial derivative of the Lagrangian are

$$\begin{array}{ll} \mathcal{L}''_{11} = -2\lambda = -2 \cdot \frac{1}{2} = -1 & \mathcal{L}''_{12} = 0 \\ \mathcal{L}''_{21} = 0 & \mathcal{L}''_{22} = -2\lambda = -1 \end{array}$$

Thus the bordered Hessian is

$$\begin{vmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \mathcal{L}''_{11} & \mathcal{L}''_{12} \\ \frac{\partial g}{\partial x_2} & \mathcal{L}''_{21} & \mathcal{L}''_{22} \end{vmatrix} = \begin{vmatrix} 0 & 2 & 6 \\ 2 & -1 & 0 \\ 6 & 0 & -1 \end{vmatrix} = 40 > 0$$

So $(1, 3)$ is a local maximum.

8. Introduction to differential equations

Reading. This lecture covers topics from Sections 5.1 and 5.3 in FMEA [**2**]. In addition it is strongly recommended to review integration, see for instance EMEA [**3**] chapter 9 or [**1**] chapter 6.

8.1. Brief review of integration. Differential equations involves derivatives, and to solve differential equations it is important to have a good understanding of both derivation and integration.

Example 8.1. Find $\frac{dx}{dt}$ in the following cases:

(1) $x(t) = 100e^{-2t}$
(2) $x(t) = \ln(2t)$
(3) $x(t) = (2t+3)^4$

Solution.

(1) $\frac{dx}{dt} = x'(t) = 100e^{-2t}(-2t) = -200e^{-2t}$
(2) $\frac{dx}{dt} = x'(t) = \frac{1}{2t} \cdot 2 = \frac{1}{t}$
(3) $\frac{dx}{dt} = x'(t) = 4(2t+3)^3 \cdot 2 = 8(2t+3)^3$

To solve differential equations, one often has to evaluate integrals. It is thus important to know the basic integration techniques.

Example 8.2. Find the integrals:

(1) $\int x^{13} dx$
(2) $\int (t^3 + 2t - 3) dt$
(3) $\int xe^x dx$
(4) $\int (x^2+1)^8 2x dx$

Solution.

(1) $\int x^{13} dx = \frac{1}{13+1} x^{13+1} + C = \frac{1}{14} x^{14} + C$
(2) $\int (t^3 + 2t - 3) dt = \frac{1}{4} t^4 + t^2 - 3t + C$
(3) To find $\int xe^x dx$ we use the formula for integration by parts
$$\int uv' dx = uv - \int u'v dx.$$
We chose $u = x$ and $v' = e^x$. Since the derivative of e^x is e^x, we get $v = e^x$ and $u' = 1$. Thus
$$\int xe^x dx = xe^x - \int 1 \cdot e^x dx = xe^x - e^x + C.$$
(4) To find $\int (x^2+1)^8 2x dx$ we use substitution. We put $u = x^2 + 1$. We get that $\frac{du}{dx} = 2x$, and deduce that $du = 2x dx$. Thus we get
$$\int (x^2+1)^8 2x dx = \int u^8 du = \frac{1}{9} u^9 + C = \frac{1}{9}(x^2+1)^9 + C.$$

8.2. What is a differential equation? Let x be an economic variable like interest rate or oil production. It is often possible to set forth models for such variables and these models often lead to a differential equation involving the variable x. In such equations one takes into account that x changes in time and one views x as a function $x(t)$ of the time t. The notation $\dot{x}$ is often used for the derivative of x with respect to t, i.e. one has

$$\dot{x} = \frac{dx}{dt}$$

Example 8.3. Consider the function $x = x(t) = 100e^{-2t}$. The derivative of x with respect to t is

$$\dot{x} = 100e^{-2t}(-2) = -2x.$$

We see that we have an identify

$$\dot{x} = -2x$$

involving the function x and is derivative. This is an example of a differential equation, and $x = x(t) = 100e^{-2t}$ is a solution of this differential equation.

Definition 8.4. A differential equation is an equation for an unknown function that relates the function to its derivatives.

The following are examples of differential equations.

Example 8.5.

(1) $\dot{x} = ax$ where a is a constant.
(2) $\dot{x} + 3x = 4$
(3) $\dot{x} + 2x = 5x^2$

The equations in the example are all what we call first order ordinary differential equations.

Example 8.6. Show that $x = Ce^{-2t} + 4$ (where C is a constant) is a solution of the differential equation

$$\dot{x} + 2x = 8.$$

Solution.
Differentiating $x = Ce^{-2t} + 4$, we obtain

$$\dot{x} = Ce^{-2t}(-2t) = -2Ce^{-2t}.$$

From this we have that

$$\begin{aligned}\dot{x} + 2x &= -2Ce^{-2t} + 2(Ce^{-2t} + 4)\\ &= -2Ce^{-2t} + 2Ce^{-2t} + 8\\ &= 8.\end{aligned}$$

Thus the equation is satisfied.

It should be remarked that a differential equation in general will have infinitely many solutions.

Definition 8.7. The set of all solutions of a differential equation is called the *general solution* of the equation. Any specific function that satisfies the equation is called a *particular solution.*

Example 8.8. We have that

$$x = Ce^{-2t} + 4$$

is the general solution of $\dot{x} + 2x = 8$ and

$$x = 3e^{-2t} + 4$$

is a particular solution of $\dot{x} + 2x = 8$.

Often one is interested in finding a particular solution of a differential equation that satisfies a *initial condition.*

Example 8.9. Find the particular solution of

$$\dot{x} + 2x = 8$$

that satisfies the *initial condition* $x(0) = 2$.

Solution.
The general solution is $x = Ce^{-2t} + 4$. Putting $t = 0$, we get

$$x(0) = Ce^{-2\cdot 0} + 4 = C + 4$$

From $x(0) = C + 4 = 2$, we conclude that $C = -2$. Thus the particular solution satisfying $x(0) = 2$, is

$$x(t) = -2e^{-2t} + 4.$$

8.3. Separable differential equations. Many differential equations are impossible to solve. In this course we will consider some classes of equations where it is possible to find a solution.

Definition 8.10. A first order ordinary differential equation is one that can be written as

$$\dot{x} = F(t, x)$$

for a function $F(t, x)$ in the variables x and t. It is said to be *separable* if $\dot{x} = f(t)g(x)$ for functions $f(t)$ and $g(x)$ in one variable.

We will see how to solve separable differential equations.

Example 8.11. Determine which of the following differential equations are separable.

(1) $\dot{x} = xt$
(2) $\dot{x} = x + t$
(3) $\dot{x} = xt + 2t$
(4) $\dot{x} = xt^2 + x^2t^2$

Solution.

(1) This is separable since we have $\dot{x} = f(t)g(x)$ with $f(t) = t$ and $g(x) = x$.
(2) This is not separable.
(3) Since the equation may be written as $\dot{x} = t(x + 2)$, this equation is separable.
(4) Since $\dot{x} = t^2(x + x^2)$, this equation is separable.

We now explain how to solve ordinary first order separable equations:

(1) Write as

$$\frac{dx}{dt} = f(t)g(x).$$

(2) Separate as

$$\frac{dx}{g(x)} = f(t)dt$$

(3) Integrate:

$$\int \frac{dx}{g(x)} = \int f(t)dt$$

Example 8.12. Find the general solution of

$$\dot{x} = \frac{2t}{3x^2}$$

Solution.
We have

$$\frac{dx}{dt} = 2t \cdot \frac{1}{3x^2}.$$

We separate as

$$3x^2 dx = 2tdt.$$

Integrating, we get

$$\int 3x^2 dx = \int 2tdt \implies x^3 = t^2 + C.$$

Taking the third root, we obtain

$$x = \sqrt[3]{t^2 + C}.$$

Example 8.13. Solve the following differential equation

$$\dot{x} = x(1 - x).$$

Solution.
We have
$$\frac{dx}{dt} = x(1-x).$$
Separating, we get
$$\frac{dx}{x(1-x)} = dt.$$
We thus need to integrate
$$\int \frac{dx}{x(1-x)} = \int dt.$$
To evaluate the first integral, we note that
$$\frac{1}{x} + \frac{1}{1-x} = \frac{1}{x(1-x)}$$
Thus we have
$$\int (\frac{1}{x} + \frac{1}{1-x})dx = \int dt.$$
Integrating, we obtain
$$\ln|x| - \ln|1-x| = t + C.$$
This may be rewritten as
$$\ln|\frac{x}{1-x}| = t + C$$
Taking the exponential function on each side, we obtain
$$|\frac{x}{1-x}| = e^{t+C} = e^t e^C.$$
We get
$$\frac{x}{1-x} = \pm e^C e^t$$
or
$$\frac{x}{1-x} = Ke^t$$
where K is a constant. This may be rewritten as
$$x = Ke^t(1-x)$$
or
$$x + xKe^t = Ke^t.$$
From this we get
$$x(1+Ke^t) = Ke^t \implies x = \frac{Ke^t}{1+Ke^t} = \frac{1}{\frac{1}{K}e^{-t}+1}$$
Putting $k = \frac{1}{K}$, we get
$$x = \frac{1}{ke^{-t}+1}$$
where k is a constant.

9. Linear first order and exact differential equations

Reading. This lecture covers topics from Sections 5.4, and 5.5 in FMEA [**2**].

9.1. First order linear differential equations. In this section we consider the class of linear first order differential equations.

Definition 9.1. A first order linear differential equation is one that can be written in the form
$$\dot{x} + a(t)x = b(t)$$
where $a(t)$ and $b(t)$ are functions of t.

We will soon learn how to solve these equations, so it is important to be able to distinguish the linear first order differential equations from other differential equations.

Example 9.2.

(1) $\dot{x} + 2tx = 4t$ is linear.
(2) $\dot{x} - x = e^{2t}$ is linear.
(3) $(t^2+1)\dot{x} + e^t x = t\ln t$ is linear because it can be written as $\dot{x} + \frac{e^t}{t^2+1}x = \frac{t\ln t}{t^2+1}$.
(4) $\dot{x} - x^2 = 0$ is not linear because of the term x^2.
(5) $\dot{x} - e^x = 2t$ is not linear because of the term e^x.

We will now see how to solve first order linear differential equations. This will be done by multiplying the equation by a factor called the *integrating factor* in order to write the left hand side of the equation as the derivative of a product. To understand this, we need to remember the rule for the derivation of a product:

$$(uv)' = u'v + uv'.$$

For example

$$\begin{aligned}\frac{d}{dt}(xe^{2t}) &= \dot{x}e^{2t} + xe^{2t}2 \\ &= \dot{x}e^{2t} + 2xe^{2t}.\end{aligned}$$

Example 9.3. Find the general solution of
$$\dot{x} + 2x = 7.$$

Solution.
The trick is to multiply the equation with e^{2t} where 2 is the coefficient of x. The factor e^{2t} is called the *integrating factor*. We get

$$\dot{x}e^{2t} + 2xe^{2t} = 7e^{2t}$$

We see that the left hand side is exactly

$$\frac{d}{dt}(xe^{2t}),$$

so that the equation may be written as

$$\frac{d}{dt}(xe^{2t}) = 7e^{2t}.$$

We can now integrate with respect to t to obtain

$$xe^{2t} = \int 7e^{2t}dt = \frac{7}{2}e^{2t} + C$$

To solve for x, we divide both sides by e^{2t} to obtain

$$x = \frac{7}{2} + Ce^{-2t}$$

as the general solution.

The method used in the example can be generalized.

Proposition 9.4. The differential equation

$$\dot{x} + ax = b$$

where a and b are constants, has the general solution

$$x(t) = Ce^{-at} + \frac{b}{a}.$$

One is often interested in what happens when t tends to infinity. If $a > 0$ then $e^{-at} \to 0$ as $t \to \infty$. This means that the solution of $\dot{x} + ax = b$ tends to $\frac{b}{a}$ when $t \to \infty$. In this case the solution is said to be stable, and $x = \frac{b}{a}$ is called the equilibrium state. When $a < 0$, then $e^{-at} \to \infty$ as $t \to \infty$. In this case (and if $C \neq 0$) the solution is said to be unstable.

Example 9.5. Suppose the price $P = P(t)$, the demand $D = D(t)$ and the supply $S = S(t)$ of a certain commodity, is governed by the following model:

$$D = a - bP$$
$$S = \alpha + \beta P$$
$$\dot{P} = \lambda(D - S)$$

where a, b, β and λ are constants. Find P as a function of t.

Solution.
Combining, we get

$$\begin{aligned}\dot{P} &= \lambda(D - S)\\ &= \lambda((a - bP) - (\alpha + \beta P))\\ &= \lambda(a - \alpha) - \lambda(b + \beta)P\end{aligned}$$

This gives

$$\dot{P} + \lambda(b + \beta)P = \lambda(a - \alpha)$$

which is linear with constant coefficients. Thus the general solution is

$$P = Ce^{-\lambda(b+\beta)t} + \frac{a - \alpha}{b + \beta}.$$

Example 9.6. Assume

$$D = 5000 - 4P$$
$$S = 1000 + 6P$$
$$\dot{P} = 0.5(D - S)$$

and that $P(0) = 900$. Find $P(0.5)$.

Solution.

$$\begin{aligned}\dot{P} &= 0.5(5000 - 4P - (100 + 6P)) \\ &= 0.5(4000 - 10P) = 2000 - 5P\end{aligned}$$

Thus we get the linear first order equation

$$\dot{P} + 5P = 2000.$$

Multiplying with the integrating factor e^{5t}, we obtain

$$\dot{P}e^{5t} + Pe^{5t} \cdot 5 = 2000e^{5t}.$$

Thus we have

$$\frac{d}{dt}(Pe^{5t}) = 2000e^{5t}$$

and integrating, we obtain

$$Pe^{5t} = 2000 \int e^{5t} dt = 2000\frac{1}{5}e^{5t} + C = 400e^{5t} + C.$$

Multiplying with e^{-5t}, we get

$$P = 400 + Ce^{-5t}$$

as the general solution.
We have that

$$P(0) = 400 + Ce^{-5 \cdot 0} = 400 + C = 900 \implies C = 500.$$

Thus we obtain the particular solution

$$P(t) = 400 + 500e^{-5t}.$$

From this we get $P(0.5) = 400 + 500e^{-5 \cdot 0.5} = 441.04$.

We now consider linear first order equations with non-constant coefficients.

Proposition 9.7. The equation

$$\dot{x} + a(t)x = b(t)$$

has integrating factor

$$e^{\int a(t) dt}.$$

Before we give a general formula for the solution of $\dot{x} + a(t)x = b(t)$, we find the solution in some examples.

Example 9.8. Find the general solution of

$$\dot{x} - 2tx = t.$$

Solution.
We have
$$\int -2t dt = -t^2 + C$$
so the integrating factor is e^{-t^2}. Multiplying with the integrating factor, we obtain
$$\dot{x}e^{-t^2} + xe^{-t^2}(-2t) = te^{-t^2}$$
Thus we have
$$\frac{d}{dt}(e^{-t^2}x) = te^{-t^2}.$$
We integrate to obtain
$$e^{-t^2}x = \int te^{-t^2}dt.$$
To calculate the integral $\int te^{-t^2}dt$ we substitute $u = -t^2$. We obtain that $\frac{du}{dt} = -2t$ or $\frac{1}{-2}du = tdt$, so we have
$$\begin{aligned}\int te^{-t^2}dt &= \int e^u(\frac{1}{-2})du = -\frac{1}{2}e^u + C \\ &= -\frac{1}{2}e^{-t^2} + C.\end{aligned}$$
We have
$$e^{-t^2}x = -\frac{1}{2}e^{-t^2} + C$$
which we multiply with e^{t^2} to obtain
$$x = -\frac{1}{2} + Ce^{t^2}$$
as the general solution.

We also consider an example where we find a particular solution.

Example 9.9. Solve the initial value problem:
$$\dot{x} + 3t^2x = e^{-t^3}, \; x(0) = 2.$$

Solution.
We have $\int 3t^2dt = t^3 + C$, so e^{t^3} is an integrating factor. Multiplying the equation by this, we obtain
$$\dot{x}e^{t^3} + xe^{t^3}(3t^2) = e^{-t^3}e^{t^3} = 1.$$
So we have that
$$\frac{d}{dt}(xe^{t^3}) = 1.$$
Integrating, we obtain
$$xe^{t^3} = \int dt = t + C.$$
Multiplying with e^{-t^3}, we get
$$x = (t + C)e^{-t^3}$$
as the general solution. To find the particular solution, we have that
$$x(0) = (0 + C)e^0 = C = 2.$$
So
$$x(t) = (t + 2)e^{-t^3}$$
is the particular solution.

We have the following general formula.

Proposition 9.10. The differential equation

$$\dot{x} + a(t)x = b(t)$$

has the general solution

$$x(t) = e^{-\int a(t)dt}(C + \int b(t)e^{\int a(t)dt}dt).$$

9.2. Exact equations. We now look at another class of differential equations where we can find solutions.

Definition 9.11. The first order differential equation

$$f(t,x) + g(t,x)\dot{x} = 0$$

is said to be exact if

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial t}.$$

Example 9.12. Determine which of the following equations are exact.

(1) $1 + tx^2 + t^2x\dot{x} = 0$
(2) $1 + (t + 2x)\dot{x} = 0$

Solution.
(1) We identify the equations as $f(t,x)+g(t,x)\dot{x} = 0$ with $f(t,x) = 1+tx^2$ and $g(t,x) = t^2x$. We have

$$\frac{\partial f}{\partial x} = 2tx \text{ and} \frac{\partial g}{\partial t} = 2tx$$

This shows that the equation is exact.
(2) We identify the equation as $f(t,x) + g(t,x)\dot{x} = 0$ with $f(t,x) = 1$ and $g(t,x) = t + 2x$. We get

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial g}{\partial t} = 1.$$

This shows that the equation is not exact.

The reason that we may find solutions to exact differential equations is that when $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial t}$, we can find a function $h(t,x)$ such that

$$\frac{\partial h}{\partial t} = f \text{ and } \frac{\partial h}{\partial x} = g.$$

For such function $h = h(t,x)$, and $x = x(t)$ a function of t, we get by the chain rule

$$\begin{aligned}\frac{d}{dt}(h(x,t)) &= \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\frac{dx}{dt} \\ &= f(t,x) + g(t,x)\dot{x} = 0.\end{aligned}$$

Thus the solution is given by

$$h(x,t) = C.$$

Let us consider an example.

Example 9.13. Let $h(x,t) = t + \frac{1}{2}t^2x^2$. Then we get

$$\begin{aligned}\frac{d}{dt}(h(x,t)) &= \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\frac{dx}{dt} \\ &= 1 + tx^2 + t^2x\dot{x}.\end{aligned}$$

This means that the exact equation

$$1 + tx^2 + t^2x\dot{x} = 0$$

has solution

$$h(x,t) = t + \frac{1}{2}t^2x^2 = C$$

which defines x implicitly as a function of t.

The procedure for solving an exact first order differential equation, is to find a function $h(t,x)$ such that

$$\frac{\partial h}{\partial t} = f \text{ and } \frac{\partial h}{\partial x} = g.$$

This can always be done, and we indicate how in an example.

Example 9.14. Verify that

$$1 + 3x^2\dot{x} = 0$$

is exact and solve it.

Solution.
We identify $1 + 3x^2\dot{x} = 0$ as $f(t,x) + g(t,x)\dot{x} = 0$ with $f(t,x) = 1$ and $g(t,x) = 3x^2$. We get

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial g}{\partial t} = 0.$$

We want to find $h = h(t,x)$ such that

$$\frac{\partial h}{\partial t} = 1 \text{ and } \frac{\partial h}{\partial x} = 3x^2$$

When we look at $\frac{\partial h}{\partial x} = 3x^2$, we see by integration that

$$h = x^3 + \alpha(t)$$

where $\alpha(t)$ is an expression involving t and not x. From this expression for h, we get

$$\frac{\partial h}{\partial t} = 0 + \alpha'(t)$$

Thus we get that $\alpha'(t) = 1$ and by integration we have that

$$\alpha = t + C$$

where C is a constant. So

$$h = x^3 + t + C.$$

The equation may now be written as

$$\frac{dh}{dt} = 0,$$

so we obtain

$$x^3 + t = C \implies x(t) = \sqrt[3]{C - t}$$

as the general solution.

10. Second-order differential equations

Reading. This lecture covers topics from Sections 6.1, 6.2 and 6.3 in FMEA [**2**]. You should also read Section 5.7, but due to time constraints, this material will not be lectured.

10.1. General second order differential equations. Second order differential equations are equations in an unknown function relating the function to is derivatives of the second and the first order.

Definition 10.1. A second order differential equation is one that can be written
$$\ddot{x} = F(t, x, \dot{x})$$
where F is a function in three variables.

Some second order differential equations are easy to solve.

Example 10.2. Find all solutions of
$$\ddot{x} = 1.$$

Solution.
From $\ddot{x} = 1$ we obtain that
$$\dot{x} = \int 1 dt = t + C_1$$
where C_1 is a constant. From $\ddot{x} = 1$ we obtain that
$$x = \int (t + C_1) dt = \frac{1}{2}t^2 + C_1 t + C_2$$
where C_2 is a constant.

Some second-order equations can be transformed into first-order equations by a substitution.

Example 10.3. Solve the differential equation
$$\ddot{x} = \dot{x} + t.$$

Solution.
If we substitute $u = \dot{x}$, we get $\dot{u} = \ddot{x}$ and the equation becomes

$$\dot{u} = u + t$$

This is a linear first-order differential equation and may be written

$$\dot{u} - u = t.$$

An integrating factor is e^{-t}, and multiplying with this the equation becomes

$$\dot{u}e^{-t} + ue^{-t}(-1) = te^{-t} \Leftrightarrow \frac{d}{dt}(ue^{-t}) = te^{-t}$$

Integrating, we obtain

$$ue^{-t} = \int te^{-t}dt = -te^{-t} - e^{-t} + C_1.$$

Multiplying the equation by e^t, one obtains

$$u = t - 1 + C_1e^t.$$

In other words, we have

$$\dot{x} = t - 1 + C_1e^t.$$

Integrating we get,

$$\dot{x} = -\frac{1}{2}t^2 - t + C_1e^t + C_2$$

as the general solution.

10.2. Linear homogenous second-order differential equations. We now turn to linear second-order differential equations.

Definition 10.4. A linear second-order differential equation is one that can be written as

$$\ddot{x} + a(t)\dot{x} + b(t)x = f(t)$$

where $a(t)$, $b(t)$ and $f(t)$ are functions of t. The equation is said to be *homogenous* if $f(t) = 0$.

Example 10.5. Consider the differential equation

$$\ddot{x} - 5\dot{x} + 6x = 0 \tag{6}$$

and let

$$r = \frac{5 \pm \sqrt{5^2 - 4 \cdot 6}}{2} = \frac{5 \pm 1}{2} = \begin{cases} \frac{6}{2} = 3 \\ \frac{4}{2} = 2 \end{cases}$$

be a solution of

$$r^2 - 5r + 6 = 0.$$

Show that

$$x = e^{rt}$$

is a solution of (6).

Solution.
From $x = e^{rt}$ we obtain $\dot{x} = e^{rt} \cdot r = re^{rt}$ and $\ddot{x} = re^{rt} \cdot r = r^2e^{rt}$. Substituting this into the left hand side of the differential equation, we obtain

$$\ddot{x} - 5\dot{x} + 6x = r^2e^{rt} - 5re^{rt} + 6e^{rt} = e^{rt}(r^2 - 5r + 6)$$

Thus we see that this is equal to zero if and only if

$$r^2 - 5r + 6 = 0$$

Definition 10.6. Let

$$\ddot{x} + a\dot{x} + bx = 0 \tag{7}$$

be a second order linear differential equation with constant coefficients. The quadratic equation

$$r^2 + ar + b = 0$$

is called the characteristic equation of (7).

Recall that a quadratic equation $r^2+ar+b=0$ has a solution if and only if $a^2-4b \geq 0$, and that in this case, the solution is given by

$$r = \frac{-a \pm \sqrt{a^2-4b}}{2}.$$

Proposition 10.7. If $x_1(t)$ and $x_2(t)$ are two solutions of

$$\ddot{x} + a\dot{x} + bx = 0$$

then for general constants A and B

$$x(t) = Ax_1(t) + Bx_2(t)$$

is also a solution.

Proof. From $x(t) = Ax_1(t) + Bx_2(t)$ we obtain $\dot{x} = A\dot{x}_1 + B\dot{x}_2$ and $\ddot{x} = A\ddot{x}_1 + B\ddot{x}_2$. Substituting this into the left hand side of the equation, we get

$$\begin{aligned}\ddot{x} + a\dot{x} + bx &= (A\ddot{x}_1 + B\ddot{x}_2) + a(A\dot{x}_1 + B\dot{x}_2) + b(Ax_1 + Bx_2)\\ &= A(\ddot{x}_1 + a\dot{x}_1 + bx_1) + B(\ddot{x}_2 + a\dot{x}_2 + bx_2)\\ &= A\cdot 0 + B \cdot 0 = 0\end{aligned}$$

□

Using this proposition and reasoning as in the example above, one can prove the following theorem.

Theorem 10.8. The general solution of

$$\ddot{x} + a\dot{x} + bx = 0$$

is:

(1) If the characteristic equation has two distinct roots $r_1 \neq r_2$:

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

(2) if the characteristic equation has one real root r:

$$x(t) = (A + Bt)e^{rt}$$

(3) if the characteristic equation has no real roots:

$$x(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$$

where $\alpha = -\frac{1}{2}a$ and $\beta = \sqrt{b - \frac{1}{4}a^2}$.

We consider an example.

Example 10.9. Find the general solution of

$$\ddot{x} - 7\ddot{x} + 12x = 0.$$

Solution.
The characteristic equation is
$$r^2 - 7r + 12 = 0$$
which has the solutions $r = 3$ and $r = 4$. Thus the general solution is
$$x(t) = Ae^{3t} + Be^{4t}.$$

10.3. Non-homogenous linear second-order differential equations. We now turn to non-homogenous linear second-order differential equations.

Proposition 10.10. Assume that $x_{\mathrm{p}}(t)$ is any solution of
$$\ddot{x} + a\dot{x} + bx = f(t) \tag{8}$$
where $f(t)$ is a function of t. Then the general solution of (8) is given as
$$x(t) = x_{\mathrm{h}}(t) + x_{\mathrm{p}}(t)$$
where $x_{\mathrm{h}}(t)$ is the general solution of the homogenous equation
$$\ddot{x} + a\dot{x} + bx = 0.$$

Example 10.11. Show that $x_{\mathrm{p}}(t) = t$ is a solution of
$$\ddot{x} - 7\dot{x} + 12x = 12t - 7$$
and find the general solution of the equation.

Solution.
From $x_{\mathrm{p}}(t) = t$, we get $\dot{x}_{\mathrm{p}} = 1$ and $\ddot{x}_{\mathrm{p}}(t) = 0$. Substituting this into the left hand side of the equation, we obtain
$$\ddot{x}_{\mathrm{p}} - 7\dot{x}_{\mathrm{p}} + 12x_{\mathrm{p}} = 0 - 7 \cdot 1 + 12t = 12t - 7.$$
This shows that $x_{\mathrm{p}}(t) = t$ is a solution of $\ddot{x} - 7\dot{x} + 12x = 12t - 7$. The general solution of the homogenous equation
$$\ddot{x} - 7\dot{x} + 12x = 0$$
was found to be
$$x_{\mathrm{h}}(t) = Ae^{3t} + Be^{4t}.$$
From the Proposition, the general solution of $\ddot{x} - 7\dot{x} + 12x = 12t - 7$, is
$$x(t) = x_{\mathrm{p}} + x_{\mathrm{h}} = t + Ae^{3t} + Be^{4t}.$$

To solve a non-homogenous equation on the form
$$\ddot{x} + a\dot{x} + bx = f(t)$$
we will try to find on a particular solution and then add with the general solution of the homogenous equation. It is not always easy to find a solution of the non-homogenous equation. We will have to relay on some kind of guessing. We will guess that the equation has a solution of the same form as $f(t)$.

Example 10.12. Find the general solution of
$$\ddot{x} - 4\dot{x} + 4x = t^2 + 2.$$

Solution.
First we try to find a particular solution of the equation. Since $f(t) = t^2 + 2$ is a polynomial of degree 2, we seek a solution on the form

$$x_{\mathrm{p}} = At^2 + Bt + C.$$

From this we get $\dot{x}_{\mathrm{p}} = 2At + B$ and $\ddot{x}_{\mathrm{p}} = 2A$. Substituting this into the left hand side of the equation, we get

$$\begin{aligned}\ddot{x}_{\mathrm{p}} - 4\dot{x}_{\mathrm{p}} + 4x_{\mathrm{p}} &= 2A - 4(2At + B) + 4(At^2 + Bt + C)\\ &= 4At^2 + (4B - 8A)t + (2A - 4B + 4C)\end{aligned}$$

equating this to $t^2 + 2$, we obtain

$$4A = 1,\ \ 4B - 8A = 0 \text{ and } 2A - 4B + 4C = 2.$$

From this we obtain $A = \frac{1}{4}$, $B = 2A = \frac{1}{2}$ and $C = \frac{1}{2} - \frac{1}{2}A + B = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} = \frac{7}{8}$. Thus we have a particular solution

$$x_{\mathrm{p}}(t) = \frac{1}{4}t^2 + \frac{1}{2}t + \frac{7}{8}.$$

To find the general solution, we find the general solution of the homogenous equation

$$\ddot{x} - 4\dot{x} + 4x = 0.$$

This has $r^2 - 4r + 4 = 0$ as characteristic polynomial. The only solution is $r = 2$, so the general solution of the homogenous equation is

$$x_{\mathrm{h}}(t) = (A + Bt)e^{2t}.$$

Thus the general solution is

$$x(t) = x_{\mathrm{p}} + x_{\mathrm{h}} = \frac{1}{4}t^2 + \frac{1}{2}t + \frac{7}{8} + (A + Bt)e^{2t}.$$

Example 10.13. Find a particular solution of

$$\ddot{x} - 7\dot{x} + 12x = e^{2t}$$

Solution.
We try to find a solution on the form

$$x_{\mathrm{p}} = re^{2t}.$$

We get

$$\dot{x}_{\mathrm{p}} = 2re^{2t} \implies \ddot{x}_{\mathrm{p}} = 4re^{2t}.$$

substituting this into the left hand side of the equation, gives

$$\begin{aligned}\ddot{x}_{\mathrm{p}} - 7\dot{x}_{\mathrm{p}} + 12x_{\mathrm{p}} &= 4re^{2t} - 7 \cdot 2re^{2t} + 12 \cdot re^{2t}\\ &= re^{2t}(4 - 24 + 12)\\ &= 2re^{2t}\end{aligned}$$

From this we see that $r = \frac{1}{2}$ gives a solution. Thus we have found the particular solution

$$x_{\mathrm{p}}(t) = \frac{1}{2}e^{2t}$$

11. Difference equations

Reading. This lecture covers topics from Sections 11.1, 11,2 and 11.3 in FMEA [**2**].

11.1. First-order difference equations. Difference equations arise naturally in many contexts.

Example 11.1. You borrow an mount K. The interest rate per period is r. The repayments are of equal amounts s. Then the outstanding balance b_t in period t satisfies

$$b_{t+1} = (1+r)b_t - s.$$

The equality $b_{t+1} = (1+r)b_t - s$ is an example of a difference equation. It is said to be a difference equation of first order, since it relates b_{t+1} to b_t and the difference between the indices is $t+1-t=1$.

Example 11.2. Consider the first-order difference equation

$$x_{t+1} = 2x_t$$

and assume that $x_0 = 1$. Find x_3.

Solution.
We have

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 2x_0 = 2\cdot 1 = 2 \\ x_2 &= 2x_1 = 2\cdot 2 = 4 \\ x_3 &= 2x_2 = 2\cdot 4 = 8. \end{aligned}$$

Thus $x_3 = 8$.

Example 11.3. Consider the difference equation

$$x_{t+1} = 3x_t.$$

Assume that $x_0 = 5$. Find a formula for x_t.

Solution.
We have

$$\begin{aligned} x_0 &= 5 \\ x_1 &= 3x_0 = 3\cdot 5 \\ x_2 &= 3x_1 = 3^2\cdot 5 \\ x_3 &= 3x_2 = 3^3\cdot 5 \\ &\vdots \\ x_t &= 3^t\cdot 5 \end{aligned}$$

When we have obtained a formula for x_t, we say that we have solved the difference equation.

Problem 11.1.
Assume that

$$x_{t+1} = ax_t$$

Find a formula for x_t that is not recursive.

Problem 11.2.
Consider the sequence

$$1, 7, 31, 128, \dots$$

Find a difference equation that describes this sequence.

Example 11.4. Solve the difference equation

$$x_{t+1} = ax_t + b$$

Solution.
We have

$$\begin{aligned}
x_1 &= ax_0 + b \\
x_2 &= ax_1 + b = a(ax_0 + b) + b = a^2x_0 + (a+1)b \\
x_3 &= ax_2 + b = a(a^2x_0 + (a+1)b) + b = a^3x_0 + (a^2 + a + 1)b \\
&\vdots \\
x_t &= a^tx_0 + (a^{t-1} + a^{t-2} + \cdots + a + 1)b
\end{aligned}$$

We know that (geometric series)

$$a^{t-1} + a^{t-2} + \cdots + a + 1 = \frac{1 - a^t}{1 - a}$$

when $a \neq 0$, so we get

$$x_t = a^tx_0 + \frac{1 - a^t}{1 - a}b = a^t(x_0 - \frac{b}{1 - a}) + \frac{b}{1 - a} \text{ when } a \neq 0.$$

In the case $a = 0$, we get

$$x_t = x_0 + (1 + 1 + \cdots + 1)b = x_0 + tb.$$

Proposition 11.5. The first order difference equation

$$x_{t+1} = ax_t + b$$

has the solution

(1)
$$x_t = a^t(x_0 - \frac{b}{1 - a}) + \frac{b}{1 - a} \text{ when } a \neq 1$$

(2)
$$x_t = x_0 + tb \text{ when } a = 1.$$

Using the proposition above, we may find a formula for the outstanding balance in the first example of this lecture.

Example 11.6. Solve

$$b_{t+1} = (1 + r)b_t - s, \quad b_0 = K.$$

Solution.
Using the formula in the proposition with $a = 1 + r$ and $b = -s$, we obtain

$$\begin{aligned} b_t &= (1+r)^t(b_0 - \frac{-s}{1-(1+r)}) + \frac{-s}{1-(1+r)} \\ &= (1+r)^t(K - \frac{s}{r}) + \frac{s}{r} \end{aligned}$$

We consider the following problem.

Problem 11.3.
Solve

$$x_{t+1} = \frac{1}{2}x_t + 1, \quad x_0 = 6$$

and find $\lim_{t\to\infty} x_t$

11.2. Second-order difference equations. In a second-order difference equation some variable x_t depending on discrete time, is related to x_{t-1} and x_{t-2}.

Definition 11.7. A second order difference equation can be written as

$$x_{t+2} = f(t, x_t, x_{t+1}), \quad t = 0, 1, 2 \ldots$$

Try to solve the following problem.

Problem 11.4.
Consider the sequence and find the next term:
(i) $1, 3, 4, 7, 11, \ldots$
(ii) $2, 3, 4, 4, 0, \ldots$

Some second-order difference equations are easy to solve.

Proposition 11.8. The *general solution* of

$$x_{t+2} + ax_{t+1} + bx_t = 0 \quad (b \neq 0)$$

where a and b are constants, is as follows:

(1) If $a^2 - 4b > 0$, then

$$x_t = Ar_1^t + Br_2^t$$

where r_1 and r_2 are the two distinct solutions of the *characteristic equation* $r^2 + ar + b = 0$.

(2) If $a^2 - 4b = 0$, then

$$x_t = (A + Bt)r^t$$

where $r = -\frac{1}{2}a$ is the only solution of the *characteristic equation* $r^2 + ar + b = 0$.

We consider an example.

Example 11.9. Find the general solution of

$$x_{t+2} = 4x_{t+1} - 4x_t.$$

Solution.
The difference equation may be rewritten as

$$x_{t+2} = 4x_{t+1} - 4x_t \iff x_{t+2} - 4x_{t+1} + 4x_t = 0.$$

The characteristic equation is

$$r^2 - 4r + 4 = 0.$$

It has only one solution $r = 2$. Thus we obtain

$$x_t = (A + Bt)2^t$$

as the general solution.

The solution is general since it contains all possible solutions of the difference equation. When we specify for instance x_0 and x_1, we get a *particular solution.*

Example 11.10. Find a formula for x_t when $x_{t+2} = 4x_{t+1} - 4x_t$ and $x_0 = 2$, $x_1 = 3$.

Solution.
The general solution is

$$x_t = (A + Bt)2^t.$$

We get $x_0 = (A + B \cdot 0)2^0 = A = 2$, and $x_1 = (A + B \cdot 1)2^1 = 4 + 2B = 3$. From this we obtain $2B = 3 - 4 = -1$ and $B = -\frac{1}{2}$. Thus we have

$$x_t = (2 - \frac{1}{2}t)2^t = 2^{t+1} - t2^{t-1}.$$

We consider another example.

Example 11.11. Find the general solution of

$$x_{t+2} = x_{t+1} + x_t.$$

Solution.
The equation can be rewritten as

$$x_{t+2} = x_{t+1} + x_t \iff x_{t+2} - x_{t+1} - x_t = 0.$$

The characteristic equation is

$$r^2 - r - 1 = 0.$$

The solutions are $r_1 = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ and $r_2 = \frac{1}{2} + \frac{1}{2}\sqrt{5}$. The general solution is thus given as

$$x_t = A\left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^t + B\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^t.$$

Example 11.12. Find the solution of

$$x_{t+2} = x_{t+1} + x_t$$

with $x_0 = 1$ and $x_1 = 3$.

Solution.
We have found the general solution

$$x_t = A\left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^t + B\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^t.$$

We get

$$\begin{aligned} x_0 &= A\left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^0 + B\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^0 \\ &= A + B = 1 \implies B = 1 - A \end{aligned}$$

and

$$\begin{aligned} x_1 &= A\left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^1 + B\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^1 \\ &= \frac{A}{2}(1 - \sqrt{5}) + \frac{1 - A}{2}(1 + \sqrt{5}) = 3 \end{aligned}$$

We obtain

$$A = \frac{1 - \sqrt{5}}{2} \text{ and } B = \frac{1 + \sqrt{5}}{2}.$$

Thus

$$x_t = \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^{t+1} + B\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^{t+1}.$$

It is interesting to note that although the formula contains fractions and square roots, x_t is always an integer.

Example 11.13. Consider the following system of difference equations:

$$\begin{aligned} x_{t+1} &= x_t + 2y_t \\ y_{t+1} &= 3x_t \end{aligned}$$

with $x_0 = 1$ and $y_0 = 0$. Derive a second order difference equation for x_t and solve this equation and the system.

12. More on difference equations

Reading. This lecture covers topics from Sections 11.3 and 11.4 in FMEA [**2**].

12.1. Non-homogenous second order difference equations. We start by reviewing how to solve second order homogenous difference equations.

Problem 12.1.
Find the general solution of the following homogenous difference equation:

$$x_{t+2} + 2x_{t+1} + x_t = 0$$

We now turn to non-homogenous equations.

Theorem 12.1. The general solution of the non-homogenous difference equation

$$x_{t+2} + ax_{t+1} + bx_t = c_t \qquad (b \neq 0) \tag{9}$$

can be found as

$$x_t = x_t^{(h)} + x_t^{(p)}$$

where $x_t^{(h)}$ is the general solution of the corresponding homogenous equation $x_{t+2} + ax_{t+1} + bx_t = 0$ and where $x_t^{(p)}$ is any particular solution of (9).

We consider some examples.

Example 12.2. Find the general solution of

$$x_{t+2} + 2x_{t+1} + x_t = 5.$$

Solution.
The general solution of the homogenous equation is

$$x_t^{(h)} = (A + Bt)(-1)^t$$

We find a particular solution by using *the method of undetermined coefficients:* We guess on a solution on the form

$$x_t^{(p)} = c$$

for some constant c. Substituting into the left hand side of the equation, we obtain

$$x_{t+2}^{(p)} + 2x_{t+1}^{(p)} + x_t^{(p)} = c + 2c + c = 4c$$

Thus we see that $4c = 5$ and hence $c = \frac{5}{4}$. We obtain

$$x_t^{(p)} = \frac{5}{4}$$

and

$$x_t = (A + Bt)(-1)^t + \frac{5}{4}$$

You should try to solve the following problem.

Problem 12.2.
Find the solution of

$$x_{t+2} - 7x_{t+1} + 12x_t = 1$$

that satisfies $x_0 = 0$ and $x_1 = 0$.

We also consider an example where the right hand side is non-constant.

Example 12.3. Find the general solution of

$$x_{t+2} - 5x_{t+1} + 6x_t = 4^t$$

Solution.
We first find the general solution of the homogenous equation:

$$r^2 - 5r + 6 = 0$$

gives $r = 2$ or $r = 3$. Thus

$$x_t^{(h)} = A \cdot 2^t + B \cdot 3^t$$

We seek a particular solution on the form

$$x_t^{(p)} = c \cdot 4^t$$

Substituting this into the left hand side of the equation, we obtain

$$\begin{aligned} x_{t+2}^{(h)} - 5x_{t+1}^{(h)} + 6x_t^{(h)} &= c4^{t+2} - 5c4^{t+1} + 6c4^t \\ &= 16c4^t - 20c4^t + 6c4^t \\ &= 2c4^t \end{aligned}$$

To have a solution, we get $2c = 1$. Thus $c = \frac{1}{2}$ and $x_t^{(p)} = \frac{1}{2} \cdot 4^t$. The general solution is then

$$x_t = A \cdot 2^t + B \cdot 3^t + \frac{1}{2} \cdot 4^t$$

In the final example, we briefly consider the asymptotic behavior.

Example 12.4. Find the general solution of

$$x_{t+2} + x_{t+1} + \frac{1}{4}x_t = 0$$

and find $\lim_{t\to\infty} x_t$

Solution.
The characteristic equation

$$r^2 + r + \frac{1}{4} = 0$$

has the single solution $r = -\frac{1}{2}$. Thus

$$x_t = (A + tB)(-\frac{1}{2})^t$$

We get that

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} (A + tB)(-\frac{1}{2})^t = 0$$

since it is known that $\lim_{t\to\infty} t^n a^t = 0$ for any n and a with $|a| < 0$.

We have the following definition:

Definition 12.5. The difference equation

$$x_{t+2} + ax_{t+1} + bx_t = c_t \quad (b \neq 0)$$

is said to be *globally asymptotically sable* if

$$\lim_{t\to\infty} x_t^{(h)} = 0.$$

Part 2

Exercise Problems with Solutions

CHAPTER 2

Exercise Problems

This chapter contains exercise problems for GRA6035 Mathematics. Each section contains exercise problems for the corresponding lecture in Chapter 1. Some of the problems are from EMEA [**3**] and FMEA [**2**], and are labeled accordingly. Solutions to all exercise problems are given in Chapter 3.

— Exercise Problems —
0. Review of matrix algebra and determinants

Problem 0.1.
Let
$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 6 \\ 7 & 0 \end{pmatrix} \text{ and } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
Compute $4A + 2B$, AB, BA, BI_2 and I_2A

Problem 0.2.
A general property of matrix multiplication states that $(AB)^T = B^T A^T$. Prove this for 2×2-matrices.

Problem 0.3.
Simplify the following matrix expressions.
(a) $AB(BC - CB) + (CA - AB)BC + CA(A - B)C$
(b) $(A - B)(C - A) + (C - B)(A - C) + (C - A)^2$

Problem 0.4.
Let
$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix}, B = \begin{pmatrix} 4 & 0 & 0 \\ 7 & 8 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(1) Compute $A + 3B$, AB, BA and $A(I_3B)$.
(2) Compute the determinants $|A|$, $|B|$, $|AB|$ and $|BA|$. Verify that $|A||B| = |AB| = |BA|$.
(3) Compute $|A|$ using two different cofactor expansions, and verify that you get the same result.
(4) Verify the following:
  (a) $(A^T)^T = A$
  (b) $(A + B)^T = A^T + B^T$
  (c) $(3A)^T = 3A^T$
  (d) $(AB)^T = B^T A^T$

Problem 0.5.
A general $m \times n$-matrix is often written $A = (a_{ij})_{m \times n}$ where a_{ij} is the entry of A at row i and column j. Assume that $m = n$ and $a_{ij} = a_{ji}$. Explain that $A = A^T$. Give a concrete example of such matrix, and explain why it is reasonable to call a matrix with the property that $A = A^T$, for symmetric.

Problem 0.6.
Let D be the matrix
$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
Compute D^2, D^3 and D^n.

Problem 0.7.
Show that the following system of linear equations

$$\begin{aligned} 3x_1 + x_2 + 5x_3 &= 4 \\ 5x_1 - 3x_2 + 2x_3 &= -2 \\ 4x_1 - 3x_2 - x_3 &= -1 \end{aligned}$$

can be written as

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 3 & 1 & 5 \\ 5 & -3 & 2 \\ 4 & -3 & -1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}.$$

Problem 0.8.
(EMEA section 15.3, ex. 3) Initially, three firms A, B and C (numbered 1, 2 and 3) share the market for a certain commodity. Firm A has 20% of the marked, B has 60% and C has 20%. In course of the next year, the following changes occur:

$$\begin{cases} \text{A keeps 85\% of its customers, while losing 5\% to B and 10\% to C} \\ \text{B keeps 55\% of its customers, while losing 10\% to A and 35\% to C} \\ \text{C keeps 85\% of its customers, while losing 10\% to A and 5\% to B} \end{cases}$$

We can represent market shares of the three firms by means of *a market share vector,* defined as a column vector $\mathbf{s}$ whose components are all nonnegative and sum to 1. Define the matrix $\mathbf{T}$ and the initial share vector $\mathbf{s}$ by

$$T = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \text{ and } \mathbf{s} = \begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix}.$$

The matrix T is called the *transition matrix.* Compute the vector $T\mathbf{s}$, show that it is also a market share vector, and give an interpretation. What is the interpretation of $T^2\mathbf{s}$ and $T^3\mathbf{s}$?
Let

$$\mathbf{q} = \begin{pmatrix} 0.4 \\ 0.1 \\ 0.5 \end{pmatrix}$$

Compute $T\mathbf{q}$. Give an interpretation.

Problem 0.9.
Let A and B be 3×3-matrices. Assume that $|A| = 2$ and $|B| = -5$.

(1) Find $|AB|$, $|-3A|$ and $|-2A^T|$.
(2) Let C be the matrix obtained from B by interchanging two rows. Find $|C|$.

Problem 0.10.
Compute

$$\begin{vmatrix} 3 & 1 & 5 \\ 9 & 3 & 15 \\ -3 & -1 & -5 \end{vmatrix}$$

by using that the determinant is unchanged when adding a multiple of one row to another.

Problem 0.11.
(EMEA 16.4.11) Without computing the determinants, show that

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$

Problem 0.12.
On a matrix A one may perform one of the following *elementary row operations:*

(1) Interchange two rows.
(2) Multiply a row by a nonzero constant.
(3) Add a multiple of one row to another row.

If A and B are matrices of the same size such that we can obtain B from A by a succession of elementary row operations, we say that A and B are row equivalent. This is often denoted $A \sim B$.
Show that A and B are row equivalent.

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 5 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

— Exercise Problems —

1. The inverse matrix and linear dependence

Problem 1.1.
Determine which of the following matrices are invertible. For each invertible matrix, find the inverse using the formula for 2×2-matrices.

$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Problem 1.2.
If $A = (a_{ij})_{n\times n}$ is an $n \times n$-matrix, then its determinant may be computed by

$$|A| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

where the cofactor A_{ij} is $(-1)^{i+j}$ times the determinant obtained from A by deleting row i and column j. This is called cofactor expansion along the first row. Similarly one may compute $|A|$ by cofactor expansion along any row or column.

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}.$$

First calculate $|A|$ using cofactor expansion along the first column, and then calculate $|A|$ again using cofactor expansion along the third row. Check that you get the same answer. Is A invertible?

Problem 1.3.
Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}.$$

Compute the cofactor matrix $\mathrm{cof}(A)$, the adjoint (adjugate) matrix $\mathrm{adj}(A)$ and the inverse matrix A^{-1}. Verify that $A\,\mathrm{adj}(A) = |A|I_3$ and that $AA^{-1} = I_3$. Answer the same questions for

$$B = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where b is any number.

Problem 1.4.
Solve the linear system of equations

$$\begin{aligned} 5x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 4 \end{aligned}$$

by writing it on matrix form $A\mathbf{x} = \mathbf{b}$ and finding A^{-1}.

Problem 1.5.
(FMEA 1.2.2 (1. ed.), 1.9.2 (2. ed.)) Compute the following matrix product using partitioning. Check the result by ordinary matrix multiplication.

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 0 & -1 \end{array}\right)\left(\begin{array}{cc} 2 & -1 \\ 0 & 1 \\ \hline 1 & 1 \end{array}\right)$$

Problem 1.6.
(FMEA 1.3.1 (1. ed.), 1.2.1 (2. ed.)) Express $\begin{pmatrix} 8 \\ 9 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Draw the three vectors in the plane.

Problem 1.7.
(FMEA 1.3.2 (1. ed.), 1.2.2 (2. ed.)) Determine which of the following pairs of vectors are linearly independent:

$$\text{(a)} \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{(c)} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Draw the vectors in (a) in the plane and explain geometrically.

Problem 1.8.
(FMEA 1.3.4 (1. ed.), 1.2.4 (2. ed.)) Prove that $(1, 1, 1)$, $(2, 1, 0)$, $(3, 1, 4)$ and $(1, 2, -2)$ are linearly dependent.

Problem 1.9.
(FMEA 1.3.5 (1. ed.), 1.2.5 (2. ed.)) Assume that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent vectors in $\mathbb{R}^m$, prove that $\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}$ and $\mathbf{a}+\mathbf{c}$ are linearly independent. Is the same true of $\mathbf{a}-\mathbf{b}, \mathbf{b}+\mathbf{c}$ and $\mathbf{a}+\mathbf{c}$?

Problem 1.10.
Prove that n vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$ are linearly dependent if and only if at least one of the vectors can be written as a linear combination of the others.

Problem 1.11.
Recall the three *elementary row operations:*

(1) Interchange two rows.
(2) Multiply a row by a nonzero constant.
(3) Add a multiple of one row to another row.

There is an efficient way of finding the inverse of a square matrix using row operations. Suppose we want to find the inverse of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 5 & 7 \end{pmatrix}.$$

To do this we form the partitioned matrix $[A|I]$:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 2 & 5 & 7 & 0 & 0 & 1 \end{array}\right)$$

We saw in a problem last week, that A was row equivalent to the identity matrix. We now perform the needed row operations. First we take (-1) times the first row and add it to the second row:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 2 & 5 & 7 & 0 & 0 & 1 \end{array}\right)$$

Then we take (-2) times the first row and add it to the last row:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right)$$

Take (-1) times the second row and add it to the third:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

Take (-3) the last row and add it to the first:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

Take (-2) times the second row and add it to the first.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 1 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

We now have the partitioned matrix $[I|A^{-1}]$ and thus

$$A^{-1} = \begin{pmatrix} 6 & 1 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

Use the same technique to find the inverse of the following matrices:

(a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

— Exercise Problems —

2. The rank of a matrix and applications

Problem 2.1.
Write the following systems of linear equations as a vector equation and then as a matrix equation. Write down the coefficient matrix and the augmented matrix for each system.

(1)

$$\begin{array}{rrrcl} 3x_1 & & -x_3 & = & 0 \\ -5x_1 & +2x_2 & +12x_3 & = & 0 \\ & 6x_2 & -5x_3 & = & 0 \end{array}$$

(2)

$$\begin{array}{rrrrcl} 5x_1 & & -6x_3 & +3x_4 & = & 0 \\ -2x_1 & -10x_2 & +7x_3 & & = & 0 \\ 2x_1 & & -5x_3 & +5x_4 & = & 0 \end{array}$$

Problem 2.2 (FMEA(2ed) 1.2.3 / (1ed) 1.3.3).
Prove that the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

are linearly independent.

Problem 2.3.
Describe all minors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}.$$

Problem 2.4 (FMEA(2ed) 1.3.1 / (1ed) 1.4.1).
Determine the ranks of the following matrices:

$(a)\ \begin{pmatrix} 1 & 2 \\ 8 & 16 \end{pmatrix}$ $\quad (b)\ \begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix}$ $\quad (c)\ \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{pmatrix}$

$(d)\ \begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{pmatrix}$ $\quad (e)\ \begin{pmatrix} 2 & 1 & 3 & 7 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 11 \end{pmatrix}$ $\quad (f)\ \begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ -2 & -5 & -2 & 0 \end{pmatrix}$

Problem 2.5 (FMEA(2ed) 1.3.2ab / (1ed) 1.4.2 ab).

$(a)\ \begin{pmatrix} x & 0 & x^2-2 \\ 0 & 1 & 1 \\ -1 & x & x-1 \end{pmatrix}$ $\qquad (b)\ \begin{pmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -6 \\ 1 & 1 & t+4 \end{pmatrix}$

Problem 2.6 (FMEA(2ed) 1.3.3 / (1ed) 1.4.3).
Give an example where $\mathrm{rk}(AB) \neq \mathrm{rk}(BA)$. (Hint: Try some 2×2 matrices).

Problem 2.7.
Using the definition of rank of a matrix, explain way m vectors in $\mathbb{R}^n$ must be linearly dependent if $m > n$.

Problem 2.8.
Show that

$$\begin{pmatrix} 3 \\ 4 \\ -1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

are linearly independent by computing a 2×2-minor.

Problem 2.9 (FMEA(2ed) 1.4.1 / (1ed) 1.5.1).
Use minors to determine if the systems have solutions. If they do, determine the degrees of freedom. Find all solutions. Check the results.

$$(a)\quad \begin{array}{rcrcrcl} -2x_1 & - & 3x_2 & + & x_3 & = & 3 \\ 4x_1 & + & 6x_2 & - & 2x_3 & = & 1 \end{array}$$

$$(b)\quad \begin{array}{rcrcrcrcl} x_1 & + & x_2 & - & x_3 & + & x_4 & = & 2 \\ 2x_1 & - & x_2 & + & x_3 & - & 3x_4 & = & 1 \end{array}$$

$$(c)\quad \begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1 \\ 2x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & 3 \\ x_1 & + & 5x_2 & - & 8x_3 & + & x_4 & = & 1 \\ 4x_1 & + & 5x_2 & - & 7x_3 & + & 7x_4 & = & 7 \end{array}$$

$$(d)\quad \begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 5 \\ 2x_1 & + & 3x_2 & - & x_3 & - & 2x_4 & = & 2 \\ 4x_1 & + & 5x_2 & + & 3x_3 & & & = & 7 \end{array}$$

Problem 2.10 (FMEA(2ed) 1.4.3 / (1ed) 1.5.3).
Discuss the number of solutions of the following system for all values of a and b:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 1 \\ -x & + & ay & - & 21z & = & 2 \\ 3x & + & 7y & + & az & = & b \end{array}$$

Problem 2.11 (FMEA(2ed) 1.4.4 / (1ed) 1.5.4).
Let $A\mathbf{x} = \mathbf{b}$ be a linear system of equations in matrix form. Prove that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions of the system, then so is $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ for every real number λ. Use this fact to prove that a linear system of equations that is consistent has either one solution or infinitely many solutions.

Problem 2.12 (FMEA(1ed) 1.5.5a).
For what values of the constants p and q does the following system have a unique solution, several solutions, or no solutions?

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 2q \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 4q \\ 3x_1 & - & 2x_2 & + & px_3 & = & q \end{array}$$

Problem 2.13 (FMEA(2ed) 1.4.6ab / (1ed) 1.5.7ab).
Let A_t be the matrix given by

$$A_t = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{pmatrix}$$

(1) Find the rank of the matrix A_t for all real numbers t
(2) When $t = -3$, find all vectors $\mathbf{x}$ that satisfy the vector equation $A_{-3}\mathbf{x} = \begin{pmatrix} 11 \\ 3 \\ 6 \end{pmatrix}$.

Problem 2.14 (Optional).
Let $\mathbf{a}$ and $\mathbf{b}$ be column vectors of the same size. The *scalar* product $\mathbf{a}$ and $\mathbf{b}$ of is defined as

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$$

Compute $\mathbf{a} \cdot \mathbf{b}$.
(a) $\mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.
(b) $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$.
Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be *orthogonal* if $\mathbf{a} \cdot \mathbf{b} = 0$. Determine if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.
(c) $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.
A set of n vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are said to be *orthogonal* if $\mathbf{a}_i \cdot \mathbf{a}_j = 0$ whenever $i \neq j$.
(d) Show that the following vectors are orthogonal:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(e) Show that if $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are *orthogonal* (and nonzero), then they are linearly independent.

Problem 2.15 (Optional).
The *norm* (or length) of a vector $\mathbf{a}$ is defined as $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. In $\mathbf{a}$ is a 2-vector represented by an arrow in the plane. Explain why it is reasonable to call this the length of the vector $\mathbf{a}$.

Problem 2.16 (Optional).
A theorem states that for a matrix A then

$$r(A) = r(A^T A).$$

(The matrix $A^T A$ is called the *Gram* matrix of A.) Use this theorem to find the rank of the following matrices:

(a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 6 \end{pmatrix}$

Assume that A is an $m \times n$ matrix. The set of n-vectors such that $A\mathbf{x} = 0$ is called the *null space* of A.

(c) Prove that the null space of $A^T A$ is equal to the null space of A.

(d) Assume that A is an $m \times n$ matrix with $m \leq n$. Prove the following special case of the formula $r(A) = r(A^T A)$: The rank of A is less than m, then $|A^T A| = 0$.

Problem 2.17.

The most efficient way of computing the rank of a matrix is usually to reduce the matrix with elementary row operations. Recall the three *elementary row operations:*

(1) Interchange two rows.
(2) Multiply a row by a nonzero constant.
(3) Add a multiple of one row to another row.

Assume we which to find the rank of a matrix A, say

$$A = \begin{pmatrix} 1 & -2 & 3 & 4 & 2 \\ 1 & 1 & -2 & -4 & 1 \\ 2 & -1 & 1 & 0 & 3 \end{pmatrix}.$$

One first reduce the matrix A to a matrix on *row echelon.* **Step 1:** One starts with the left most nonzero entry in the matrix and move the row it sits in to the top. In our particular example this is already done for us. **Step 2:** We make this into a 1 by dividing (again this is not necessary in our example). The we use the first row to obtain zeros in the column. In our example we take (-1) times the first row and add to the second row

$$\begin{pmatrix} 1 & -2 & 3 & 4 & 2 \\ 0 & 3 & -5 & -8 & -1 \\ 2 & -1 & 1 & 0 & 3 \end{pmatrix}$$

Then we take (-2) times the first row and add it to the last row:

$$\begin{pmatrix} 1 & -2 & 3 & 4 & 2 \\ 0 & 3 & -5 & -8 & -1 \\ 0 & 3 & -5 & -8 & -1 \end{pmatrix}.$$

Step 3: Now we forget about the first row in the matrix, and use step 1 and 2 on the remaining matrix:

$$\begin{pmatrix} 1 & -2 & 3 & 4 & 2 \\ 0 & 3 & -5 & -8 & -1 \\ 0 & 3 & -5 & -8 & -1 \end{pmatrix}$$

We thus make 3 into a 1 by multiplying the row with $\frac{1}{3}$:

$$\begin{pmatrix} 1 & -2 & 3 & 4 & 2 \\ 0 & 1 & \frac{-5}{3} & -\frac{8}{3} & -\frac{8}{3} \\ 0 & 3 & -5 & -8 & -1 \end{pmatrix}$$

Then we take -3 times this row and add to the last row:

$$\begin{pmatrix} 1 & -2 & 3 & 4 & 2 \\ 0 & 1 & \frac{-5}{3} & -\frac{8}{3} & -\frac{8}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This is a matrix on row echelon form.

(Actually it is not strictly necessary to divide by 3. One could proceed from

$$\begin{pmatrix} 1 & -2 & 3 & 4 & 2 \\ 0 & 3 & -5 & -8 & -1 \\ 0 & 3 & -5 & -8 & -1 \end{pmatrix}$$

by take 1 times the second tow and add it to the last to obtain

$$\begin{pmatrix} 1 & -2 & 3 & 4 & 2 \\ 0 & 3 & -5 & -8 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

)

The rank of A is the number of non-zero rows in the row echelon form of A. Thus the rank of A is 2.

Compute the rank of the matrices in Problem 4, using this method.

— Exercise Problems —
3. Eigenvalues and diagonalization

Problem 3.1.
Find the eigenvalues of the following matrices:

(1) $\begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}$ (2) $\begin{pmatrix} 2 & 4 \\ -2 & 6 \end{pmatrix}$ (3) $\begin{pmatrix} 1 & 4 \\ 6 & -1 \end{pmatrix}$

(4) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ (5) $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$

Problem 3.2.
Find the eigenvectors corresponding to each eigenvalue for the matrices in the previous problem.

Problem 3.3.
(FMEA 1.5.7 in the second edition and 1.6.3 in the first edition.) Suppose that A is a square matrix and let λ be an eigenvalue of A. Prove that if $|A| \neq 0$ then $\lambda \neq 0$. In this case show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Problem 3.4.
Let

$$A = \begin{pmatrix} 1 & 18 & 30 \\ -2 & -11 & -10 \\ 2 & 6 & 5 \end{pmatrix}.$$

Verify that $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ are eigenvectors for A and find the corresponding eigenvalues.

Problem 3.5.
For the matrix

$$A = \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}$$

in Problem 1, find an invertible matrix P such that $D = P^{-1}AP$ is diagonal.

Problem 3.6.
Show that the matrix

$$A = \begin{pmatrix} 3 & 5 \\ 0 & 3 \end{pmatrix}$$

is not diagonalizable.

Problem 3.7.

Initially, two firms A and B (numbered 1 and 2) share the market for a certain commodity. Firm A has 20% of the marked and B has 80%. In course of the next year, the following changes occur:

$$\begin{cases} \text{A keeps 85\% of its customers, while losing 15\% to B} \\ \text{B keeps 55\% of its customers, while losing 45\% to A} \end{cases}$$

We can represent market shares of the two firms by means of *a market share vector,* defined as a column vector $\mathbf{s}$ whose components are all nonnegative and sum to 1. Define the matrix $\mathbf{T}$ and the initial share vector $\mathbf{s}$ by

$$T = \begin{pmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{pmatrix} \text{ and } \mathbf{s} = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}.$$

The matrix T is called the *transition matrix.*

(1) Compute the vector $T\mathbf{s}$, and show that it is also a market share vector.

(2) Find the eigenvalues of T by solving the characteristic equation.

(3) Show that

$$\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are eigenvectors for T.

(4) Write down a matrix P such that

$$D = P^{-1}TP$$

is diagonal.

(5) Show that $T^n = PD^nP^{-1}$. Compute $\lim_{n\to\infty} D^n$ and use this to find $\lim_{n\to\infty} T^n\mathbf{s}$.

(6) Explain that the we will approach a situation where A's and B's market shares are constant. What are these shares.

Problem 3.8.
Answer the following multiple choice questions.

Question 1

Assume that A is an $n \times m$-matrix and that B is an $t \times s$-matrix.

When is the product AB defined?

A. The product AB is defined only when $m = t$.
B. The product AB is defined only when A and B are square matrices.
C. The product AB is defined only when $s = m$ and in this case AB is an $n \times s$-matrix.
D. The product AB is defined when A and B are square matrices, but AB need not be a square matrix.
E. I prefer not to answer.

Question 2

Assume that A is a 3×3-matrix, and that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors such that $A\mathbf{u} = \mathbf{0}$, $A\mathbf{v} = 2\mathbf{v}$ and $A\mathbf{w} = 2\mathbf{w}$.

Can we conclude that A is diagonalizable?

A. Yes.
B. No, because $\mathbf{u}$ is not an eigenvector.
C. No, because A has only one eigenvalue.
D. No, because $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ need not be linearly independent.
E. I prefer not to answer.

Question 3

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Are u and v eigenvectors for A?

A. Yes, $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for A.
B. No, only $\mathbf{v}$ is an eigenvector for A.
C. No, only $\mathbf{u}$ is an eigenvector for A.
D. No, neither $\mathbf{u}$ nor $\mathbf{v}$ is an eigenvector for A.
E. I prefer not to answer this question.

The following problems are optional.

Problem 3.9.
A system of linear equations can be solved using row reductions. Consider for instance:

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 5 \\ 2x_1 & + & 3x_2 & - & x_3 & - & 2x_4 & = & 2 \\ 4x_1 & + & 5x_2 & + & 3x_3 & & & = & 12 \end{array}$$

This system has the augmented matrix

$$A_{\mathbf{b}} = \begin{pmatrix} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 12 \end{pmatrix}$$

(a) Show that $A_{\mathbf{b}}$ can be reduced to

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

using row reductions.
(b) Using one more elementary row operations show that it can be reduced to

$$\begin{pmatrix} 1 & 0 & 7 & 5 & 13 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix is on so-called *reduced row echelon form:* In every row the first non-zero entry is 1 (called leading 1) and this is the only non-zero entry in its column.
(c) The matrix in (b) is the augmented matrix of the following system of linear equations

$$\begin{array}{rcrcrcrcr} x_1 & & & + & 7x_3 & + & 5x_4 & = & 13 \\ & & x_2 & - & 5x_3 & - & 4x_4 & = & -8 \end{array}$$

Since elementary row operation does not change the solutions of a system of linear equations, this system has the same solutions has the original system. Show that the solutions may be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 13 \\ -8 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -7 \\ 5 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$

Problem 3.10.
If one whish to find the eigenvectors of a matrix on might have to solve several systems of linear equations. Use the method indicated in the previous problem when you answer this problem: Determine if the matrix

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 5 \end{pmatrix}$$

is diagonalizable, and if this is the case, find a matrix P such that $P^{-1}AP$ is diagonal.

— Exercise Problems —
4. Quadratic forms and concave/convex functions

Problem 4.1.
(1.8.2 in the first edtion of FMEA and 1.7.3 in the second edtion of FMEA) Write the following quadratic forms as $\mathbf{x}^T A\mathbf{x}$ with A symmetric:
(a) $x^2 + 2xy + y^2$ (b) $ax^2 + bxy + cy^2$ (c) $3x_1^2 - 2x_1x_2 + 3x_1x_3 + x_2^2 + 3x_3^2$

Problem 4.2.
Write the following quadratic forms as $\mathbf{x}^T A\mathbf{x}$ and determine the definiteness.
(a) $Q(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + 5x_3^2$
(b) $Q(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + 3x_2^2 + 5x_3^2$

Problem 4.3.
Let
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
be any two by two matrix. Show that the characteristic equation may be written as
$$\lambda^2 - \mathrm{tr}(A)\lambda + |A| = 0$$
where $\mathrm{tr}(A)$ is the trace of A defined as the sum of the elements on the diagonal. Show that if λ_1 and λ_2 are the eigenvalues of A, then $|A| = \lambda_1\lambda_2$.

Problem 4.4.
(2.2.1 in the first edition of FMEA and 2.2.1 in the secon edition of FMEA.) (Do not bother about strictly convex/concave.)

Problem 4.5.
Draw the line segment
$$[\mathbf{x}, \mathbf{y}] = \{\mathbf{z} : \text{ there exists } s \in [0.1] \text{ such that } \mathbf{z} = s\mathbf{x} + (1-s)\mathbf{y}\}$$
in the plane where
(a) $\mathbf{x} = (0, 0)$ and $\mathbf{y} = (2, 2)$
(b) $\mathbf{x} = (-1, 1)$ and $\mathbf{y} = (3, 4)$
Mark the points corresponding to $s = 0, 1$ and $\frac{1}{2}$ on each line segment.

Problem 4.6.
FMEA 2.2.2 abcd

Problem 4.7.
FMEA 2.3.1 (Do not bother about strictly convex/concave.)

Problem 4.8.
Let f be defined for all x and y by $f(x, y) = x - y - x^2$. Show that f is concave.

Problem 4.9.
Answer the following multiple choice questions.

Question 1

Consider the three vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}.$$

Are the vectors, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, linearly independent?

A. Yes, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent since $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$ has only the trivial solution.
B. Yes, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent since $\mathbf{u}$ is not a linear combination of $\mathbf{v}$ and $\mathbf{w}$.
C. No, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent since $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$ has only the trivial solution.
D. No, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent since $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
E. I prefer not to answer this question.

Question 2

Compute the matrix product

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 3 \end{pmatrix}.$$

if it is defined.

A. The matrix product is not defined.

B. The answer is 5.

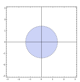
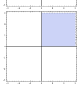
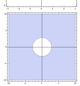
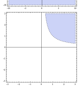
C. The answer is $\begin{pmatrix} -1 & 3 \\ -2 & 6 \end{pmatrix}$.

D. The answer is $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$.

E. I prefer not to answer.

Question 3

Which of the following subsets (in blue/shaded) of the plane is *not* convex?

A. 
B. 
C. 
D. 
E. I prefer not to answer this question.

— Exercise Problems —
5. The Hessian matrix

Problem 5.1.
Use principal minors to determine the definiteness of the quadratic forms in Problem 1.8.3abd in the first edition of FMEA or Problem 1.7.3abd in the second edition of FMEA.

Problem 5.2.
2.3.3 in the first and second edition of FMEA

Problem 5.3.
2.3.4 in the first and second edition of FMEA

Problem 5.4.
Consider the function

$$f(x, y) = x^4 + 16y^4 + 32xy^3 + 8x^3y + 24x^2y^2.$$

Find the Hessian matrix. Show that f is convex.

Problem 5.5.
3.1.1 in the first and second edition of FMEA

Problem 5.6.
3.1.2 in the first and second edition of FMEA.

Problem 5.7.
Consider the function f defined on the subset $S = \{(x, y, z) : z > 0\}$ of $\mathbb{R}^3$ by

$$f(x, y, z) = 2xy + x^2 + y^2 + z^3.$$

Show that S is convex. Find the stationary points of f. Find the Hessian matrix. Is f concave or convex? Does f have a global extreme point?

Problem 5.8.
Show that

$$f(x, y, z) = x^4 + y^4 + z^4 + x^2 + y^2 - xy + zy + z^2.$$

is convex.

— Exercise Problems —
6. Local extreme points and the Lagrange problem

Problem 6.1.
(3.2.1 in both editions of FMEA)

Problem 6.2.
Classify the stationary points of

$$f(x, y, z) = -2x^3 + 15x^2 - 36x + 2y - 3z + \int_y^z e^{t^2}\, dt$$

Problem 6.3.
(3.2.3 in first edition and 3.2.2 in second edition of FMEA) Let f be defined for all (x, y) by $f(x, y) = x^3 + y^3 - 3xy$.
(a) Show that $(0, 0)$ and $(1, 1)$ are the only stationary points, and compute the Hessian matrix at these points.
(b) Determine the definiteness of the Hessian matrix at each stationary point and use this to determine the nature of each point.
(c) Use the usual second derivative test in two variables to classify the stationary points.

Problem 6.4.
(3.3.1 in both editions of FMEA)
(a) Solve the problem

$$\max(100 - x^2 - y^2 - z^2) \text{ subject to } x + 2y + z = a.$$

(b) Let $(x^*(a), y^*(a), z^*(a))$ be the maximum point and let $\lambda(a)$ be the corresponding Lagrange multiplier. Let $f^*(a) = f(x^*(a), y^*(a), z^*(a))$. Show that

$$\frac{\partial(f^*(a))}{\partial a} = \lambda(a)$$

Problem 6.5.
(3.3.2 in both editions of FMEA)
(a) Solve the problem

$$\max f(x, y, z) = x + 4y + z \text{ subject to } x^2 + y^2 + z^2 = b_1 = 216 \text{ and } x + 2y + 3z = b_2 = 0.$$

(b) (x^*, y^*, z^*) be the maximum point and let λ_1 and λ_2 be the corresponding Lagrange multipliers. Let $f^* = f(x^*, y^*, z^*)$. Change the first constraint to $x^2 + y^2 + z^2 = 215$ and the second to $x + y + 2y + 3z = 0.1$. It can be shown that corresponding change in the f^* is approximately equal to

$$\lambda_1 \Delta b_1 + \lambda_2 \Delta b_2.$$

Use this to estimate the change in f^*.

Problem 6.6.
(3.3.3a in both editions of FMEA)

— Exercise Problems —

7. Envelope theorems and the bordered Hessian

Problem 7.1.
FMEA 3.1.4 (1. and 2. ed.)

Problem 7.2.
FMEA 3.1.5 (1. and 2. ed.)

Problem 7.3.
FMEA 3.3.4 (1. and 2. ed.)

Problem 7.4.
FMEA 3.3.5 (1. ed.), 3.3.6 (2. ed.)

Problem 7.5.
FMEA 3.4.1 (1. and 2. ed.)

— Exercise Problems —
8. Introduction to differential equations

Problem 8.1.
Find $\dot{x}$.
(a) $x = \frac{1}{2}t - \frac{3}{2}t^2 + 5t^3$
(b) $x = (2t^2 - 1)(t^4 - 1)$
(c) $x = (\ln t)^2 - 5\ln t + 6$
(d) $x = \ln(3t)$
(e) $x = 5e^{-3t^2+t}$
(f) $x = 5t^2e^{-3t}$

Problem 8.2.
Find the integrals.
(a) $\int t^3 dt$
(b) $\int_0^1 (t^3 + t^5 + \frac{1}{3})dt$
(c) $\int \frac{1}{t}dt$
(d)$\int te^{t^2}dt$
(e)$\int \ln t dt$

Problem 8.3.
The following differential equations may be solved by integrating the right hand side. Find the general solution, and the particular solution satisfying $x(0) = 1$.
(a) $\dot{x} = 2t$.
(b) $\dot{x} = e^{2t}$
(c) $\dot{x} = (2t + 1)e^{t^2+t}$
(d) $\dot{x} = \frac{2t+1}{t^2+t+1}$.

Problem 8.4.
FMEA 5.1.1 in both editions.

Problem 8.5.
(FMEA 5.1.2 in both editions.) Show that $x = Ct^2$ is a solution of $t\dot{x} = 2x$ for all choices of the constant C. Find the particular solution satisfying $x(1) = 2$.

Problem 8.6.
FMEA 5.3.1 in both editions.

Problem 8.7.
FMEA 5.3.2 in both editions.

Problem 8.8.
FMEA 5.3.3 in both editions.

The following problems will be discussed at the plenary problem session 22. October. They are related to the following setup:

$$\text{maximize } f(x_1, \ldots, x_n) \text{ subject to } \begin{cases} g_1(x_1, \ldots, x_n) \leq b_1 \\ \vdots \\ g_m(x_1, \ldots, x_n) \leq b_m \end{cases}$$

To solve this problem the Lagrangian is defined as before

$$\mathcal{L}(\mathbf{x}) = f(\mathbf{x}) - \lambda_1 g_1(\mathbf{x}) - \cdots - \lambda_m g_m(\mathbf{x}),$$

and the partial derivatives are set to zero:

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \text{ for } i = 1, \ldots, n.$$

In addition we introduce the *complementary slackness conditions*

$$\lambda_j \geq 0 \text{ and } \lambda_j = 0 \text{ if } g_j(\mathbf{x}) < b_j \text{ for } j = 1, \ldots, m.$$

The first order conditions together with the complementary slackness conditions are often called the *Kuhn-Tucker conditions.* In addition to the Kuhn-Tucker conditions, the constraints must be satisfied.

Problem 8.9.
Maximize the function

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_2 - 1$$

subject to $g(x_1, x_2) = x_1^2 + x_2^2 \leq 1$.

Problem 8.10.
FMEA 3.5.1 (1. and 2. ed.)

Problem 8.11.
FMEA 3.5.3 (1. and 2. ed.)

— Exercise Problems —
9. Linear first order and exact differential equations

Problem 9.1.
(FMEA 5.4.1 in both editions.) Find the general solution of $\dot{x} + \frac{1}{2}x = \frac{1}{4}$. Determine the equilibrium state of the equation. Is it stable? Draw some typical solutions.

Problem 9.2.
FMEA 5.4.2 in both editions.

Problem 9.3.
(FMEA 5.4.4 in both editions.) Find the general solutions of the following differential equations, and in each case, find the particular solution satisfying $x(0) = 1$.

(a) $\dot{x} - 3x = 5$
(b) $3\dot{x} + 2x + 16 = 0$
(c) $\dot{x} + 2x = t^2$

Problem 9.4.
Problem 5.4.7 in the first edition of FMEA and 5.4.6 in the second edition.

Problem 9.5.
Determine which of the following equations are exact:
(a) $(2x + t)\dot{x} + 2 + x = 0$
(b) $x^2\dot{x} + 2t + x = 0$
(c) $(t^5 + 6x^2)\dot{x} + (5xt^4 + 2) = 0$

Problem 9.6.
Solve the exact equations in the previous problem.

Problem 9.7.
FMEA 5.5.1 in both editions.

— Exercise Problems —

10. Second-order differential equations

Problem 10.1.
FMEA 6.1.1ac in both editions.

Problem 10.2.
FMEA 6.1.2 in both editions.

Problem 10.3.
FMEA 6.1.3 in both editions.

Problem 10.4.
FMEA 6.3.1 in both editions.

Problem 10.5.
FMEA 6.3.2bc in both editions.

Problem 10.6.
FMEA 6.3.3 in both editions.

Problem 10.7.
FMEA 6.3.6 in both editions.

Problem 10.8.
6.3.7 in first edition of FMEA and 6.3.8 in second edition.

Problem 10.9.
6.3.8 in first edition of FMEA and 6.3.9 in second edition.

— Exercise Problems —

11. Difference equations

Problem 11.1.
FMEA 11.1.1 in both editions.

Problem 11.2.
FMEA 11.2.1 in both editions.

Problem 11.3.
FMEA 11.3.1 in both editions.

Problem 11.4.
FMEA 11.3.2 in both editions.

Problem 11.5.
FMEA 11.4.1ab in both editions.

Problem 11.6.
FMEA 11.4.5 in both editions.

— Exercise Problems —
12. More on difference equations

Problem 12.1.
Find the general solution of the difference equation $3x_{t+2} - 12x_t = 4$.

Problem 12.2.
FMEA 11.4.2a in both editions.

Problem 12.3.
FMEA 11.4.7b in both editions.

CHAPTER 3

Solutions to Exercise Problems

This chapter contains solutions to exercise problems for GRA6035 Mathematics. Each section contains solutions to the exercise problems in the corresponding section in Chapter 2.

— Solutions to Exercise Problems —

0. Review of matrix algebra and determinants

Problem 0.1.

Solution.

$$4A + 2B = 4\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} + 2\begin{pmatrix} 2 & 6 \\ 7 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 24 \\ 30 & 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 6 \\ 7 & 0 \end{pmatrix} = \begin{pmatrix} 25 & 12 \\ 15 & 24 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 6 \\ 7 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 28 & 12 \\ 14 & 21 \end{pmatrix}$$

$$BI_2 = B$$

$$I_2A = A$$

Problem 0.2.
A general property of matrix multiplication states that $(AB)^T = B^T A^T$. Prove this for 2×2-matrices.

Solution.
$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$

$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax+bz & bw+ay \\ cx+dz & dw+cy \end{pmatrix} \Longrightarrow$

$(AB)^T = \begin{pmatrix} ax+bz & bw+ay \\ cx+dz & dw+cy \end{pmatrix}^T = \begin{pmatrix} ax+bz & cx+dz \\ bw+ay & dw+cy \end{pmatrix}$

$A^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$B^T = \begin{pmatrix} x & y \\ z & w \end{pmatrix}^T = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$

$B^T A^T = \begin{pmatrix} x & z \\ y & w \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} ax+bz & cx+dz \\ bw+ay & dw+cy \end{pmatrix}$

Problem 0.3.
Simplify the following matrix expressions.
(a) $AB(BC - CB) + (CA - AB)BC + CA(A - B)C$
(b) $(A - B)(C - A) + (C - B)(A - C) + (C - A)^2$

Solution.
(a) $AB(BC-CB)+(CA-AB)BC+CA(A-B)C = ABBC-ABCB+CABC-ABBC+CAAC-CABC$
$= -ABCB + CAAC = -ABCB + CA^2C$
(b) $(A-B)(C-A)+(C-B)(A-C)+(C-A)^2 =$
$AC - A^2 - BC + BA + CA - C^2 - BA + BC + C^2 - CA - AC + A^2 = 0$

Problem 0.4.

Solution.

(1)

$$A+3B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix} + 3 \begin{pmatrix} 4 & 0 & 0 \\ 7 & 8 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 2 & 0 \\ 21 & 27 & 4 \\ 1 & 2 & 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 7 & 8 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 16 & 2 \\ 21 & 24 & 4 \\ 18 & 16 & 3 \end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & 0 & 0 \\ 7 & 8 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 0 \\ 8 & 40 & 9 \\ 1 & 2 & 1 \end{pmatrix}$$

$$A(I_3B) = AB = \begin{pmatrix} 18 & 16 & 2 \\ 21 & 24 & 4 \\ 18 & 16 & 3 \end{pmatrix}$$

(2)

$$|A| = 3,\ |B| = 32,\ |AB| = |BA| = 96$$

(a)

$$A+B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 0 \\ 7 & 8 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 0 \\ 7 & 11 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

(b)

$$\begin{aligned} B^T A^T &= \begin{pmatrix} 4 & 0 & 0 \\ 7 & 8 & 1 \\ 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix}^T \\ &= \begin{pmatrix} 4 & 7 & 0 \\ 0 & 8 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 18 & 21 & 18 \\ 16 & 24 & 16 \\ 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 18 & 16 & 2 \\ 21 & 24 & 4 \\ 18 & 16 & 3 \end{pmatrix}^T \end{aligned}$$

Problem 0.5.

Solution.
The matrix

$$A = \begin{pmatrix} 13 & 3 & 2 \\ 3 & -2 & 4 \\ 2 & 4 & 3 \end{pmatrix}$$

is symmetric since $A^T = A$. We see that A mirrors along the diagonal.

Problem 0.6.

Solution.

$$D^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$D^3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}^3 = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$D^n = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-3)^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}$$

Problem 0.7.

Solution.

$$A\mathbf{x} = \begin{pmatrix} 3 & 1 & 5 \\ 5 & -3 & 2 \\ 4 & -3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 + 5x_3 \\ 5x_1 - 3x_2 + 2x_3 \\ 4x_1 - 3x_2 - x_3 \end{pmatrix}$$

Thus we see that $A\mathbf{x} = \mathbf{b}$ if and only if

$$\begin{aligned} 3x_1 + x_2 + 5x_3 &= 4 \\ 5x_1 - 3x_2 + 2x_3 &= -2 \\ 4x_1 - 3x_2 - x_3 &= -1 \end{aligned}$$

Problem 0.8.

Solution.

$$T\mathbf{s} = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.35 \\ 0.4 \end{pmatrix}$$

This is the market share vector after one year.

$$T^2\mathbf{s} = T(T\mathbf{s}) = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.35 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.287\,5 \\ 0.225 \\ 0.487\,5 \end{pmatrix}$$

This is the market share vector after two years.

$$T^3\mathbf{s} = \begin{pmatrix} 0.315\,63 \\ 0.162\,5 \\ 0.521\,88 \end{pmatrix}$$

$$T\mathbf{q} = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.1 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.1 \\ 0.5 \end{pmatrix}$$

We see that if the market share vector is $\mathbf{q}$, then it does not change. We say that $\mathbf{q}$ represent an equilibrium.

Problem 0.9.

Solution. (1)

$$|AB| = |A||B| = 2 \cdot (-5) = -10$$
$$|-3A| = (-3)^3|A| = (-27) \cdot 2 = -54$$
$$|-2A^T| = (-2)^3|A^T| = (-8) \cdot |A| = (-8) \cdot 2 = -16$$

(2) When two rows are interchanged, the determinant changes sign. Thus

$$|C| = -|B| = -(-5) = 5.$$

Problem 0.10.

Solution.

If we take one times the first row and add it to the last row, we get

$$\begin{vmatrix} 3 & 1 & 5 \\ 9 & 3 & 15 \\ -3 & -1 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 5 \\ 9 & 3 & 15 \\ 0 & 0 & 0 \end{vmatrix}$$

Thus we get that the answer is 0, for instance by taking cofactor expansion along the third row.

Problem 0.11.

Solution.

Let

$$A = \begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{pmatrix}.$$

We also have that

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = |A|^2 = |A||A| = |AA| = |A^2| = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}.$$

The following problem is optional.

Problem 0.12.
On a matrix A one may perform one of the following *elementary row operations:*

(1) Interchange two rows.
(2) Multiply a row by a nonzero constant.
(3) Add a multiple of one row to another row.

If A and B are matrices of the same size such that we can obtain B from A by a succession of elementary row operations, we say that A and B are row equivalent. This is often denoted $A \sim B$.
Show that A and B are row equivalent.
(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
(b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$
(c) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 5 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution.
(a) We take -1 times the first row and add it to the second row, to get B.
(b) We first take -2 times the first row and add it to the second row. We obtain the following matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We then multiply the third row by 2, in order to get B.
(c) We first take -1 times the first row and add it to the secound row. We obtain the following matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 5 & 7 \end{pmatrix}$$

We then take -2 times the first row and add to the third row. This gives:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

We take -1 times the second row and add to the third row, and obtain:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We take -3 times the third row and add to the first, and -2 times the second and add to the first and get B.
Note that $B = I$. A theorem says that a square matrix is invertible if and only if it is row equivalent to the identity matrix.

— Solutions to Exercise Problems —

1. The inverse matrix and linear dependence

Problem 1.1.

Solution.
To determine which matrices are invertible, we calculate the determinants.

$$\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 1 \cdot 3 - 1 \cdot 3 = 0.$$

From this we see that $\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$ is not invertible. We further have:

$$\begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix} = 1 \cdot 3 - (-1) \cdot 3 = 6 \neq 0,$$

so $\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}$ is invertible. To find the inverse we use the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

From this we get that

$$\begin{aligned} \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}^{-1} &= \frac{1}{1 \cdot 3 - (-1) \cdot 3} \begin{pmatrix} 3 & -3 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix} \end{aligned}$$

For the last matrix we find that

$$\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

so it is invertible and

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

Problem 1.2.

Solution.

We calculate $|A|$ using cofactor expansion along the first column:

$$\begin{aligned}|A| &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \\ &= (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 5 & 6 \\ 0 & 8 \end{vmatrix} + (-1)^{2+1} \cdot 0 \cdot \begin{vmatrix} 2 & 3 \\ 0 & 8 \end{vmatrix} + (-1)^{3+1} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \\ &= (5 \cdot 8 - 0 \cdot 6) + 0 + (2 \cdot 6 - 5 \cdot 3) \\ &= 40 + 12 - 15 = 37\end{aligned}$$

We calculate $|A|$ using cofactor expansion along the third row.

$$\begin{aligned}|A| &= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \\ &= (-1)^{3+1} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + (-1)^{3+2} \cdot 0 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 6 \end{vmatrix} + (-1)^{3+3} \cdot 8 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} \\ &= (2 \cdot 6 - 5 \cdot 3) + 0 + 8 \cdot (1 \cdot 5 - 0 \cdot 2) \\ &= 12 - 15 + 8 \cdot 5 = 37\end{aligned}$$

Problem 1.3.

Solution.

In order to find the cofactor matrix, we must find all the cofactors of A. We get

$$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 5 & 6 \\ 0 & 8 \end{vmatrix} = 40,\ A_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & 6 \\ 1 & 8 \end{vmatrix} = 6,\ A_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 0 & 5 \\ 1 & 0 \end{vmatrix} = -5$$

$$A_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 2 & 3 \\ 0 & 8 \end{vmatrix} = -16,\ A_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 3 \\ 1 & 8 \end{vmatrix} = 5,\ A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3,\ A_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 3 \\ 0 & 6 \end{vmatrix} = -6,\ A_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = 5.$$

From this we get that

$$\mathrm{cof}(A) = \begin{pmatrix} 40 & 6 & -5 \\ -16 & 5 & 2 \\ -3 & -6 & 5 \end{pmatrix}^T.$$

The adjoint matrix is the transpose of the cofactor matrix:

$$\mathrm{adj}(A) = \mathrm{cof}(A)^T = \begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -6 \\ -5 & 2 & 5 \end{pmatrix}.$$

The determinant $|A|$ of A is 37 from the problem above. The inverse matrix is then

$$\begin{aligned} A^{-1} &= \frac{1}{|A|}\,\mathrm{adj}(A) = \frac{1}{37}\begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -6 \\ -5 & 2 & 5 \end{pmatrix} \\ &= \begin{pmatrix} \frac{40}{37} & -\frac{16}{37} & -\frac{3}{37} \\ \frac{6}{37} & \frac{5}{37} & -\frac{6}{37} \\ -\frac{5}{37} & \frac{2}{37} & \frac{5}{37} \end{pmatrix} \end{aligned}$$

We verify that

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}\begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -6 \\ -5 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 37 & 0 & 0 \\ 0 & 37 & 0 \\ 0 & 0 & 37 \end{pmatrix}$$

and that

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}\begin{pmatrix} \frac{40}{37} & -\frac{16}{37} & -\frac{3}{37} \\ \frac{6}{37} & \frac{5}{37} & -\frac{6}{37} \\ -\frac{5}{37} & \frac{2}{37} & \frac{5}{37} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, we find that

$$\mathrm{cof}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix} \text{ and } \mathrm{adj}(B) = \begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We also calculate that $|B| = 1$, so

$$B^{-1} = \begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 1.4.

Solution.
We note that

$$\begin{pmatrix} 5x_1 + x_2 \\ 2x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This means that

$$\begin{aligned} 5x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 4 \end{aligned}$$

is equivalent to

$$\begin{pmatrix} 5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

We thus have

$$A = \begin{pmatrix} 5 & 1 \\ 2 & -1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Since $|A| = 5(-1) - 2 \cdot 1 = -7 \neq 0$, A is invertible. By the formula for the inverse of an 2×2-matrix, we get

$$A^{-1} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{2}{7} & -\frac{5}{7} \end{pmatrix}.$$

If we multiply the matrix equation $A\mathbf{x} = \mathbf{b}$ on the left by A^{-1}, we obtain

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}.$$

Now, the important point is that $A^{-1}A = I_2$ and $I_2\mathbf{x} = \mathbf{x}$. Thus we get that $\mathbf{x} = A^{-1}\mathbf{b}$. From this we find the solution:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{2}{7} & -\frac{5}{7} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

In other words $x_1 = 1$ and $x_2 = -2$.

Problem 1.5.

Solution.
We write the product as

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = AC + BD$$

We find that

$$AC = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}$$

and

$$BD = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

We thus get

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 0 & -1 \end{array}\right) \left(\begin{array}{cc} 2 & -1 \\ 0 & 1 \\ \hline 1 & 1 \end{array}\right) = \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -3 & 0 \end{pmatrix}.$$

Problem 1.6.

Solution.
We must find numbers c_1 and c_2 so that

$$\begin{pmatrix} 8 \\ 9 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

We have that

$$c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2c_1 - c_2 \\ 5c_1 + 3c_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

In other words we must solve

$$\begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

Multiplying with the inverse from the left, we get that

$$\begin{aligned} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 8 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{11} & \frac{1}{11} \\ -\frac{5}{11} & \frac{2}{11} \end{pmatrix} \begin{pmatrix} 8 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \end{aligned}$$

so $c_1 = 3$ and $c_2 = -2$.

Problem 1.7.

Solution.
(a) To see if $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -6 \end{pmatrix}$ are linearly independent, we must look at the possible values for c_1 and c_2 in the equation

$$c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By inspection we see that if $c_1 = 3$ and $c_2 = 1$, the equation is satisfied (and there are other solutions as well.) From this we conclude that the vectors are linearly dependent or equivalently they are *not* linearly independent.
(b) We look at the solutions of

$$c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This can also be written as

$$\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

as we did in the previous problem. Since

$$\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = 11 \neq 0$$

the matrix is invertible, and multiplying with the inverse from the left we obtain

$$\begin{aligned} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{4}{11} & -\frac{3}{11} \\ \frac{1}{11} & \frac{2}{11} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

We see that the only possibility is $c_1 = 0$ and $c_2 = 0$. By definition this means that $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ are linearly independent.
(c) Since

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

the vectors are linearly dependent.

Problem 1.8.

Solution.
We must prove that the equation

$$c_1(1,1,1)+c_2(2,1,0)+c_3(3,1,4)+c_4(1,2,-2)=(0,0,0)$$

has nontrivial solutions. The left side is equal to $(c_1+2c_2+3c_3+c_4, c_1+c_2+c_3+2c_4, c_1+4c_3-2c_4)$. We get the three equations

$$\begin{array}{cccccc} c_1 & +2c_2 & +3c_3 & +c_4 & =0 \\ c_1 & +c_2 & +c_3 & +2c_4 & =0 \\ c_1 & & +4c_3 & -2c_4 & =0 \end{array} \tag{10}$$

To solve this system of equation, we replace it with a simpler system that has the same solutions. If we take a multiple of one equation and add it to another equation, the solutions of the system will remain the same. We may take -1 times the first equation and add it to the second equation and also to the last equation

$$\begin{array}{cccccc} c_1 & +2c_2 & +3c_3 & +c_4 & =0 \\ & -c_2 & -2c_3 & +c_4 & =0 \\ & -2c_2 & +c_3 & -3c_4 & =0 \end{array}$$

Now we take -2 times the second equation and add it to the third equation. From this we obtain:

$$\begin{array}{cccccc} c_1 & +2c_2 & +3c_3 & +c_4 & =0 \\ & -c_2 & -2c_3 & +c_4 & =0 \\ & & 5c_3 & -5c_4 & =0 \end{array}$$

We may rewrite the equations to get

$$\begin{aligned} c_1 &= -2c_2-3c_3-c_4 \\ c_2 &= -2c_3+c_4 \\ c_3 &= c_4 \end{aligned}$$

We can substitute the expression for c_3 from the last equation into the two first equations

$$\begin{aligned} c_1 &= -2c_2-3c_4-c_4 = -2c_2-4c_4 \\ c_2 &= -2c_4+c_4 = -c_4 \\ c_3 &= c_4 \end{aligned}$$

Finally we substitute the expression for c_2 given by the second equation into the first equation, and we get

$$\begin{aligned} c_1 &= -2c_4 \\ c_2 &= -c_4 \\ c_3 &= c_4 \end{aligned}$$

We see that for any choice of c_4, the system (10) has a solution. In particular not all c_i's have to be zero. This shows that the vectors are linearly dependent.

Problem 1.9.

Solution.
We must prove that
$$c_1(\mathbf{a}+\mathbf{b})+c_2(\mathbf{b}+\mathbf{c})+c_3(\mathbf{a}+\mathbf{c})=\mathbf{0}$$
has only the trivial solution $c_1 = 0$, $c_2 = 0$, $c_3 = 0$. Rewriting the equation, we have $(c_1 + c_3)\mathbf{a} + (c_1 + c_2)\mathbf{b} + (c_2 + c_3)\mathbf{c} = \mathbf{0}$. Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent, we conclude that $c_1 + c_3 = 0$, $c_1 + c_2 = 0$ and $c_2 + c_3 = 0$. From this $c_1 = .-c_3$, $c_1 = -c_2$ and $c_2 = -c_3$. Thus we have that $c_1 = c_2$ and that $c_1 = -c_2$, and $c_2 = -c_2$. This last equation, gives $c_2 + c_2 = 0$ or $2c_2 = 0$. we conclude that $c_2 = 0$. Substituting in the other equations, $c_1 = 0$, and $c_3 = 0$. This shows that $\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}$ and $\mathbf{a}+\mathbf{c}$ are linearly independent.
Similarly, we have that
$$c_1(\mathbf{a}-\mathbf{b})+c_2(\mathbf{b}+\mathbf{c})+c_3(\mathbf{a}+\mathbf{c})=\mathbf{0}$$
can be written as
$$(c_1+c_3)\mathbf{a}+(-c_1+c_2)\mathbf{b}+(c_2+c_3)\mathbf{c}=\mathbf{0}.$$
We must have $c_1 + c_3 = 0$, $-c_1 + c_2 = 0$ and $c_2 + c_3 = 0$. We get $c_1 = c_2$ and $c_3 = -c_1$. Thus choosing $c_1 = 1$, we get $c_2 = 1$, $c_3 = -1$ and this is a non-trivial solution. Thus we conclude that $\mathbf{a}-\mathbf{b}, \mathbf{b}+\mathbf{c}$ and $\mathbf{a}+\mathbf{c}$ are linearly dependent.

Problem 1.10.

Solution.
Assume that $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then the equation
$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_n\mathbf{v}_n=\mathbf{0}$$
has a solution where not all c_i's are 0. For instance if $c_2 \neq 0$, we can divide by this and we get
$$\frac{c_1}{c_2}\mathbf{v}_1+\mathbf{v}_2+\cdots+\frac{c_n}{c_2}\mathbf{v}_n=\mathbf{0}.$$
From this we get that $\mathbf{v}_2 = -\frac{c_1}{c_2}\mathbf{v}_1 - \frac{c_3}{c_2}\mathbf{v}_3 + \cdots + \frac{c_n}{c_2}\mathbf{v}_n$. Thus $\mathbf{v}_2$ can be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_3, \ldots, \mathbf{v}_n$. More generally, if $c_i \neq 0$, we may write $\mathbf{v}_i$ as a linear combination of the remaining vectors.
Assume that one of the vectors can be written as a linear combination of the others. For instance, if $\mathbf{v}_1 = a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$, then we get that
$$(-1)\mathbf{v}_1+a_2\mathbf{v}_2+\cdots+a_n\mathbf{v}_n=\mathbf{0}.$$
From this it follows that $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent.

Problem 1.11.

Solution.

(a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

— Solutions to Exercise Problems —
2. The rank of a matrix and applications

Problem 2.1.

Solution.

(1) As a vector equation, we have

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

where

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} -1 \\ 12 \\ -5 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

As matrix equation, we have

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 3 & 0 & -1 \\ -5 & 2 & 12 \\ 0 & 6 & -5 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(2) As a vector equation, we have

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = \mathbf{b}$$

where

$$\mathbf{a}_1 = \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 \\ -10 \\ 0 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} -6 \\ 7 \\ -5 \end{pmatrix}, \ \mathbf{a}_4 = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

As matrix equation, we have

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 3 & 0 & -6 & 3 \\ -2 & -10 & 7 & 0 \\ 2 & 0 & -5 & 5 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 2.2.
(FMEA 1.2.3 in 2. ed. and 1.3.3 in 1. ed.)

Solution.
See answers on page 555 in 1. ed. and 559 in 2. ed. of FMEA.

Problem 2.3.

Solution.
Removing a column gives a 3-minor. Thus there are 4 3-minors. To get a 2-minor, we must remove a row and two columns. There are $3 \cdot 4 \cdot 3/2 = 18$ ways to do this, so there are 18 2-minors. List some of them.

Problem 2.4.
(FMEA 1.3.1 in 2. ed. and 1.4.1 in 1. ed.)

Solution.
See answers on page 555 in 1. ed. and 559 in 2. ed. of FMEA.

Problem 2.5.
(FMEA 1.3.2ab in 2. ed. and 1.4.2 ab in 1. ed.)

Solution.
(a) Let

$$A = \begin{pmatrix} x & 0 & x^2-2 \\ 0 & 1 & 1 \\ -1 & x & x-1 \end{pmatrix}$$

We have

$$\begin{vmatrix} x & 0 & x^2-2 \\ 0 & 1 & 1 \\ -1 & x & x-1 \end{vmatrix} = x^2 - x - 2.$$

We have that $x^2 - x - 2 = 0$ if and only if $x = -1$ or $x = 2$, so if $x \neq -1$ and $x \neq 2$, then $r(A) = 3$. If $x = -1$, then

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & -1 & -2 \end{pmatrix}.$$

Since for instance $\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$, it follows that $r(A) = 2$. If $x = 2$, then

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix}.$$

Since for instance $\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 \neq 0$, we see that $r(A) = 2$.

(b) The rank is 3 if $t \neq -4$, $t \neq -2$, and $t \neq 2$. The rank is 2 if $t = -4$,$t = -2$, or $t = 2$.

Problem 2.6.
(FMEA 1.3.3 in 2. ed. and 1.4.3 in 1. ed.)

Solution.
See answers on page 555 in 1. ed. and 559 in 2. ed. of FMEA.

Problem 2.7.
Using the definition of rank of a matrix, explain way m vectors in $\mathbb{R}^n$ must be linearly dependent if $m > n$.

Solution.
Put the m vectors as columns in a matrix. This yield a $n \times m$-matrix A. Since the rank of A is equal to the order of the largest non-vanishing minor, we see that $r(A) \leq n$. By definition the rank of A is the largest number of linearly independent columns in A. We thus have at most n linearly independent columns in A. If $m > n$ the m vectors can not be linearly independent, hence they must be dependent.

Problem 2.8.
Show that

$$\begin{pmatrix} 3 \\ 4 \\ -1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

are linearly independent by computing a 2×2-minor.

Solution.

Since $\begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} = 3 \neq 0$, the rank of the matrix

$$\begin{pmatrix} 3 & 0 \\ 4 & 1 \\ -1 & 0 \\ 2 & 1 \end{pmatrix}$$

is 2. By definition of rank, the two vectors must be linearly independent.

Problem 2.9.
(FMEA 1.4.1 in 2. ed. and 1.5.1 in 1.ed.) In (c) and (d) you do not need to find the solutions.

Solution.
See answers on page 556 in 1.ed. and on 559 in 2. ed. of FMEA.

Problem 2.10.
(FMEA 1.4.3 in 2. ed. and 1.5.3 in 1.ed.)

Solution.
We have

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & a & -21 \\ 3 & 7 & a \end{pmatrix} \text{ and } A_{\mathbf{b}} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & a & -21 & 2 \\ 3 & 7 & a & b \end{pmatrix}$$

When $|A| = 3$, then we have $r(A) = r(A_{\mathbf{b}}) = 3$ and we have one unique solution. For instance by cofactor expansion one finds that

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ -1 & a & -21 \\ 3 & 7 & a \end{vmatrix} = a^2 - 7a = a(a-7).$$

Thus $r(A) = 3$ when $a \neq 0$ and $a \neq 7$. When $a = 0$, we have

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & -21 \\ 3 & 7 & 0 \end{pmatrix} \text{ and } A_{\mathbf{b}} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -21 & 2 \\ 3 & 7 & 0 & b \end{pmatrix}.$$

We see that $r(A) = 2$. To have solutions we must have $r(A_{\mathbf{b}}) = 2$, and in particular

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 3 & 7 & b \end{vmatrix} = 2b - 9 = 0.$$

On the other hand if $b = \frac{9}{2}$, we also have that

$$\begin{vmatrix} 1 & 3 & 1 \\ -1 & -21 & 2 \\ 3 & 0 & \frac{9}{2} \end{vmatrix} = \begin{vmatrix} 2 & 3 & 1 \\ 0 & -21 & 2 \\ 7 & 0 & \frac{9}{2} \end{vmatrix} = 0,$$

and this implies that $r(A_{\mathbf{b}}) = 2$.
Treating the case $a = 7$ similarly, one has in conclusion that

(1) There is a unique solution if $a \neq 0$ and $a \neq 7$.
(2) There are no solutions if $a = 0$ and $b \neq \frac{9}{2}$.
(3) There are infinitely many solutions if $a = 0$ and $b = \frac{9}{2}$.
(4) There are no solutions if $a = 7$ and $b \neq \frac{10}{3}$.
(5) There are infinitely many solutions if $a = 7$ and $b = \frac{10}{3}$.

Problem 2.11.
(FMEA 1.4.4 in 2. ed. and 1.5.4 in 1.ed.)

Solution.
$A(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) = \lambda A\mathbf{x}_1 + (1-\lambda)A\mathbf{x}_2 = \lambda \mathbf{b} + (1-\lambda)\mathbf{b} = \mathbf{b}$. This shows that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are different solutions, then so are all points on the straight line through $\mathbf{x}_1$ and $\mathbf{x}_2$.

Problem 2.12.
(FMEA 1.5.5a in 1.ed.)
For what values of the constants p and q does the following system have a unique solution, several solutions, or no solutions?

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 2q \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 4q \\ 3x_1 & - & 2x_2 & + & px_3 & = & q \end{array}$$

Solution.
Unique solution for $p \neq 3$. For $p = 3$ and $q = 0$, there are infinitely many solutions (1 degree of freedom). For $p = 0$ and $q \neq 3$, there are no solutions.

Problem 2.13.
(FMEA 1.4.6ab in 2. ed. and 1.5.7ab)

Solution.
See answers on page 556 in 1. ed. and 560 in 2. ed of FMEA.

The following problems are optional.

Solution.
(a) $\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}^T \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2$
(b) $\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 2 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} = 5$
(c) $\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0 \implies$ orthogonal.
(d) $\mathbf{a}_1\cdot\mathbf{a}_2 = 0$, $\mathbf{a}_1\cdot\mathbf{a}_3 = 0$ and $\mathbf{a}_2\cdot\mathbf{a}_3 = 0$. In general we have ($\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ why?)
(e) $c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = 0 \implies \mathbf{a}_1\cdot(c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n) = c_1\mathbf{a}_1\cdot\mathbf{a}_1 = 0$. Now $\mathbf{a}_1\cdot\mathbf{a}_1 \neq 0$ if $\mathbf{a}_1 \neq 0$. Thus $c_1 = 0$. In the same way, we see that $c_2 = 0, \ldots c_n = 0$ and this shows that they are linearly independent.

Solution.
By Pythagoras the norm is equal to the length.

Solution.
(a) $A^TA = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 30 & 60 \\ 60 & 120 \end{pmatrix}$. $\begin{vmatrix} 30 & 60 \\ 60 & 120 \end{vmatrix} = 0 \implies$ The rank of A^TA is 1. Thus $r(A) = 1$.
(b) $A^TA = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 6 \end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 18 \\ 18 & 41 \end{pmatrix}$. $\begin{vmatrix} 14 & 18 \\ 18 & 41 \end{vmatrix} = 250 \neq 0$. Thus $r(A) = r(A^TA) = 2$.
(c) $A^TA\mathbf{x} = 0 \implies \mathbf{x}^TA^TA\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = |A\mathbf{x}|^2 = 0 \implies A\mathbf{x} = 0$. This show that

$$A^TA\mathbf{x} = 0 \iff A\mathbf{x} = 0$$

Thus the null space of A^TA is equal to the null space of A.
(d) If $r(A) < m$ the columns which we call $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly dependent, i.e.

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = 0$$

where not all the c_i are zero. From this it is clear that

$$c_1\mathbf{a}_i^T\mathbf{a}_1 + \cdots + c_n\mathbf{a}_i^T\mathbf{a}_n = 0$$

for all i. But the columns in A^TA are

$$\begin{pmatrix} \mathbf{a}_1^T\mathbf{a}_j \\ \mathbf{a}_2^T\mathbf{a}_j \\ \vdots \\ \mathbf{a}_m^T\mathbf{a}_j \end{pmatrix}.$$

and we have

$$c_1\begin{pmatrix} \mathbf{a}_1^T\mathbf{a}_1 \\ \mathbf{a}_2^T\mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m^T\mathbf{a}_1 \end{pmatrix} + \cdots + c_m\begin{pmatrix} \mathbf{a}_1^T\mathbf{a}_m \\ \mathbf{a}_2^T\mathbf{a}_m \\ \vdots \\ \mathbf{a}_m^T\mathbf{a}_n \end{pmatrix} = 0$$

Thus the columns of A^TA are linearly dependent, and we have $|A^TA| = 0$.

Solution.

(a) $\begin{pmatrix} 1 & 2 \\ 8 & 16 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \Longrightarrow r = 1$

(b) $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 \\ 0 & -6 & -7 \end{pmatrix} \Longrightarrow r = 2$

(c) $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow r = 2$

(d) $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{pmatrix} \Longrightarrow r = 3$

(e) $\begin{pmatrix} 2 & 1 & 3 & 7 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 11 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 3 & 7 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow r = 2$

(f) $\begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ -2 & -5 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & -\frac{7}{5} & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow r = 3$

— Solutions to Exercise Problems —
3. Eigenvalues and diagonalization

Problem 3.1.

Solution.

(1) The characteristic equation of $\begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}$ is
$$\begin{vmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 5 = 0.$$
The solutions are $\lambda = -5$ and $\lambda = -1$.

(2) The characteristic equation is
$$\begin{vmatrix} 2-\lambda & 4 \\ -2 & 6-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 20 = 0.$$
This equation has no solutions, so there are no eigenvalues.

(3) The characteristic equation is
$$\begin{vmatrix} 1-\lambda & 4 \\ 6 & -1-\lambda \end{vmatrix} = \lambda^2 - 25 = 0.$$
The solutions are $\lambda = -5$ and $\lambda = 5$.

(4) The characteristic equation is
$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)(4-\lambda) = 0$$
The solutions are $\lambda = 2$, $\lambda = 3$ and $\lambda = 4$.

(5) The characteristic equation is
$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix} = 0.$$
We know that a determinant is unchanged if we take a multiple of one row an add it to an other row. If we take the second row and add it to the first, we get
$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2-\lambda & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix}$$
Since the determinant of a matrix is equal to the determinant of the transpose matrix, we may play the same game with columns. Thus if we subtract the first column from the second column, we get
$$\begin{aligned} \begin{vmatrix} 2-\lambda & 2-\lambda & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix} &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & -2 & -2-\lambda \end{vmatrix} \\ &= (2-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} \\ &= (2-\lambda)(\lambda+\lambda^2) \\ &= (2-\lambda)\lambda(1+\lambda). \end{aligned}$$
The solutions of the characteristic equations is thus $\lambda = -1$, $\lambda = 0$ and $\lambda = 2$.

Problem 3.2.

Solution.
(1) We first consider $\lambda = -1$. To find the eigenvalues corresponding to $\lambda = -1$, we have to solve the matrix equation

$$\begin{pmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Substituting $\lambda = -1$, this is

$$\begin{pmatrix} 3 & -7 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Written as a system of linear equations, this is

$$\begin{aligned} 3x_1 - 7x_2 &= 0 \\ 3x_1 - 7x_2 &= 0. \end{aligned}$$

Since $\begin{vmatrix} 3 & -7 \\ 3 & -7 \end{vmatrix} = 0$, the rank of $\begin{pmatrix} 3 & -7 \\ 3 & -7 \end{pmatrix}$ is 1 and we know that there is one superfluous equation and $2 - 1 = 1$ degree of freedom. This is seen from the linear system of equations as well since in consist of two equal equations. The system is thus equivalent to the system

$$3x_1 - 7x_2 = 0$$

consisting of one equation in two unknowns. Choosing x_1 to be the base variable an x_2 to be free, we find that

$$\begin{aligned} x_1 &= \frac{7}{3}x_2 \\ x_2 &= \text{free} \end{aligned}$$

We may assign a parameter for the free variable and write this as

$$\begin{aligned} x_1 &= \frac{7}{3}t \\ x_2 &= t \end{aligned}$$

where t can be chosen freely. This is conveniently written using vector notation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{7}{3}t \\ t \end{pmatrix} = t \begin{pmatrix} \frac{7}{3} \\ 1 \end{pmatrix}.$$

For each value of $t \neq 0$ we get an eigenvector

$$t \begin{pmatrix} \frac{7}{3} \\ 1 \end{pmatrix}$$

corresponding to $\lambda = -1$. Since all these will be linearly dependent (they are a multiple of each other) it is common to chose one of them as representing the eigenvectors corresponding to $\lambda = -1$. For instance for $t = 3$ we get

$$\begin{pmatrix} 7 \\ 3 \end{pmatrix}.$$

We now consider the eigenvalue $\lambda = -5$. Substituting $\lambda = -5$ into

$$\begin{pmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

we get

$$\begin{pmatrix} 7 & -7 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solution.
(continued) Written as a system of linear equations, this is

$$7x_1 - 7x_2 = 0$$
$$3x_1 - 3x_2 = 0.$$

There is clearly one superfluous equations and the system is equivalent to

$$x_1 - x_2 = 0.$$

We may write the solutions as

$$x_1 = x_2$$
$$x_2 = \text{free}$$

or choosing a parameter for the free variable

$$x_1 = t$$
$$x_2 = t$$

Using vector notation this may be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We choose

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as the representative for the eigenvectors corresponding to $\lambda = -5$.
(2) Since there are no eigenvalues there are no eigenvectors.
(3) We consider

$$A = \begin{pmatrix} 1 & 4 \\ 6 & -1 \end{pmatrix}.$$

We have found the eigenvalues $\lambda = -5$ and $\lambda = 5$. To find the eigenvectors we must solve

$$(A - \lambda I)\mathbf{x} = 0$$

for each eigenvalue λ. For $\lambda = -5$

$$A - \lambda I = \begin{pmatrix} 1-(-5) & 4 \\ 6 & -1-(-5) \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 6 & 4 \end{pmatrix}.$$

We must solve

$$\begin{pmatrix} 6 & 4 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is equivalent to

$$6x_1 + 4x_2 = 0$$

We get

$$x_1 = -\frac{2}{3}x_2$$
$$x_2 = \text{free}$$

Choosing a parameter and using vector notation we get

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}.$$

We may choose $t = 3$ and

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

as representative for the eigenvectors corresponding to $\lambda = -5$. Similarly we arrive at

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as a representative for the eigenvectors corresponding to $\lambda = 5$.

Solution.
(continued) (4) We consider

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

To find the eigenvectors we we must solve

$$(A - \lambda I)\mathbf{x} = 0$$

for each eigenvalue λ. For $\lambda = 2$, we get

$$A - \lambda I = \begin{pmatrix} 2-2 & 0 & 0 \\ 0 & 3-2 & 0 \\ 0 & 0 & 4-2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

We must thus solve

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is equivalent to

$$\begin{aligned} x_2 &= 0 \\ 2x_3 &= 0 \end{aligned}$$

Since there are two equations there is one degree of freedom, and we see that x_1 is the free variable. Thus

$$\begin{aligned} x_1 &= \text{free} \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

Choosing a parameter and using vector notation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and we choose

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

as a representative for the eigenvectors corresponding to $\lambda = 2$. Similarly we get

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

for $\lambda = 3$ and

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

for $\lambda = 4$.

Solution.
(continued) (5) We have

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}.$$

To find the eigenvectors we solve $(A - \lambda I)\mathbf{x} = 0$. For $\lambda = -1$ we must solve

$$\begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The rank is 2 so there is one superfluous equation. The two first rows are linearly independent, so the matrix equation is equivalent to

$$\begin{array}{rrr} 3x_1 & +x_2 & -x_3 = 0 \\ & 2x_2 & -x_3 = 0 \end{array}$$

or

$$\begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In order to simplify we multiply from the left by

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ 0 & \frac{1}{2} \end{pmatrix}$$

(using the formula from Lecture 2 for the inverse of a two by two matrix) and get

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix}.$$

Thus our system is equivalent to

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or

$$\begin{array}{rrr} x_1 & & -\frac{1}{2}x_3 = 0 \\ & x_2 & +\frac{1}{2}x_3 = 0 \end{array}$$

Rewriting

$$\begin{aligned} x_1 &= \frac{1}{2}x_3 \\ x_2 &= -\frac{1}{2}x_3 \\ x_3 &= \text{free} \end{aligned}$$

Choosing a parameter and using vector notation, we get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}.$$

For $t = 2$ we get

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

which we choose as a representative for the eigenvectors with $\lambda = -1$. For $\lambda = 0$ one gets

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

and for $\lambda = 2$ one gets

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 3.3.

Solution.
See answers in FMEA.

Problem 3.4.

Solution.
We calculate that

$$\begin{pmatrix} 1 & 18 & 30 \\ -2 & -11 & -10 \\ 2 & 6 & 5 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 15 \\ -5 \\ 0 \end{pmatrix}.$$

We then observe that

$$\begin{pmatrix} 15 \\ -5 \\ 0 \end{pmatrix} = -5 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

Thus

$$\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

is an eigenvector with eigenvalue $\lambda = -5$. We further calculate

$$\begin{pmatrix} 1 & 18 & 30 \\ -2 & -11 & -10 \\ 2 & 6 & 5 \end{pmatrix} \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 25 \\ 0 \\ -5 \end{pmatrix}$$

and observe that

$$\begin{pmatrix} 25 \\ 0 \\ -5 \end{pmatrix} = -5 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$$

so

$$\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to the eigenvalue $\lambda = -5$. Finally

$$\begin{pmatrix} 1 & 18 & 30 \\ -2 & -11 & -10 \\ 2 & 6 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ -5 \\ 5 \end{pmatrix}$$

and

$$\begin{pmatrix} 15 \\ -5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

so

$$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

is an eigenvector with eigenvalue $\lambda = 5$.

Problem 3.5.

Solution.
For
$$A = \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}$$
we have found eigenvectors
$$\begin{pmatrix} 7 \\ 3 \end{pmatrix} \text{ corresponding to } \lambda = -1$$
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponding to } \lambda = -5$$
We form the matrix P with these vectors as columns
$$P = \begin{pmatrix} 7 & 1 \\ 3 & 1 \end{pmatrix}$$
and get
$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 7 & 1 \\ 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ 3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix} = D. \end{aligned}$$

Problem 3.6.

Solution.
The characteristic equation is
$$\begin{vmatrix} 3-\lambda & 5 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0$$
and we see that $\lambda = 3$ is the only eigenvalue. To find the corresponding eigenvectors, we solve
$$\begin{pmatrix} 3-3 & 5 \\ 0 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
or equivalently
$$\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
This gives
$$5x_2 = 0$$
so x_1 has to be free:
$$\begin{aligned} x_1 &= \text{free} \\ x_2 &= 0. \end{aligned}$$
Choosing a parameter and using vector notation we get
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$
All these vectors will be linearly dependent. We thus see that it is not possible to find two linearly independent eigenvectors and this shows that A is not diagonalizable.

Problem 3.7.

Solution.
(1)
$$\begin{pmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{pmatrix}\begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.53 \\ 0.47 \end{pmatrix}$$
and $0.53 + 0.47 = 1$.
(2) The characteristic equation is
$$\begin{vmatrix} 0.85-\lambda & 0.45 \\ 0.15 & 0.55-\lambda \end{vmatrix} = \lambda^2 - 1.4\lambda + 0.4 = 0$$
which has the solutions $\lambda = 0.4$ and $\lambda = 1.0$.
(3) From
$$\begin{pmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{pmatrix}\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$$
we see that $\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$. From
$$\begin{pmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.4 \\ -0.4 \end{pmatrix}$$
we see that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 0.4$.
(4) We form the matrix P using the eigenvectors as columns
$$P = \begin{pmatrix} 0.75 & 1 \\ 0.25 & -1 \end{pmatrix}$$
and we get that
$$\begin{aligned} P^{-1}TP &= \begin{pmatrix} 0.75 & 1 \\ 0.25 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{pmatrix}\begin{pmatrix} 0.75 & 1 \\ 0.25 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1.0 & 1.0 \\ 0.25 & -0.75 \end{pmatrix}\begin{pmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{pmatrix}\begin{pmatrix} 0.75 & 1 \\ 0.25 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1.0 & 0.0 \\ 0 & 0.4 \end{pmatrix} = D \end{aligned}$$
(5) $D^n = (P^{-1}TP)^n = P^{-1}TPP^{-1}TP \cdots P^{-1}TP = P^{-1}T^nP$. Multiplying with P from the left, we get $PD^n = T^nP$. Then multiplying with P^{-1} from the left, we get $PD^nP^{-1} = T^n$.
$$\begin{aligned} \lim_{n\to\infty} D^n &= \lim_{n\to\infty}\begin{pmatrix} 1.0 & 0.0 \\ 0 & 0.4 \end{pmatrix}^n \\ &= \lim_{n\to\infty}\begin{pmatrix} (1.0)^n & 0.0 \\ 0 & (0.4)^n \end{pmatrix} \\ &= \begin{pmatrix} 1.0 & 0.0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$
We have that
$$\begin{aligned} \lim_{n\to\infty} T^n\mathbf{s} &= \begin{pmatrix} 0.75 & 1 \\ 0.25 & -1 \end{pmatrix}\begin{pmatrix} 1.0 & 0.0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1.0 & 1.0 \\ 0.25 & -0.75 \end{pmatrix}\begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}. \\ &= \begin{pmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{pmatrix}\begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} \\ &= \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix} \end{aligned}$$
(6) We see from this that we will approach the situation where A's share is 75% and B's share is 25%.

Problem 3.8.

Solution.

Question 1: When is the product AB defined? Correct answer: A.

Question 2: Can we conclude that A is diagonalizable? Correct answer: D. (B is wrong since **u** is an eigenvector with eigenvalue 0. C is wrong since A at least has the eigenvalues 0 and 2.)

Question 3: Are u and v eigenvectors for A? Correct answer: A. In fact,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

gives

$$A\mathbf{u} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

so **u** is an eigenvector corresponding to the eigenvalue 0. We have

$$A\mathbf{v} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

so **v** is an eigenvector with eigenvalue 1.

Problem 3.9.

Solution.
(a) We start with

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 12 \end{pmatrix}$$

and take (-2) times the first row an add it to the second row:

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 4 & 5 & 3 & 0 & 12 \end{pmatrix}.$$

We then take (-4) times the first row and add it to the last row:

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 1 & -5 & -4 & -8 \end{pmatrix}$$

We take (-1) times the second row and add it to the last row to obtain the answer.
(b) We start with

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and take (-1) times the second row and add it to the first row to obtain:

$$\begin{pmatrix} 1 & 0 & 7 & 5 & 13 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) From the equations we get

$$\begin{aligned} x_1 &= 13 - 7x_3 - 5x_4 \\ x_2 &= 8 + 5x_3 + 4x_4 \end{aligned}$$

so x_3 and x_4 are free. Using vector notation we get

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 13 - 7x_3 - 5x_4 \\ -8 + 5x_3 + 4x_4 \\ x_3 \\ x_4 \end{pmatrix} \\ &= \begin{pmatrix} 13 \\ -8 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -7x_3 \\ 5x_3 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_4 \\ 4x_4 \\ 0 \\ x_4 \end{pmatrix} \\ &= \begin{pmatrix} 13 \\ -8 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -7 \\ 5 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 4 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

Problem 3.10.

Solution.

The eigenvalues are 3 and 6. We need to find the eigenvectors.

$\lambda = 3:$

$$A - 3I = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 5 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

To solve the system $(A - 3I)\mathbf{x} = 0$ we reduce the augmented matrix

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From

$$x_1 + x_2 + 2x_3 = 0$$

we get $x_1 = -x_2 - 2x_3$. Using vector notation:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

$\lambda = 6:$

$$A - 6I = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 5 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 2 \\ 0 & -3 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

To solve the system $(A - 6I)\mathbf{x} = 0$ we reduce the augmented matrix:

$$\begin{pmatrix} -2 & 1 & 2 & 0 \\ 0 & -3 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 0 & 0 \\ -2 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 0 & 0 \\ -2 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From this we get

$$\begin{array}{rrrl} x_1 & & -x_3 & = 0 \\ & x_2 & & = 0 \end{array}$$

or

$$x_1 = x_3$$
$$x_2 = 0$$

Using vector notation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Thus we get 3 linearly independent eigenvectors, and

$$P = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

gives

$$P^{-1}AP = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 5 \end{pmatrix}\begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

— Solutions to Exercise Problems —
4. Quadratic forms and concave/convex functions

Problem 4.1.
(1.8.2 in the first edtion of FMEA and 1.7.3 in the second edtion of FMEA) Write the following quadratic forms as $\mathbf{x}^T A\mathbf{x}$ with A symmetric:
(a) $x^2+2xy+y^2$ (b) $ax^2+bxy+cy^2$ (c) $3x_1^2-2x_1x_2+3x_1x_3+x_2^2+3x_3^2$

Solution.
(a) $Q=x^2+2xy+y^2$. The coefficients in front of x^2 and y^2 are placed on the diagonal. The coefficient in front of xy is divided by 2 and placed in the remaining two entries. We get that

$$Q=x^2+2xy+y^2=\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(b) $Q=ax^2+bxy+cx^2$. The coefficients in front of x^2 and y^2 are placed on the diagonal. The coefficient in front of xy is divided by 2 and placed in the remaining two entries. We get that

$$Q=x^2+2xy+y^2=\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(c) $Q=3x_1^2-2x_1x_2+3x_1x_3+x_2^2+3x_3^2$. The coefficients in front of the squares goes to the diagonal. The coefficient in front of x_1x_2 is equal divided between the $(1,2)$ and the $(2,1)$ entry of the matrix, similarly for the coefficient in front of x_1x_3. We get that

$$Q=\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}\begin{pmatrix} 3 & -1 & \frac{3}{2} \\ -1 & 1 & 0 \\ \frac{3}{2} & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Problem 4.2.
Write the following quadratic forms as $\mathbf{x}^T A\mathbf{x}$ and determine their definiteness.

(a) $Q(x_1,x_2,x_3)=x_1^2+3x_2^2+5x_3^2$
(b) $Q(x_1,x_2,x_3)=x_1^2+2x_1x_2+3x_2^2+5x_3^2$

Solution.
(a) We get

$$Q = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For a diagonal matrix, the eigenvalues are read of the diagonal. We get $\lambda = 1$, $\lambda = 2$ and $\lambda = 5$.
(b) We get

$$Q = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

We find the eigenvalues from the characteristic equation

$$\begin{aligned} \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} &= (5-\lambda)(-1)^{3+3} \begin{vmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} \\ &= (5-\lambda)(\lambda^2 - 4\lambda + 2) = 0 \end{aligned}$$

We get that $\lambda = 5$ or $\lambda^2 - 4\lambda + 2 = 0$. The eigenvalues are thus $2 - \sqrt{2}$, $\sqrt{2} + 2$ and 5. In particular they are all positive, so Q is positive definite.

Problem 4.3.

Solution.
The characteristic equation is

$$\begin{aligned}\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} &= ad - bc - a\lambda - d\lambda + \lambda^2 \\ &= \lambda^2 - (a+d)\lambda + (ad-bc) \\ &= \lambda^2 - \mathrm{tr}(A)\lambda + |A| = 0.\end{aligned}$$

We now assume that the characteristic equation has the solutions λ_1 and λ_2. We may factor the characteristic polynomial $\lambda^2 - \mathrm{tr}(A)\lambda + |A| = (\lambda - \lambda_1)(\lambda - \lambda_2)$. Multiplying out we get

$$\lambda^2 - \mathrm{tr}(A)\lambda + |A| = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

From this we conclude that $\mathrm{tr}(A) = \lambda_1 + \lambda_2$ and $|A| = \lambda_1\lambda_2$.

Problem 4.4.
FMEA 2.2.1 (Do not bother about strictly convex/concave.)

Solution.
See answers in FMEA.

Problem 4.5.
Draw the line segment

$$[\mathbf{x}, \mathbf{y}] = \{\mathbf{z} : \text{ there exists } s \in [0.1] \text{ such that } \mathbf{z} = s\mathbf{x} + (1-s)\mathbf{y}\}$$

in the plane where

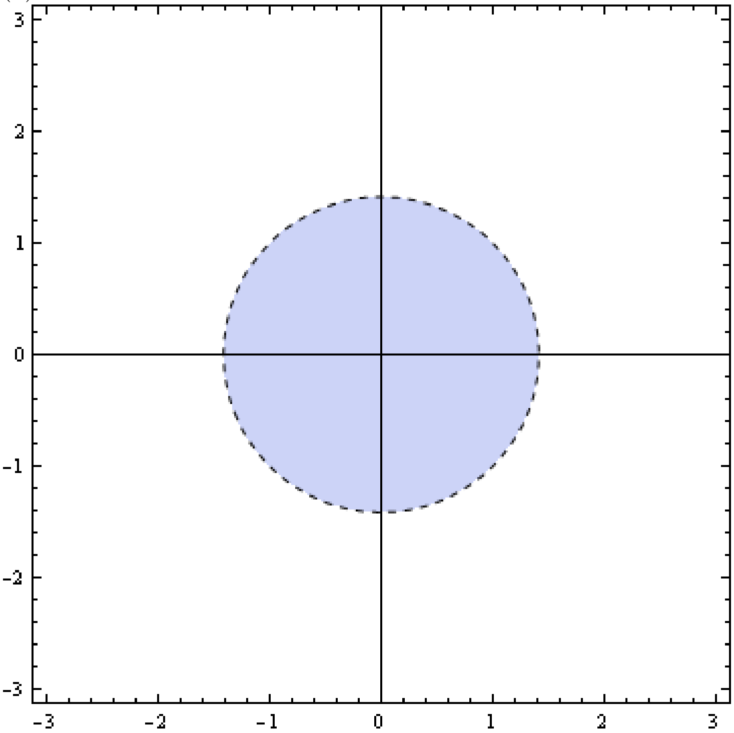
(a) $\mathbf{x} = (0,0)$ and $\mathbf{y} = (2,2)$
(b) $\mathbf{x} = (-1,1)$ and $\mathbf{y} = (3,4)$

Mark the points corresponding to $s = 0, 1$ and $\frac{1}{2}$ on each line segment.

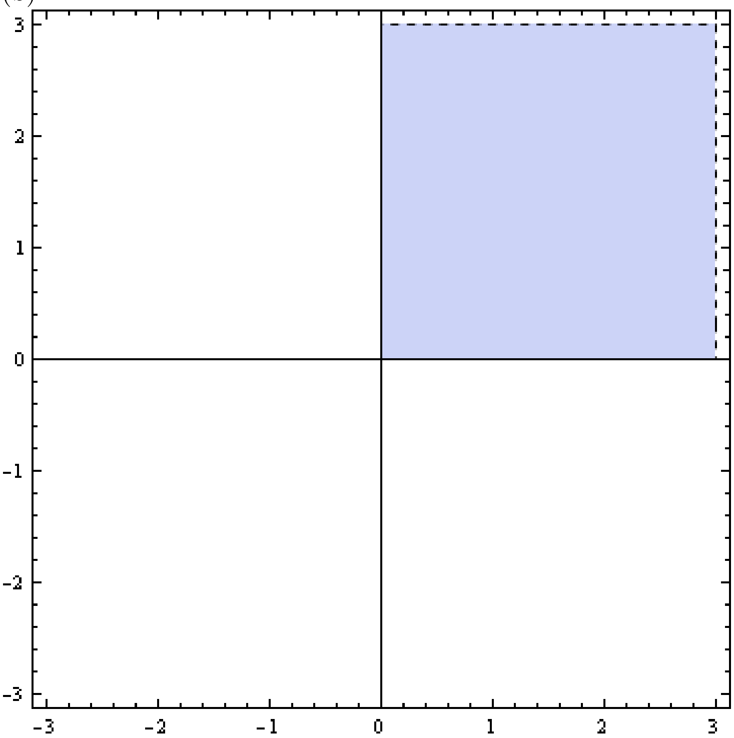
Solution.
(a) $s = 0$ gives $(2,2)$, $s = 1$ gives $(0,0)$ and $s = \frac{1}{2}$ gives $\mathbf{z} = \frac{1}{2}\mathbf{x} + (1 - \frac{1}{2})\mathbf{y} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} = \frac{1}{2}(0,0) + \frac{1}{2}(2,2) = (1,1)$.
(b) $s = 0$ gives $(3,4)$, $s = 1$ gives $(-1,1)$ and $s = \frac{1}{2}$ gives $\mathbf{z} = \frac{1}{2}\mathbf{x} + (1 - \frac{1}{2})\mathbf{y} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} = \frac{1}{2}(-1,1) + \frac{1}{2}(3,4) = (1, \frac{5}{2})$.

Problem 4.6.
FMEA 2.2.2 abcd

Solution.
(a)



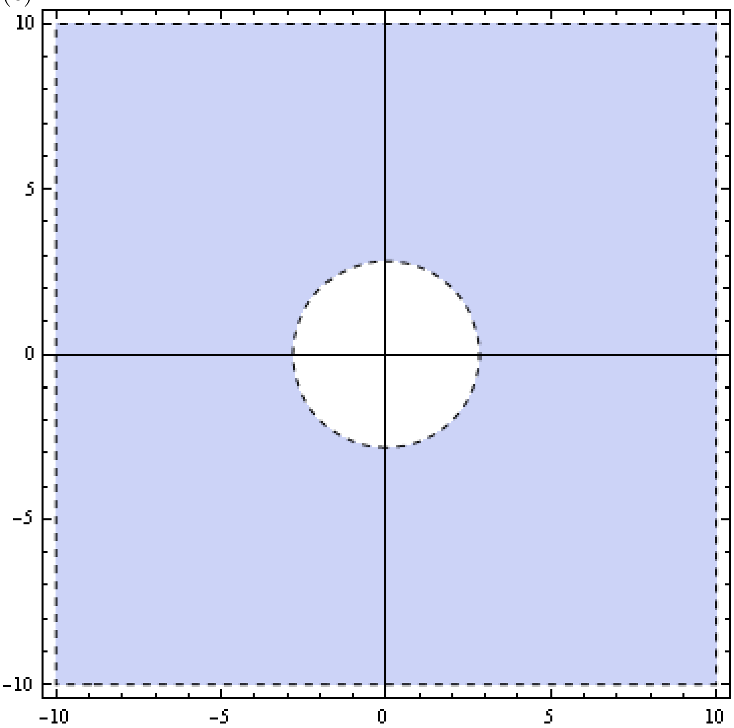
This is convex.
(b)



This is convex.

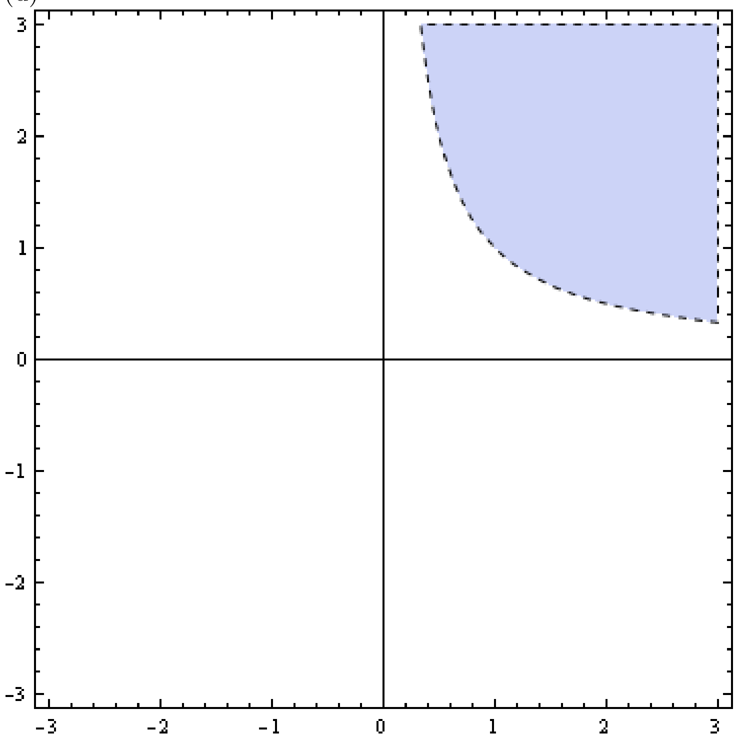
Solution.

(c)



This is not convex.

(d)



This is convex.

Problem 4.7.

FMEA 2.3.1 (Do not bother about strictly convex/concave.)

Solution.
See answers in FMEA.

Problem 4.8.
Let f be defined for all x and y by $f(x, y) = x - y - x^2$. Show that f is concave.

Solution.
It was shown in Lecture 5 that the function $g(x, y) = x^2$ was convex. This means that the function $g(x, y) = -x^2$ is concave. In the Lecture it was also shown in an example that a linear function is both concave and convex. In the same way one finds that $h(x, y) = x - y$ is both concave and convex. By a theorem in Lecture 5, we get that $f = h + g$ is concave.

Problem 4.9.

Solution.

Question 1: Correct answer: D.
Question 2: Correct answer: C.
Question 3: Correct answer: C.

— Solutions to Exercise Problems —
5. The Hessian matrix

Problem 5.1.

Solution.
See answers in FMEA.

Problem 5.2.

Solution.
(a) We have

$$f(x,y) = ax^2 + 2bxy + cy^2 + px + qy + r.$$

The Hessian matrix is

$$\mathbf{f}'' = \begin{pmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ 2b & 2c \end{pmatrix}.$$

The leading principal minors are

$$D_1 = 2a \text{ and } D_2 = \begin{vmatrix} 2a & 2b \\ 2b & 2c \end{vmatrix} = 4(ac - b^2).$$

So if $a > 0$ and $ac - b^2 > 0$ the Hessian is positive definite, so the function is strictly convex. If $a < 0$ and $ac - b^2 > 0$ we have $(-1)^1 D_1 > 0$ and $(-1)^2 D_2 > 0$, so the Hessian is negative definite. This means that the function is strictly concave.
(b) We know from a theorem in Lecture 6 that a function is convex if and only if the Hessian is positive semidefinite. We also know that a symmetric matrix is positive semidefinite if all principal minors are positive or zero. The principal minors are

$$\begin{vmatrix} 2a & 2b \\ 2b & 2c \end{vmatrix}, 2a \text{ and } 2c.$$

Thus the Hessian is positive semidefinite if and only if $ac - b^2 \geq 0$, $a \geq 0$ and $c \geq 0$. Thus this is a sufficient and a necessary condition for the function f to be convex. Similarly we get that the Hessian is negative definite if and only if $ac - b^2 \geq 0$ and $a \leq 0$ and $c \leq 0$, so this is a necessary and sufficient condition for the function f to be concave.

Problem 5.3.

Solution.
One approach is to use the result of the previous problem. We will however solve the problem directly.
The function
$$f(x,y) = -6x^2 + (2a+4)xy - y^2 + 4ay$$
has the following partial derivatives:
$$\begin{aligned} f_1' &= -12x + (2a+4)y \\ f_2' &= (2a+4)x - 2y + 4a. \end{aligned}$$
The Hessian matrix becomes
$$\mathbf{f}'' = \begin{pmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{pmatrix} = \begin{pmatrix} -12 & 2a+4 \\ 2a+4 & -2 \end{pmatrix}.$$
The principal minors are
$$-12, -2 \text{ and } D_2 = \begin{vmatrix} -12 & 2a+4 \\ 2a+4 & -2 \end{vmatrix} = 8 - 4a^2 - 16a$$
Since principal minors of order 1 are negative, the function is never convex. The function is concave if and only if
$$8 - 4a^2 - 16a \geq 0 \Longleftrightarrow -2 - \sqrt{6} \leq a \leq -2 + \sqrt{6}.$$

Problem 5.4.
Consider the function
$$f(x,y) = x^4 + 16y^4 + 32xy^3 + 8x^3y + 24x^2y^2.$$
Find the Hessian matrix. Show that f is convex.

Solution.
The Hessian matrix is
$$\mathbf{f}'' = \begin{pmatrix} 48xy + 12x^2 + 48y^2 & 96xy + 24x^2 + 96y^2 \\ 96xy + 24x^2 + 96y^2 & 192xy + 48x^2 + 192y^2 \end{pmatrix}$$
Completing the squares we see that
$$\mathbf{f}'' = \begin{pmatrix} 12(x+2y)^2 & 24(x+2y)^2 \\ 24(x+2y)^2 & 48(x+2y)^2 \end{pmatrix}.$$
The principal minors of order one are $12(x+2y)^2 \geq 0$ and $48(x+2y)^2 \geq 0$. The principal minor of order two is
$$\begin{vmatrix} 12(x+2y)^2 & 24(x+2y)^2 \\ 24(x+2y)^2 & 48(x+2y)^2 \end{vmatrix} = 0.$$
Thus all principal minors are non-negative, thus the function is convex (but not strictly convex).

Problem 5.5.

Solution.
See answers in FMEA.

Problem 5.6.

Solution.
(a) $\pi(x,y) = 13x + 8y - C(x,y) = 9x + 6y - 0.04x^2 + 0.01xy - 0.01y^2 - 500$.
(b) $\pi_1' = 9 - 0.08x + 0.01y$ and $\pi_2' = 6 + 0.01x - 0.02y$. The Hessian matrix is

$$\begin{pmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{pmatrix}.$$

The leading principal minors are

$$D_1 = -0.08 \text{ and } D_2 = \begin{vmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{vmatrix} = 0.0015 > 0$$

Thus the function concave.
Solving $\pi_1' = 9 - 0.08x + 0.01y = 0$ and $\pi_2' = 6 + 0.01x - 0.02y = 0$, we get that $(160, 380)$ as the only stationary point which has to be a maximum.

Problem 5.7.
Consider the function f defined on the subset $S = \{(x,y,z) : z > 0\}$ of $\mathbb{R}^3$ by

$$f(x,y,z) = 2xy + x^2 + y^2 + z^3.$$

Show that S is convex. Find the stationary points of f. Find the Hessian matrix. Is f concave or convex? Does f have a global extreme point?

Solution.
If (x_1, y_1, z_1) and (x_2, y_2, z_2) are two points in S. We must show that the line segment

$$[(x_1, y_1, z_1), (x_2, y_2, z_2)] = \{(x,y,z) | (x,y,z) = s(x_1, y_1, z_1) + (1-s)(x_2, y_2, z_2),\ s \in [0,1]\}$$

is in S. For a point on the line segment we have $z = sz_1 + (1-s)z_2$. Since either $s, z_1, (1-s)$ and z_2 are non negative, z is non negative. Since either s or $(1-s)$ is positive, z must be positive. This shows that the line segment is contained in S.
The first order conditions are

$$\begin{aligned} f_1' &= 2x + 2y = 0 \\ f_2' &= 2x + 2y = 0 \\ f_3' &= 3z^2 = 0 \end{aligned}$$

From this we see that

$$\begin{aligned} x &= -y \\ z &= 0 \end{aligned}$$

We conclude that there are no stationary points in S.
The Hessian matrix is

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 6z \end{pmatrix}.$$

Solving

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 6z-\lambda \end{vmatrix} = -\lambda(\lambda - 4)(\lambda - 6z)$$

The eigenvalues are thus $\lambda = 0, \lambda = 4$ and $\lambda = 6z$. Since all eigenvalues are ≥ 0, we conclude that the Hessian is positive semidefinite, hence the function is convex. Since there are no stationary points in S (and since S is open), f does not have a global minimum or maximum.

Problem 5.8.

Solution.
The function
$$f(x, y, z) = x^4 + y^4 + z^4 + x^2 + y^2 - xy + zy + z^2$$
has the Hessian matrix,
$$\begin{pmatrix} 12x^2 + 2 & -1 & 0 \\ -1 & 12y^2 + 2 & 1 \\ 0 & 1 & 12z^2 + 2 \end{pmatrix}.$$
The leading principal minors are
$$D_1 = 12x^2 + 2 > 0$$
$$D_2 = \begin{vmatrix} 12x^2 + 2 & -1 \\ -1 & 12y^2 + 2 \end{vmatrix}$$
$$= 24x^2 + 24y^2 + 144x^2y^2 + 3 > 0$$
$$D_3 = \begin{vmatrix} 12x^2 + 2 & -1 & 0 \\ -1 & 12y^2 + 2 & 1 \\ 0 & 1 & 12z^2 + 2 \end{vmatrix}$$
$$= 36x^2 + 48y^2 + 36z^2 + 288x^2y^2 + 288x^2z^2 + 288y^2z^2 + 1728x^2y^2z^2 + 4 > 0$$
So the Hessian is positive definite, and hence f is convex.

— Solutions to Exercise Problems —
6. Local extreme points and the Lagrange problem

Problem 6.1.

Solution.

$$\begin{aligned}
f_1'(\mathbf{x}) &= 2x_1 - x_2 + 2x_3 = 0 \\
f_2'(\mathbf{x}) &= 2x_2 - x_1 + x_3 = 0 \\
f_3'(\mathbf{x}) &= 6x_3 + 2x_1 + x_2 = 0
\end{aligned}$$

Rearranging, we get the following *linear* system of equations:

$$\begin{aligned}
2x_1 - x_2 + 2x_3 &= 0 \\
-x_1 + 2x_2 + x_3 &= 0 \\
2x_1 + x_2 + 6x_3 &= 0
\end{aligned}$$

This may be written on matrix form as follows:

$$\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since

$$\begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{vmatrix} = 4$$

the only solution is the trivial solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus $(0,0,0)$ is the only stationary point. To classify this point, we look at the *Hessian matrix:*

$$\begin{pmatrix} f''_{11}(\mathbf{x}) & f''_{12}(\mathbf{x}) & f''_{13}(\mathbf{x}) \\ f''_{21}(\mathbf{x}) & f''_{22}(\mathbf{x}) & f''_{23}(\mathbf{x}) \\ f''_{31}(\mathbf{x}) & f''_{32}(\mathbf{x}) & f''_{33}(\mathbf{x}) \end{pmatrix}.$$

We have that

$$\begin{aligned}
f''_{11}(\mathbf{x}) &= \frac{\partial}{\partial x_1}(f_1'(\mathbf{x})) = \frac{\partial}{\partial x_1}(2x_1 - x_2 + x_3) \\
&= 2
\end{aligned}$$

Likewise, we find all the second order partial derivatives and we get

$$\mathbf{f}''(\mathbf{x}) = (f''_{ij}(\mathbf{x}))_{3\times 3} = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{pmatrix}.$$

Solution.
(continued) We can now calculate the *leading principal minors* of $\mathbf{f}''(\mathbf{x})$:

$$\begin{aligned}
D_1(\mathbf{x}) &= 2 \\
D_2(\mathbf{x}) &= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 2 \cdot 2 - (-1) \cdot (-1) = 3 \\
D_3(\mathbf{x}) &= \begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{vmatrix} = 4
\end{aligned}$$

Since all leading principal minors are positive, we conclude that $(0,0,0)$ is a minimum point.
Note: One may also note that the function $f(x_1,x_2,x_3) = x_1^2+x_2^2+3x_3^2-x_1x_2+2x_1x_3+x_2x_3$ is a quadratic form and use the theory of quadratic forms to prove that $(0,0,0)$ is a minimum point.

Problem 6.2.

Solution.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -6x^2 + 30x - 36 = 0 \\
\frac{\partial f}{\partial y} &= 2 - e^{y^2} = 0 \\
\frac{\partial f}{\partial z} &= -3 + e^{z^2} = 0
\end{aligned}$$

We get from $-6x^2+30x-36=0$, that $x = 2$ or 3. From $2 - e^{y^2} = 0$, we get $e^{y^2} = 2 \iff y^2 = \ln 2 \iff y = \pm\sqrt{\ln 2}$. Similarly we get $z = \pm\sqrt{\ln 3}$. Thus we get 8 stationary points $(2, \pm\sqrt{\ln 2}, \pm\sqrt{\ln 3})$ and $(3, \pm\sqrt{\ln 2}, \pm\sqrt{\ln 3})$. The Hessian matrix

$$\begin{aligned}
\mathbf{f}'' &= \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} \\
&= \begin{pmatrix} -12x+30 & 0 & 0 \\ 0 & -2ye^{y^2} & 0 \\ 0 & 0 & 2ze^{z^2} \end{pmatrix}
\end{aligned}$$

Solution.
(continues) From this we get

$$\mathbf{f}''(2, \pm\sqrt{\ln 2}, \pm\sqrt{\ln 3}) = \begin{pmatrix} -12\cdot 2+30 & 0 & 0 \\ 0 & -2(\pm\sqrt{\ln 2})e^{(\pm\sqrt{\ln 2})^2} & 0 \\ 0 & 0 & 2(\pm\sqrt{\ln 3})e^{(\pm\sqrt{\ln 3})^2} \end{pmatrix}$$
$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & \mp 4\sqrt{\ln 2} & 0 \\ 0 & 0 & \pm 6\sqrt{\ln 3} \end{pmatrix}$$

and

$$\mathbf{f}''(3, \pm\sqrt{\ln 2}, \pm\sqrt{\ln 3}) = \begin{pmatrix} -12\cdot 3+30 & 0 & 0 \\ 0 & -2(\pm\sqrt{\ln 2})e^{(\pm\sqrt{\ln 2})^2} & 0 \\ 0 & 0 & 2(\pm\sqrt{\ln 3})e^{(\pm\sqrt{\ln 3})^2} \end{pmatrix}$$
$$= \begin{pmatrix} -6 & 0 & 0 \\ 0 & \mp 4\sqrt{\ln 2} & 0 \\ 0 & 0 & \pm 6\sqrt{\ln 3} \end{pmatrix}$$

We get

Stationary point	Type
$(2, -\sqrt{\ln 2}, -\sqrt{\ln 3})$	saddle
$(2, -\sqrt{\ln 2}, \sqrt{\ln 3})$	minimum
$(2, \sqrt{\ln 2}, \sqrt{\ln 3})$	saddle
$(2, \sqrt{\ln 2}, -\sqrt{\ln 3})$	saddle
$(3, -\sqrt{\ln 2}, -\sqrt{\ln 3})$	saddle
$(3, -\sqrt{\ln 2}, \sqrt{\ln 3})$	saddle
$(3, \sqrt{\ln 2}, \sqrt{\ln 3})$	saddle
$(3, \sqrt{\ln 2}, -\sqrt{\ln 3})$	maximum

Problem 6.3.
Let f be defined for all (x, y) by $f(x, y) = x^3 + y^3 - 3xy$.
(a) Show that $(0, 0)$ and $(1, 1)$ are the only stationary points, and compute the Hessian matrix at these points.
(b) Determine the definiteness of the Hessian matrix at each stationary point and use this to determine the nature of each point.
(c) Use the usual second derivative test in two variables to classify the stationary points.

Solution.

(a) We have that $f_1'(x,y) = 3x^2 - 3y$ and $f_2'(x,y) = 3y^2 - 3x$. To find the stationary points, we thus have to solve

$$3x^2 - 3y = 0$$
$$3y^2 - 3x = 0$$

From these equations, we get that $x^2 = y$ and $x = y^2$. We thus have that $x = x^4 \iff x^4 - x = 0 \iff x(x^3 - 1) = 0$. The only solutions are $x = 0$ and $x = 1$. If $x = 0$ we get $y = 0$, and if $x = 1$, we get $y = 1$. Thus $(0,0)$ and $(1,1)$ are the only stationary points.

(b) The Hessian is given by

$$\begin{aligned}\mathbf{f}''(x,y) &= \begin{pmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{pmatrix} \\ &= \begin{pmatrix} 6x_1 & -3 \\ -3 & 6x_2 \end{pmatrix}.\end{aligned}$$

For $(x_1, x_2) = (0,0)$, we get

$$\mathbf{f}''(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}.$$

We find the eigenvalues by solving the characteristic equation

$$\begin{vmatrix} 0-\lambda & -3 \\ -3 & 0-\lambda \end{vmatrix} = \lambda^2 - 9 = 0.$$

We get that $\lambda = \lambda_1 = -3$ or $\lambda = \lambda_2 = 3$. Thus the Hessian is indefinite. We conclude that $(0,0)$ is a saddle point.

For $(x_1, x_2) = (1,1)$, we get

$$\mathbf{f}''(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$$

and we find the eigenvalues by solving

$$\begin{vmatrix} 6-\lambda & -3 \\ -3 & 6-\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 27 = 0.$$

The solution is $\lambda = \lambda_1 = 3$ and $\lambda = \lambda_2 = 9$. Thus the Hessian is positive definite. We conclude that $(1,1)$ is a minimum.

Problem 6.4.

(a) Solve the problem

$$\max(100 - x^2 - y^2 - z^2) \text{ subject to } x + 2y + z = a.$$

(b) Let $(x^*(a), y^*(a), z^*(a))$ be the maximum point and let $\lambda(a)$ be the corresponding Lagrange multiplier. Let $f^*(a) = f(x^*(a), y^*(a), z^*(a))$. Show that

$$\frac{\partial(f^*(a))}{\partial a} = \lambda(a)$$

Solution.
We let $f(x,y,z) = 100 - x^2 - y^2 - z^2$, and let $g(x,y,z) = x + 2y + z$. We will find the maximum of $f(x,y,z)$ such that $g(x,y,z) = a$. We have that

$$\begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$$

Since this has rank one, the NDCQ condition is satisfied.
We define $\mathcal{L}(x,y,z) = f(x,y,z) - \lambda g(x,y,z)$. The necessary first-order conditions for optimality are then:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = -2x - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = -2y - 2\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial z} &= \frac{\partial f}{\partial z} - \lambda \frac{\partial g}{\partial z} = -2z - \lambda = 0 \end{aligned}$$

We get

$$\begin{aligned} \lambda &= -2x \\ \lambda &= -y \\ \lambda &= -2z \end{aligned}$$

We thus get

$$\begin{aligned} x + 2y + z &= a \\ 2x - y &= 0 \\ -y + 2z &= 0 \end{aligned}$$

This is a system of linear equations, with the following augmented matrix

$$\begin{pmatrix} 1 & 2 & 1 & a \\ 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix}$$

Solving, we get $x^* = \frac{1}{6}a$, $y^* = \frac{1}{3}a$, $z^* = \frac{1}{6}a$. Since the Lagrangian

$$\mathcal{L}(x,y,z) = f(x,y,z) - \lambda g(x,y,z) = 100 - x^2 - y^2 - z^2 - \lambda(x + 2y + z)$$

is a sum of concave functions, it is concave and we get that $(x^*, y^*, z^*) = (\frac{1}{6}a, \frac{1}{3}a, \frac{1}{6}a)$ must be a maximum. (We get $\lambda = -y^* = -\frac{1}{3}a$)
(b) We have that the optimal value function is

$$\begin{aligned} f^*(a) = f(x^*, y^*, z^*) &= f(\frac{1}{6}a, \frac{1}{3}a, \frac{1}{6}a) \\ &= 100 - (\frac{1}{6}a)^2 - (\frac{1}{3}a)^2 - (\frac{1}{6}a)^2 \\ &= 100 - \frac{1}{6}a^2 \end{aligned}$$

The left hand side of $\frac{\partial f^*(a)}{\partial a} = \lambda(a)$ is thus

$$\frac{\partial f^*(\mathbf{a})}{\partial a} = -\frac{1}{6}2a = -\frac{1}{3}a$$

The right hand side of $\frac{\partial f^*(a)}{\partial a} = \lambda(a)$ is $\lambda = \lambda(a) = -\frac{1}{3}a$ and this verifies the equation $\frac{\partial f^*(a)}{\partial a} = \lambda(a)$.

Problem 6.5.

(a) Solve the problem

$\max f(x,y,z) = x + 4y + z$ subject to $x^2 + y^2 + z^2 = b_1 = 216$ and $x + 2y + 3z = b_2 = 0$.

(b) Let (x^*, y^*, z^*) be the maximum point and let λ_1 and λ_2 be the corresponding Lagrange multipliers. Let $f^* = f(x^*, y^*, z^*)$. Change the first constraint to $x^2 + y^2 + z^2 = 215$ and the second to $x + y + 2y + 3z = 0.1$. It can be shown that corresponding change in the f^* is approximately equal to

$$\lambda_1 \Delta b_1 + \lambda_2 \Delta b_2.$$

Use this to estimate the change in f^*.

Solution.
(a) We have

$$g_1(x,y,z) = x^2 + y^2 + z^2$$
$$g_2(x,y,z) = x + 2y + 3z$$

and

$$\begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x & 2y & 2z \\ 1 & 2 & 3 \end{pmatrix}$$

This has rank less than 2 if and only if $(x,y,z) = t(1,2,3)$ for some t, but this would violate the constraints.
The Lagrangian is

$$\begin{aligned} \mathcal{L}(x,y,z) &= f(x,y,z) - \lambda_1 g_1(x,y,z) - \lambda_2 g_2(x,y,z) \\ &= x + 4y + z - \lambda_1(x^2 + y^2 + z^2) - \lambda_2(x + 2y + 3z). \end{aligned}$$

The necessary first-order conditions for optimalitty are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} - \lambda_1 \frac{\partial g_1}{\partial x} - \lambda_2 \frac{\partial g_1}{\partial x} = 1 - 2\lambda_1 x - \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} - \lambda_1 \frac{\partial g_1}{\partial y} - \lambda_2 \frac{\partial g_2}{\partial y} = 4 - 2\lambda_1 y - 2\lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial z} &= \frac{\partial f}{\partial z} - \lambda_1 \frac{\partial g_1}{\partial z} - \lambda_2 \frac{\partial g_2}{\partial z} = 1 - 2\lambda_1 z - 3\lambda_2 = 0 \end{aligned}$$

From these equations, we get

$$\begin{aligned} 1 - \lambda_2 &= 2\lambda_1 x \\ 4 - 2\lambda_2 &= 2\lambda_1 y \\ 1 - 3\lambda_2 &= 2\lambda_1 z \end{aligned} \tag{11}$$

We also have the constraints

$$g_1(x,y,z) = x^2 + y^2 + z^2 = 216$$
$$g_2(x,y,z) = x + 2y + 3z = 0$$

We see from the equations that we can not have $\lambda_1 = 0$. Multiplying $x + 2y + 3z = 0$ with $-2\lambda_1$ we get

$$2\lambda_1 x + 2 \cdot 2\lambda_1 y + 3 \cdot 2\lambda_2 z = 0$$

Substituting (11), we get

$$(1 - \lambda_2) + 2(4 - 2\lambda_2) + 3(1 - 3\lambda_2) = 0$$

The solution of this equation is $\lambda_2 = \frac{6}{7}$. Substituting into (11), we get

$$\begin{aligned} 2\lambda_1 x &= 1 - \lambda_2 = 1 - \frac{6}{7} = \frac{1}{7} \implies x = \frac{1}{14\lambda_1} \\ 2\lambda_1 y &= 4 - 2\lambda_2 = 4 - \frac{12}{7} = \frac{16}{7} \implies y = \frac{8}{7\lambda_1} \\ 2\lambda_1 z &= 1 - 3\lambda_2 = 1 - 3 \cdot \frac{6}{7} = -\frac{11}{7} \implies z = \frac{-11}{14\lambda_1} \end{aligned}$$

Solution.

(continued.) We substitute this into $x^2 + y^2 + z^2 = 216$ and we get

$$(\frac{1}{14\lambda_1})^2 + (\frac{8}{7\lambda_1})^2 + (\frac{-11}{14\lambda_1})^2 = 216$$

Simplifying, we get $\frac{27}{14\lambda_1^2} = 216$, which has the two solutions: $\lambda_1 = -\frac{1}{28}\sqrt{7}, \lambda_1 = \frac{1}{28}\sqrt{7}$. When $\lambda_1 = \pm\frac{1}{28}\sqrt{7}$, we get

$$\mathcal{L}(x, y, z) = x + 4y + z \pm \frac{1}{28}\sqrt{7}(x^2 + y^2 + z^2) - \frac{6}{7}(x + 2y + 3z).$$

Since a linear function is both convex and concave, and since $-\frac{1}{28}\sqrt{7}(x^2+y^2+z^2)$ is concave and $\frac{1}{28}\sqrt{7}(x^2 + y^2 + z^2)$ is convex, we get a maximum when $\lambda_1 = \frac{1}{28}\sqrt{7}$ (and a minimum when $\lambda_1 = -\frac{1}{28}\sqrt{7}$). Thus

$$\begin{aligned}(x^*, y^*, z^*) &= (\frac{1}{14\lambda_1}, \frac{8}{7\lambda_1}, \frac{-11}{14\lambda_1}) \\ &= (\frac{2}{7}\sqrt{7}, \frac{32}{7}\sqrt{7}, -\frac{22}{7}\sqrt{7})\end{aligned}$$

solves the maximum problem.

(b) We get

$$\begin{aligned}\Delta f^* &\approx \lambda_1(x, y, z)db_1 + \lambda_2(x, y, z)db_2 \\ &\approx \frac{1}{28}\sqrt{7} \cdot (-1) + \frac{6}{7} \cdot 0.1 \\ &\approx -0.009\end{aligned}$$

Problem 6.6.

Solution.

The Lagrangian is

$$\begin{aligned}\mathcal{L}(x,y,z) &= f(x,y,z) - \lambda_1 g_1(x,y,z) - \lambda_2 g_2(x,y,z) \\ &= e^x + y + z - \lambda_1(x+y+z) - \lambda_2(x^2+y^2+z^2)\end{aligned}$$

The necessary first-order conditions for optimality are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} - \lambda_1 \frac{\partial g_1}{\partial x} - \lambda_2 \frac{\partial g_1}{\partial x} = e^x - \lambda_1 - 2\lambda_2 x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} - \lambda_1 \frac{\partial g_1}{\partial y} - \lambda_2 \frac{\partial g_2}{\partial y} = 1 - \lambda_1 - 2\lambda_2 y = 0 \\ \frac{\partial \mathcal{L}}{\partial z} &= \frac{\partial f}{\partial z} - \lambda_1 \frac{\partial g_1}{\partial z} - \lambda_2 \frac{\partial g_2}{\partial z} = 1 - \lambda_1 - 2\lambda_2 z = 0\end{aligned}$$

From the last two equations we get

$$\begin{aligned}1 - \lambda_1 &= 2\lambda_2 y \\ 1 - \lambda_1 &= 2\lambda_2 z\end{aligned}$$

so we get that $2\lambda_2 y = 2\lambda_2 z \iff 2\lambda_2(y-z) = 0$. Thus either $\lambda_2 = 0$ or $y = z$.
If $\lambda_2 = 0$ we get $\lambda_1 = 1$. From $e^x - \lambda_1 - 2\lambda_2 x = 0$ we then have that $e^x = 1 \implies x = \ln 1 = 0$.
From $x + y + z = 1$ and $x^2 + y^2 + z^2 = 1$, we then get

$$\begin{aligned}y + z &= 1 \\ y^2 + z^2 &= 1\end{aligned}$$

From the first equation, we get $z = 1 - y$. Substituting this into the second equation, we get

$$y^2 + (1-y)^2 = 1 \iff 2y^2 - 2y = 0 \iff y(y-1) = 0$$

Thus we get the two solutions $y = 0$ and $y = 1$. We have $y = 0 \implies z = 1 - 0 = 1$ and $y = 1 \implies z = 1 - 1 = 0$, we get two candidates for the maximum:

$$(0,0,1) \text{ and } (0,1,0) \text{ with } \lambda_1 = 1 \text{ and } \lambda_2 = 0.$$

Solution.
(continued.) If $y = z$, we get from $x + y + z = 1$ and $x^2 + y^2 + z^2 = 1$, that

$$x + 2y = 1 \text{ and } x^2 + 2y^2 = 1$$

From the first we get $x = 1 - 2y$. Substituting this into the second equation, we get

$$(1 - 2y)^2 + 2y^2 = 1 \iff 6y^2 - 4y = 0 \iff y(3y - 2) = 0$$

We get again two solutions, $y = 0$ and $y = \frac{2}{3}$. We have that $y = 0 \implies x = 1 - 2 \cdot 0 = 1$ and $y = \frac{2}{3} \implies x = 1 - 2 \cdot \frac{2}{3} = -\frac{1}{3}$. From

$$e^x - \lambda_1 - 2\lambda_2 x = 0 \text{ and } 1 - \lambda_1 - 2\lambda_2 y = 0$$

we then get:

$$\begin{aligned} x = 1, y = 0 &\implies e - \lambda_1 - 2\lambda_2 = 0 \text{ and } 1 - \lambda_1 = 0 \\ &\implies \lambda_1 = 1 \text{ and } e - 1 = 2\lambda_2 \\ &\implies \lambda_1 = 1 \text{ and } \lambda_2 = \frac{e-1}{2} \end{aligned}$$

In this case we get from $1 - \lambda_1 - 2\lambda_2 z = 0$ that $z = 0$.
This gives the candidate

$$(1, 0, 0) \text{ with } \lambda_1 = 1 \text{ and } \lambda_2 = \frac{e-1}{2}$$

We get that

$$\begin{aligned} x = -\frac{1}{3}, y = \frac{2}{3} &\implies e^{-\frac{1}{3}} - \lambda_1 - 2\lambda_2(-\frac{1}{3}) = 0 \text{ and } 1 - \lambda_1 - 2\lambda_2 \frac{2}{3} = 0 \\ &\implies \lambda_1 = e^{-\frac{1}{3}} + \frac{2}{3}\lambda_2 \text{ and } \lambda_1 = 1 - \frac{4}{3}\lambda_2 \\ &\implies e^{-\frac{1}{3}} + \frac{2}{3}\lambda_2 \ = 1 - \frac{4}{3}\lambda_2 \end{aligned}$$

Solving this one gets $\lambda_2 = \frac{1}{2} - \frac{1}{2}e^{-\frac{1}{3}}$ and $\lambda_1 = 1 - \frac{4}{3}\lambda_2 = 1 - \frac{4}{3}(\frac{1}{2} - \frac{1}{2}e^{-\frac{1}{3}}) = \frac{2}{3}e^{-\frac{1}{3}} + \frac{1}{3}$.
From $1 - \lambda_1 - 2\lambda_2 z = 0$, we get $z = \frac{2}{3}$
This gives the candidate

$$(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \text{ with } \lambda_1 = \frac{2}{3}e^{-\frac{1}{3}} + \frac{1}{3} \text{ and } \lambda_2 = \frac{1}{2} - \frac{1}{2}e^{-\frac{1}{3}}.$$

Thus we have four candidates for the maximum.
Check the four points to see which gives the largest value.

— Solutions to Exercise Problems —

7. Envelope theorems and the bordered Hessian

Problem 7.1.

Solution.
The first-order conditions are that

$$f_1' = -2x - y + 2r = 0$$
$$f_2' = -x - 4y + 2r = 0$$

Solving, we get the solution

$$x^* = \frac{6r}{7} \text{ and } y^* = \frac{2r}{7}.$$

The function f is concave, so this gives the maximum. The optimal value function is

$$f^*(r) = f(\frac{6r}{7}, \frac{2r}{7}, r) = \frac{8r^2}{7}.$$

We get

$$\frac{d}{dr}(f^*(r)) = \frac{16r}{7}.$$

On the other hand we have

$$\frac{\partial f}{\partial r} = 2x + 2y \implies \left(\frac{\partial f}{\partial r}\right)_{x=\frac{6r}{7}, y=\frac{2r}{7}} = 2 \cdot \frac{6r}{7} + 2 \cdot \frac{2r}{7} = \frac{16r}{7}$$

Problem 7.2.

Solution.

$$f(x, y, r, s) = r^2x + 3s^2y - x^2 - 8y^2$$

The first order conditions are

$$\frac{\partial f}{\partial x} = r^2 - 2x = 0$$
$$\frac{\partial f}{\partial y} = 3s^2 - 16y = 0$$

We obtain

$$x^* = \frac{r^2}{2}$$
$$y^* = \frac{3s^2}{16}$$

The Hessian is

$$\begin{pmatrix} -2 & 0 \\ 0 & -26 \end{pmatrix}$$

and negative definite. Thus f is concave and (x^*, y^*) is a maximum.

Solution.
(continued) The optimal value function is

$$
\begin{aligned}
f^*(r,s) &= r^2(\frac{r^2}{2}) + 3s^2(\frac{3s^2}{16}) - (\frac{r^2}{2})^2 - 8(\frac{3s^2}{16})^2 \\
&= \frac{1}{4}r^4 + \frac{9}{32}s^4
\end{aligned}
$$

We get

$$\frac{\partial f^*}{\partial r} = r^3 \text{ and } \frac{\partial f^*}{\partial s} = \frac{9}{8}s^3.$$

On the other hand

$$
\begin{aligned}
\frac{\partial f}{\partial r} = 2rx \implies \left(\frac{\partial f}{\partial r}\right)_{x=x^*,y=y^*} &= 2r(\frac{r^2}{2}) = r^3 \\
\frac{\partial f}{\partial s} = 6sy \implies \left(\frac{\partial f}{\partial s}\right)_{x=x^*,y=y^*} &= 6s(\frac{3s^2}{16}) = \frac{9}{8}s^3
\end{aligned}
$$

Thus we see that

$$
\begin{aligned}
\frac{\partial f^*}{\partial r} &= \left(\frac{\partial f}{\partial r}\right)_{x=x^*,y=y^*} \\
\frac{\partial f^*}{\partial s} &= \left(\frac{\partial f}{\partial s}\right)_{x=x^*,y=y^*}
\end{aligned}
$$

Problem 7.3.

Solution.
(a) $x^*(m) = \frac{1}{3}(m+2)$, $y^*(m) = \frac{1}{9}(m-4)$, $\lambda = \frac{3}{4(m+5)}$
(b) $U^*(m) = \frac{3}{4}\ln(m+5) - \ln 3$, $\frac{dU^*}{dm} = \frac{3}{4}\cdot\frac{1}{m+5} = \lambda$

Problem 7.4.

Solution.
We seek to maximize

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to

$$\begin{aligned} g_1(x, y, z) &= x^2 + y^2 + 4z^2 = 1 \\ g_2(x, y, z) &= x + 3y + 2z = 0. \end{aligned}$$

We define the Lagrange function

$$\begin{aligned} \mathcal{L} &= f - \lambda_1 g_1 - \lambda_2 g_2 \\ &= x^2 + y^2 + z^2 - \lambda_1(x^2 + y^2 + 4z^2) - \lambda_2(x + 3y + 2z). \end{aligned}$$

The first order conditions are

$$\begin{aligned} \mathcal{L}_1' &= 2x - 2\lambda_1 x - \lambda_2 = 2x(1 - \lambda_1) - \lambda_2 = 0 \\ \mathcal{L}_2' &= 2y - 2\lambda_1 y - 3\lambda_2 = 2y(1 - \lambda_1) - 3\lambda_2 = 0 \\ \mathcal{L}_3' &= 2z - 8\lambda_1 z - 2\lambda_2 = 2z(1 - 4\lambda_1) - 2\lambda_2 = 0 \end{aligned}$$

We get

$$\lambda_2 = 2x(1 - \lambda_1) = \frac{1}{3} \cdot 2y(1 - \lambda_1) = z(1 - 4\lambda_1)$$

If $\lambda_1 \neq 1$, we get that $y = 3x$. Substituting this into $x + 3y + 2z = 0$, we get

$$x + 3 \cdot (3x) + 2z = 0 \implies 10x + 2z = 0.$$

From this we get $z = -5x$. Substituting $y = 3x$ and $z = -5x$ into $x^2 + y^2 + 4z^2 = 1$, we get

$$x^2 + (3x)^2 + 4(-5x)^2 = 1$$

Solution.
(continued) From this we get the equation

$$110x^2 = 1$$

which has the solutions $x = \pm\frac{1}{\sqrt{110}}$. From this we get the two points

$$(\pm\frac{1}{\sqrt{110}}, \pm\frac{3}{\sqrt{110}}, \mp\frac{5}{\sqrt{110}}).$$

We get

$$\begin{aligned} f(\pm\frac{1}{\sqrt{110}}, \pm\frac{3}{\sqrt{110}}, \mp\frac{5}{\sqrt{110}}) &= \left(\pm\frac{1}{\sqrt{110}}\right)^2 + \left(\pm\frac{3}{\sqrt{110}}\right)^2 + \left(\mp\frac{5}{\sqrt{110}}\right)^2 \\ &= \frac{7}{22} \end{aligned}$$

If $\lambda_1 = 1$, we get $\lambda_2 = 0$ and $z = 0$. From $x + 3y + 2z = 0$, we get $x + 3y = 0$ and from $x^2 + y^2 + 4z^2 = 1$ we get $x^2 + y^2 = 1$. We get $x = -3y$ and substitute thus into $x^2 + y^2 = 1$ to obtain

$$(-3y)^2 + y^2 = 1$$

This gives $10y^2 = 1$ which has solutions $y = \pm\frac{1}{\sqrt{10}}$. From this we get the points

$$(\mp\frac{3}{\sqrt{10}}, \pm\frac{1}{\sqrt{10}}, 0) \text{ corresponding to } \lambda_1 = 1 \text{ and } \lambda_2 = 0$$

and we have

$$\begin{aligned} f(\mp\frac{3}{\sqrt{10}}, \pm\frac{1}{\sqrt{10}}, 0) &= \left(\mp\frac{3}{\sqrt{10}}\right)^2 + \left(\pm\frac{1}{\sqrt{10}}\right)^2 + 0^2 \\ &= 1 \end{aligned}$$

Comparing, we see that this is the maximum value.
(b) We redefine

$$\begin{aligned} g_1(x, y, z; a) &= x^2 + y^2 + 4z^2 - a \\ g_2(x, y, z; a) &= x + 3y + 2z - a \end{aligned}$$

and we want to maximize

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to

$$\begin{aligned} g_1(x, y, z; a) &= x^2 + y^2 + 4z^2 - a = 1 \\ g_2(x, y, z; a) &= x + 3y + 2z - a = 0. \end{aligned}$$

Let $f^*(a)$ be the corresponding optimal value function. From (a), we know that

$$f^*(0) = 1$$

since we solved the problem in the case $a = 0$. We would like to find $f^*(0.05)$, and we estimate this by linear approximation

$$f^*(0.05) \cong f^*(0) + \left(\frac{df^*}{da}\right)_{a=0} \cdot 0.05.$$

By the envelope theorem we have

$$\left(\frac{df^*}{da}\right)_{a=0} = \left(\frac{\partial\mathcal{L}}{\partial a}\right)_{x=x^*, y=y^*, \lambda=\lambda^*}.$$

We have

$$\frac{\partial\mathcal{L}}{\partial a} = \lambda_1 + \lambda_2$$

so

$$\left(\frac{\partial\mathcal{L}}{\partial a}\right)_{x=x^*, y=y^*, \lambda=\lambda^*} = 1 + 0 = 1.$$

So that

$$\begin{aligned} f^*(0.05) &\cong f^*(0) + 1 \cdot 0.05. \\ &= 1 + 0.05 = 1.05 \end{aligned}$$

Problem 7.5.

Solution.
See answers in FMEA.

— Solutions to Exercise Problems —
8. Introduction to differential equations

Problem 8.1.
Find $\dot{x}$.
(a) $x = \frac{1}{2}t - \frac{3}{2}t^2 + 5t^3$
(b) $x = (2t^2 - 1)(t^4 - 1)$
(c) $x = (\ln t)^2 - 5\ln t + 6$
(d) $x = \ln(3t)$
(e) $x = 5e^{-3t^2+t}$
(f) $x = 5t^2e^{-3t}$

Solution.
(a) $\dot{x} = \frac{1}{2} - 3t + 15t^2$
(b) $\dot{x} = 4t(t^4 - 1) + (2t^2 - 1)4t^3 = 12t^5 - 4t^3 - 4t$
(c) $\dot{x} = 2(\ln t)\frac{1}{t} - 5\frac{1}{t}$
(d) $\dot{x} = \frac{1}{t}$
(e) $\dot{x} = 5e^{-3t^2+t}(-6t + 1)$
(f) $\dot{x} = 10te^{-3t} - 15t^2e^{-3t}$

Problem 8.2.
Find the integrals.
(a) $\int t^3 dt$
(b) $\int_0^1 (t^3 + t^5 + \frac{1}{3})dt$
(c) $\int \frac{1}{t} dt$
(d)$\int te^{t^2} dt$
(e)$\int \ln t dt$

Solution.
(a) $\int t^3 dt = \frac{1}{4}t^4 + C$
(b) $\int_0^1 (t^3 + t^5 + \frac{1}{3})dt = \frac{3}{4}$
(c) $\int \frac{1}{t} dt = \ln|t| + C$
(d) To find the integral $\int te^{t^2} dt$ we substitute $u = t^2$. This gives $\frac{du}{dt} = 2t$ or $\frac{du}{2} = tdt$. We get

$$\int te^{t^2} dt = \int e^u \frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{t^2} + C$$

(e) We use integration by parts

$$\int uv' dt = uv - \int u'v dx.$$

We write $\int \ln t dt$ as $\int (\ln t) \cdot 1 dt$ and let $u = \ln t$ and $v' = 1$. Thus $u' = \frac{1}{t}$ and $v = t$, and

$$\begin{aligned}\int \ln t dt &= (\ln t)t - \int \frac{1}{t} t dt \\ &= t\ln t - \int 1 dt \\ &= t\ln t - t + C\end{aligned}$$

Problem 8.3.
The following differential equations may be solved by integrating the right hand side. Find the general solution, and the particular solution satisfying $x(0) = 1$.
(a) $\dot{x} = 2t$.
(b) $\dot{x} = e^{2t}$
(c) $\dot{x} = (2t+1)e^{t^2+t}$
(d) $\dot{x} = \frac{2t+1}{t^2+t+1}$.

Solution.
(a) $x = \int 2t dt = t^2 + C$. The general solution is $x = t^2 + C$. We get $x(0) = C = 1$, so $x = t^2 + 1$ is the particular solution satisfying $x(0) = 1$.
(b) $x = \frac{1}{2}e^{2t} + C$ is the general solution. We get $x(0) = \frac{1}{2}e^{2\cdot 0} + C = \frac{1}{2} + C = 1 \implies C = \frac{1}{2}$. Thus $x(t) = \frac{1}{2}e^{2t} + \frac{1}{2}$ is the particular solution.
(c) To find the integral $\int (2t+1)e^{t^2+t}dt$, we substitute $u = t^2 + t$. We get $\frac{du}{dt} = 2t + 1 \implies du = (2t+1)dt$, so

$$\int (2t+1)e^{t^2+t}dt = \int e^u du = e^u + C = e^{t^2+t} + C.$$

The general solution is $x = e^{t^2+t} + C$. This gives $x(0) = 1 + C = 1 \implies C = 0$. The particular solution is $x = e^{t^2+t}$.
(d) We substitute $u = t^2+t+1$ in $\int \frac{2t+1}{t^2+t+1}dt$ to find the general solution $x = \ln(t^2+t+1)+C$. We get $x(0) = \ln 1 + C = C = 1$. The particular solution is $x(t) = \ln(t^2 + t + 1) + 1$.

Problem 8.4.
FMEA 5.1.1 in both editions.

Solution.
$x(t) = Ce^{-t} + \frac{1}{2}e^t \implies \dot{x} = -Ce^{-t} + \frac{1}{2}e^t$. From this we get

$$\dot{x} + x = -Ce^{-t} + \frac{1}{2}e^t + Ce^{-t} + \frac{1}{2}e^t = e^t$$

so we see that $\dot{x} + x = e^t$ is satisfied when $x = Ce^{-t} + \frac{1}{2}e^t$.

Problem 8.5.
(FMEA 5.1.2 in both editions.) Show that $x = Ct^2$ is a solution of $t\dot{x} = 2x$ for all choices of the constant C. Find the particular solution satisfying $x(1) = 2$.

Solution.
$x = Ct^2 \implies \dot{x} = 2Ct$. We have

$$t\dot{x} = t \cdot 2Ct = 2Ct^2 = 2x.$$

Problem 8.6.
FMEA 5.3.1 in both editions.

Solution.
The equation $x^2\dot{x} = t + 1$ is separable:
$$x^2 \frac{dx}{dt} = t + 1$$
gives
$$\begin{aligned} \int x^2 dx &= \int (t+1) dt \\ \frac{1}{3}x^3 &= \frac{1}{2}t^2 + t + C \\ x^3 &= \frac{3}{2}t^2 + 3t + 3C \end{aligned}$$
Taking third root and renaming the constant
$$x(t) = \sqrt[3]{\frac{3}{2}t^2 + 3t + K}$$
We want the particular solution with $x(1) = 1$. We have
$$\begin{aligned} x(1) &= \sqrt[3]{\frac{3}{2}1^2 + 3 + K} \\ &= \sqrt[3]{K + \frac{9}{2}} = 1 \implies K + \frac{9}{2} = 1 \end{aligned}$$
We get $K = -\frac{7}{2}$. Thus
$$x(t) = \sqrt[3]{\frac{3}{2}t^2 + 3t - \frac{7}{2}}$$
is the particular solution.

Problem 8.7.
FMEA 5.3.2 in both editions.

Solution.
(a) $\dot{x} = t^3 - 1$ gives
$$x = \int (t^3 - 1)dt$$
We get
$$x = \frac{1}{4}t^4 - t + C.$$
(b) We must evaluate the integral $\int (te^t - t)dt$. To evaluate $\int te^t dt$ we use integration by parts
$$\int uv'dt = uv - \int u'vdt.$$
with $v' = e^t$ and $u = t$. We get $u' = 1$ and $v = e^t$. Thus
$$\int te^t dt = te^t - \int e^t dt = te^t - e^t + C$$
We get
$$x = \int (te^t - t)dt = te^t - e^t - \frac{1}{2}t^2 + C$$
(c) $e^x \dot{x} = t + 1$ is separated as
$$e^x dx = (t+1)dt \implies \int e^x dx = \int (t+1)dt$$
Thus we get
$$e^x = \frac{1}{2}t^2 + t + C.$$
Taking the natural logarithm on each side, we get
$$x(t) = \ln(\frac{1}{2}t^2 + t + C).$$

Problem 8.8.
FMEA 5.3.3 in both editions.

Solution.
(a) $t\dot{x} = x(1-t)$ is separated as

$$\frac{dx}{x} = \frac{1-t}{t}dt \implies \int \frac{dx}{x} = \int \frac{1-t}{t}dt$$

Note that $\frac{1-t}{t} = \frac{1}{t} - 1$, so

$$\ln|x| = \ln|t| - t + C$$

From this we get

$$e^{\ln|x|} = e^{\ln|t|-t+C} = e^{\ln|t|}e^{-t}e^{C} \implies |x| = |t|e^{-t}e^{C}$$

From this we deduce that

$$x(t) = te^{-t}K$$

where K is a constant as the general solution. We will find the particular solution with $x(1) = \frac{1}{e}$. We get

$$x(1) = e^{-1}K = e^{-1} \implies K = 1.$$

The particular solution is

$$x(t) = te^{-t}.$$

(b) The equation $(1+t^3)\dot{x} = t^2 x$ is separated as

$$\frac{dx}{x} = \frac{t^2}{1+t^3}dt \implies \int \frac{dx}{x} = \int \frac{t^2}{1+t^3}dt$$

We get

$$\ln|x| = \frac{1}{3}\ln|1+t^3| + C = \ln|1+t^3|^{\frac{1}{3}} + C$$

This gives

$$e^{\ln|x|} = e^{\ln|1+t^3|^{\frac{1}{3}}+C}$$

This gives

$$|x| = |1+t^3|^{\frac{1}{3}}e^{C}$$

from which we deduce the general solution

$$x(t) = K(1+t^3)^{\frac{1}{3}}$$

where K is a constant. We which to find the particular solution with $x(0) = 2$. We get

$$x(0) = K = 2.$$

Thus the particular solution is

$$x(t) = 2(1+t^3)^{\frac{1}{3}}.$$

(c) $x\dot{x} = t$ is separated as

$$xdx = tdt \implies \int xdx = \int tdt$$

The general solution is

$$x^2 = t^2 + C$$

where x is define implicitly. We want the particular solution where $x(\sqrt{2}) = 1$. We get

$$1^2 = (\sqrt{2})^2 + C \implies 1 = 2 + C \implies C = -1$$

We have

$$x^2 = t^2 - 1 \implies x = \pm\sqrt{t^2-1}$$

since $x(\sqrt{2}) < 0$ we have

$$x(t) = \sqrt{t^2-1}$$

as the particular solution.

Solution.
(continued)
(d) $e^{2t}\frac{dx}{dt} - x^2 - 2x = 1$, is separated as follows:

$$e^{2t}\dot{x} - x^2 - 2x = 1 \implies e^{2t}\dot{x} = 1 + x^2 + 2x = (x+1)^2 \implies$$
$$\frac{dx}{(x+1)^2} = e^{-2t}dt \implies \int \frac{dx}{(x+1)^2} = \int e^{-2t}dt$$

To solve the integral
$$\int \frac{dx}{(x+1)^2}$$
we substitute $u = x + 1$. We get $\frac{du}{dx} = 1 \implies dx = du$. Thus
$$\int \frac{dx}{(x+1)^2} = \int \frac{1}{u^2}du = \int u^{-2}du = \frac{1}{-1}u^{-2+1} + C = -u^{-1} + C = -\frac{1}{(x+1)} + C$$
Thus we get
$$-\frac{1}{(x+1)} = \frac{1}{-2}e^{-2t} + C = -\frac{1}{2}e^{-2t} + C \implies -x - 1 = \frac{1}{-\frac{1}{2}e^{-2t} + C}$$
From this we get
$$x(t) = \frac{-1}{-\frac{1}{2}e^{-2t} + C} - 1$$
as the general solution. We want the particular solution with $x(0) = 0$. We get
$$x(0) = \frac{-1}{-\frac{1}{2}e^{0} + C} - 1 = 0$$
From this we get $C = -\frac{1}{2}$. Thus the particular solution is
$$\begin{aligned} x(t) &= \frac{-1}{-\frac{1}{2}e^{-2t} - \frac{1}{2}} - 1 \\ &= \frac{1 - e^{-2t}}{1 + e^{-2t}}. \end{aligned}$$

Problem 8.9.
Maximize the function
$$f(x_1, x_2) = x_1^2 + x_2^2 + x_2 - 1$$
subject to $g(x_1, x_2) = x_1^2 + x_2^2 \leq 1$.

Solution.
The Lagrangian is
$$\mathcal{L}(\mathbf{x}) = x_1^2 + x_2^2 + x_2 - 1 - \lambda(x_1^2 + x_2^2 - 1),$$
and the Kuhn-Tucker conditions are
$$\mathcal{L}_1' = 2x_1 - 2\lambda x_1 = 2x_1(1-\lambda) = 0$$
$$\mathcal{L}_2' = 2x_2 + 1 - 2\lambda x_2 = 2x_2(1-\lambda) + 1 = 0$$
and
$$\lambda \geq 0 \text{ and } \lambda = 0 \text{ if } x_1^2 + x_2^2 < 1.$$
From $\mathcal{L}_1' = 0$ we obtain $2x_1(1-\lambda) = 0$. Thus $x_1 = 0$ or $\lambda = 1$. If $\lambda = 1$, then $\mathcal{L}_2' = 2x_2 + 1 - 2x_2 = 1 \neq 0$, so we conclude that $x_1 = 0$.

CASE $x_1^2 + x_2^2 = 1$ **:**

From $x_1^2 + x_2^2 = 1$ and $x_1 = 0$ we obtain $x_2^2 = 1$ or $x_2 = \pm 1$. We have $\mathcal{L}_2' = 2x_2(1-\lambda) + 1 = \pm 2(1-\lambda) + 1 = 0 \implies (1-\lambda) = \frac{-1}{\pm 2} = \mp\frac{1}{2} \implies \lambda = 1 \pm \frac{1}{2}$. Thus
$$(0,-1) \text{ corresponding to } \lambda = \frac{1}{2} \text{ and}$$
$$(0,1) \text{ corresponding to } \lambda = \frac{3}{2}$$
are candidates for maximum.

CASE $x_1^2 + x_2^2 < 1$ **:**

From $x_1 = 0$ we get $x_2^2 < 1$. This is the same as to say $-1 < x_2 < 1$. Since $\lambda = 0$, $\mathcal{L}_2' = 2x_2 + 1 = 0$ gives $x_2 = -\frac{1}{2}$. We conclude that
$$(0, -\frac{1}{2}) \text{ corresponding to } \lambda = 0$$
is a candidate for maximum. We compute
$$f(0,1) = 1$$
$$f(0,-1) = -1 \text{ and}$$
$$f(0, -\frac{1}{2}) = -\frac{5}{4}$$
and conclude that $f(0,1) = 1$ is the maximal value.

Problem 8.10.

Solution.
See answers in FMEA.

Problem 8.11.
Minimize
$$4\ln(x^2+2) + y^2 \text{ subject to } \begin{cases} x^2 + y \geq 2 \\ x \geq 1 \end{cases}$$

Solution.

We will use the following general method of solving

$$\max f(x_1,\dots,x_n) \text{ subject to } \begin{cases} g_1(x_1,\dots,x_n) \le b_1 \\ \vdots \\ g_m(x_1,\dots,x_n) \le b_m \end{cases}$$

by applying the following steps:

(1) $\mathcal{L} = f - \lambda_1 g_1 - \cdots - \lambda_m g_m$

(2) $\mathcal{L}'_1 = 0, \mathcal{L}'_2 = 0, \dots, \mathcal{L}'_n = 0$ (FOC's)

(3) $\lambda_j \ge 0$ and $\lambda_j = 0$ if $g_j(x_1,\dots,x_n) < b_j$

(4) Require $g_j(x_1,\dots,x_n) \le b_j$

To transform the problem into this setting, we define

$$f(x,y) = -(4\ln(x^2+2) + y^2)$$

since minimizing $4\ln(x^2+2)+y^2$ is the same as maximizing $-(4\ln(x^2+2)+y^2)$. We also rewrite the constraints as

$$\begin{aligned} g_1(x,y) &= -x^2 - y \le -2 \\ g_2(x,y) &= -x \le -1 \end{aligned}$$

We define the Lagrange function:

$$\begin{aligned} \mathcal{L} &= -(4\ln(x^2+2)+y^2) - \lambda_1(-x^2-y) - \lambda_2(-x) \\ &= -4\ln(x^2+2) - y^2 + \lambda_1(x^2+y) + \lambda_2 x \end{aligned}$$

The first order conditions are the

$$\begin{aligned} \mathcal{L}'_1 &= -4\frac{1}{x^2+2}\cdot 2x + 2\lambda_1 x + \lambda_2 = \frac{-8x}{x^2+2} + 2\lambda_1 x + \lambda_2 = 0 \\ \mathcal{L}'_2 &= -2y + \lambda_1 = 0 \end{aligned}$$

Since there are two constraints, there are four cases to consider:

$\boxed{\text{The case } -x^2 - y = -2 \text{ and } -x = -1:}$

Since $x = 1$, we deduce from $\mathcal{L}'_1 = 0$ that

$$\frac{-8\cdot 1}{1^2+2} + 2\lambda_1 \cdot 1 + \lambda_2 = 0 \iff 2\lambda_1 + \lambda_2 - \frac{8}{3} = 0$$

From $x^2 + y = 2$ and $x = 1$ we obtain that $y = 1$. From $\mathcal{L}'_2 = -2y + \lambda_1 = 0$ we the obtain that $\lambda_1 = 2$. Substituting this into $2\lambda_1 + \lambda_2 - \frac{8}{3} = 0$ we get

$$2\cdot 2 + \lambda_2 - \frac{8}{3} = 0 \iff \lambda_2 = -\frac{4}{3} < 0$$

This violates the complementary slackness conditions that says that $\lambda_2 \ge 0$ since the second constraint is active. We conclude that the case case $-x^2 - y = -2$ and $-x = -1$ does not lead to a solution.

Solution.
(continues)

$\boxed{\text{The case } -x^2 - y = -2 \text{ and } -x < -1:}$

Since the second constraint is inactive, we get $\lambda_2 = 0$. Substituting this into $\frac{-8x}{x^2+2} + 2\lambda_1 x + \lambda_2 = 0$ we get

$$\frac{-8x}{x^2+2} + 2\lambda_1 x = 0 \Longleftrightarrow 2x(\lambda_1 - \frac{4}{x^2+2}) = 0$$

Since $x > 1$ this gives

$$\lambda_1 = \frac{4}{x^2+2}$$

From $-x^2 - y = -2$ we have that $y = 2 - x^2$ and substituting this and $\lambda_1 = \frac{4}{x^2+2}$ into $-2y + \lambda_1 = 0$ gives

$$-2(2-x^2) + \frac{4}{x^2+2} = 0 \Longleftrightarrow (x^2+2)(x^2-2) + 2 = 0 \Longleftrightarrow x^4 = 2$$

From this we obtain that

$$x = \pm\sqrt[4]{2} \cong \pm 1.\,189\,2$$

Since $x > 1$ we get that

$$x = \sqrt[4]{2}$$

From $y = 2 - x^2$ we obtain

$$y = 2 - \sqrt{2}$$

and from $\lambda_1 = 2y$ we get

$$\lambda_1 = 2(2 - \sqrt{2})$$

Thus we have the following candidate for optimum

$$\boxed{(\sqrt[4]{2}, 2-\sqrt{2}) \longleftrightarrow \lambda_1 = 2(2-\sqrt{2}), \lambda_2 = 0}$$

$\boxed{\text{The case } -x^2 - y < -2 \text{ and } -x = -1:}$

Since the first constraint is inactive, we get $\lambda_1 = 1$. Substituting this into $-2y + \lambda_1 = 0$ we get

$$y = 0.$$

Since $x = 1$ by assumption, we see that $-x^2 - y = -1$ which is not less that -2 so the first constraint is not satisfied. Thus the case $-x^2 - y < -2$ and $-x = -1$ does not give any solution

$\boxed{\text{The case } -x^2 - y < -2 \text{ and } -x < -1:}$

Since both constraints are inactive, we get $\lambda_1 = 0$ and $\lambda_2 = 0$. Thus we get from $-2y + \lambda_1 = 0$ that

$$y = 0$$

and from $\frac{-8x}{x^2+2} + 2\lambda_1 x + \lambda_2 = 0$ that

$$x = 0$$

But $-x = 0$ is not less that -1, so this gives no solutions.

$\boxed{\text{Conclusion:}}$

The minimum value (subject to the constraints) is given by

$$(x,y) = (\sqrt[4]{2}, 2-\sqrt{2}) \implies 4\ln(x^2+2) + y^2 = 4\ln(\sqrt{2}+2) + (2-\sqrt{2})^2 \cong 5.\,254\,9$$

— Solutions to Exercise Problems —

9. Linear first order and exact differential equations

Problem 9.1.
(FMEA 5.4.1 in both editions,) Find the general solution of $\dot{x} + \frac{1}{2}x = \frac{1}{4}$. Determine the equilibrium state of the equation. Is it stable? Draw some typical solutions.

Solution.
See solutions in FMEA.

Problem 9.2.
FMEA 5.4.2 in both editions.

Solution.

(a) $x = Ce^{-t} + 10$
(b) $x = Ce^{3t} - 9$
(c) $x = Ce^{-5t/4} + 20$

Problem 9.3.
(FMEA 5.4.4 in both editions.) Find the general solutions of the following differential equations, and in each case, find the particular solution satisfying $x(0) = 1$.
(a) $\dot{x} - 3x = 5$
(b) $3\dot{x} + 2x + 16 = 0$
(c) $\dot{x} + 2x = t^2$

Solution.

(a) $x = Ce^{3t} - 5/3$, $x(0) = 1 \implies C = 8/3$.
(b) $x = Ce^{-2t/3} - 8$, $x(0) = 1 \implies C = 9$
(c) $x = Ce^{-2t} + \frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{4}$, $x(0) = 1 \implies C = 3/4$.

Problem 9.4.
Problem 5.4.7 in the first edition of FMEA and 5.4.6 in the second edition.

Solution.
See answers in FMEA.

Problem 9.5.
Determine which of the following equations are exact:
(a) $(2x + t)\dot{x} + 2 + x = 0$
(b) $x^2\dot{x} + 2t + x = 0$
(c) $(t^5 + 6x^2)\dot{x} + (5xt^4 + 2) = 0$

Solution.

(a) $\frac{\partial}{\partial t}(2x+t) = 1, \frac{\partial}{\partial x}(2+x) = 1 \implies$ exact.
(b) $\frac{\partial}{\partial t}(x^2) = 0, \ \ \frac{\partial}{\partial x}(2t+x) = 1 \implies$ not exact.
(c) $\frac{\partial}{\partial t}(t^5+6x^2) = 5t^4, \ \frac{\partial}{\partial x}(5xt^4+2) = 5t^4 \implies$ exact.

Problem 9.6.
Solve the exact equations in the previous problem.

Solution.
(a) $h'_x = 2x + t \implies h = x^2 + tx + \alpha(t) \implies h'_t = x + \alpha'(t) = 2 + x \implies \alpha'(t) = 2 \implies \alpha(t) = 2t + C$. Thus $h = x^2 + tx + 2t + C$.
The general solution is given by

$$x^2 + tx + 2t = C$$

for some constant C, and this determines x implicitly as a function of t.
(c) $h'_x = t^5 + 6x^2 \implies h = xt^5 + 2x^3 + \alpha(t) \implies h'_t = 5xt^4 + \alpha'(t) = 5xt^4 + 2 \implies \alpha'(t) = 2 \implies \alpha = 2t + C$. Thus $h = xt^5 + 2x^3 + 2t + C$.
The general solution is given by

$$xt^5 + 2x^3 + 2t = C$$

for a constant C, and this determines x implicitly as a function of t.

Problem 9.7.
FMEA 5.5.1 in both editions.

Solution.
See solutions in FMEA.

— Solutions to Exercise Problems —

10. Second-order differential equations

Problem 10.1.
FMEA 6.1.1ac in both editions.

Solution.
(a) $\ddot{x} = t \implies \dot{x} = \frac{1}{2}t^2 + C_1 \implies x = \frac{1}{6}t^3 + C_1 t + C_2$
(c) $\ddot{x} = e^t + t^2 \implies \dot{x} = e^t + \frac{1}{3}t^3 + C_1 \implies x = e^t + \frac{1}{12}t^4 + C_1 t + C_2$

Problem 10.2.
FMEA 6.1.2 in both editions.

Solution.
$\ddot{x} = t^2 - t \implies \dot{x} = \frac{1}{3}t^3 - \frac{1}{2}t^2 + C_1 \implies x = \frac{1}{12}t^4 - \frac{1}{6}t^3 + C_1 t + C_2$
$x(0) = 1 \Leftrightarrow \frac{1}{12}0^4 - \frac{1}{6}0^3 + C_1 0 + C_2 = C_2 = 1$
$\dot{x}(0) = 2 \Leftrightarrow \frac{1}{3}0^3 - \frac{1}{2}0 + C_1 = C_1 = 2$
Particular solution: $x(t) = \frac{1}{12}t^4 - \frac{1}{6}t^3 + 2t + 1$.

Problem 10.3.
FMEA 6.1.3 in both editions.

Solution.
Substitute $u = \dot{x}$. Then $\ddot{x} = \dot{x} + t \Leftrightarrow \dot{u} = u + t \Leftrightarrow \dot{u} - u = t$. The integrating factor is e^{-t}, and we get $ue^{-t} = \int te^{-t}dt = -e^{-t} - te^{-t} + C_1$. From this: $u = (-e^{-t} - te^{-t} + C_1)e^t = C_1e^t - t - 1 \implies x = \int (Ce^t - t - 1)dt = C_1e^t - t - \frac{1}{2}t^2 + C_2$.
$x(0) = 1 \implies C_1 + C_2 = 1 \implies C_2 = 1 - C_1$
$x(1) = 2 \implies C_1e - 1 - \frac{1}{2} + C_2 = C_2 + eC_1 - \frac{3}{2} = 2 \implies 1 - C_1 + eC_1 - \frac{3}{2} = 2$. From this we get $C_1 = \frac{5}{2(e-1)}$ and $C_2 = 1 - C_1 = 1 - \frac{5}{2(e-1)} = \frac{2e-7}{2(e-1)}$.
Particular solution is: $x(t) = \frac{5}{2(e-1)}e^t - t - \frac{1}{2}t^2 + \frac{2e-7}{2(e-1)}$.

Problem 10.4.
FMEA 6.3.1 in both editions.

Solution.
(a) Characteristic equation is $r^2 - 3 = 0 \implies r = \pm\sqrt{3} \implies x(t) = C_1 e^{-\sqrt{3}t} + C_2 e^{\sqrt{3}t}$.
(b) Characteristic equation is $r^2 + 4r + 8 = 0$. This has no real solutions. Thus we put $\alpha = -\frac{1}{2}a = -\frac{1}{2}4 = -2, \beta = \sqrt{b - \frac{1}{4}a^2} = \sqrt{8 - \frac{1}{4}4^2} = 2$. From this the general solution is $x(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t) = e^{-2t}(A\cos 2t + B\sin 2t)$.
(c) $3\ddot{x} + 8\dot{x} = 0 \iff \ddot{x} + \frac{8}{3}\dot{x} = 0$. The characteristic equation is $r^2 + \frac{8}{3}r = 0 \implies r = 0$ or $r = -\frac{8}{3}$. The general solution is $x(t) = C_1 e^{0t} + C_2 e^{-\frac{8}{3}t} = C_1 + C_2 e^{-\frac{8}{3}t}$.
(d) $4\ddot{x} + 4\dot{x} + x = 0$ has characteristic equation $4r^2 + 4r + 1 = 0$. There is one solution $r = -\frac{1}{2}$. The general solution is $x(t) = (C_1 + C_2 t)e^{-\frac{1}{2}t}$.
(e) First we solve the homogenous equation $\ddot{x} + \dot{x} - 6x = 8$. The characteristic equation is $r^2 + r - 6 = 0$. It has the solutions $r = -3$ and $r = 2$. The general solution of the homogenous equation is thus

$$x_h(t) = C_1 e^{-3t} + C_2 e^{2t}.$$

Solution.
(continued.) In order to find the general solution of the non-homogenous equation $\ddot{x} + \dot{x} - 6x = 8$, we need to find a particular solution and we guess on a solution of the form $x_p(t) = A$ for some constant A. Putting this into the equation gives $A = -\frac{8}{6} = -\frac{4}{3}$. Thus the general solution is

$$x(t) = -\frac{4}{3} + C_1 e^{-3t} + C_2 e^{2t}.$$

(f) We first solve the homogenous equation $\ddot{x} + 3\dot{x} + 2x = 0$. The characteristic equation is $r^2 + 3r + 2 = 0$. The solutions are $r = -1$ and $r = -2$. The general solution of the homogenous equation is thus

$$x_h(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

To find a solution of the non-homogenous equation $\ddot{x} + 3\dot{x} + 2x = e^{5t}$, we guess on a solution of the form $x_p(t) = Ae^{5t}$. We have that

$$\dot{x}_p = 5Ae^{5t} \text{ and } \ddot{x}_p = 25Ae^{5t}.$$

Putting this into the equation we obtain

$$25Ae^{5t} + 3 \cdot 5Ae^{5t} + 2Ae^{5t} = e^{5t}$$

From this we get $42Ae^{5t} = e^{5t}$ and we must have $A = \frac{1}{42}$.Thus the solution is

$$x(t) = \frac{1}{42}e^{5t} + C_1 e^{-t} + C_2 e^{-2t}.$$

Problem 10.5.
FMEA 6.3.2bc in both editions.

Solution.
(b) We first solve $\ddot{x} - x = 0$. The characteristic equation is $r^2 - 1 = 0$. We get $x_h = C_1 e^{-t} + C_2 e^t$. To find a solution of $\ddot{x} - x = e^{-t}$, we guess on solution of the form $x_p = Ae^{-t}$. We have $\dot{x}_p = -Ae^{-t}$ and $\ddot{x}_p = Ae^{-t}$. Putting this into the left hand side of the equation, we get

$$Ae^{-t} - (Ae^{-t}) = 0.$$

So this does not work. The reason is that e^{-t} is a solution of the homogenous equation. We try something else: $x_p = Ate^{-t}$. This gives

$$\begin{aligned}\dot{x}_p &= A(e^{-t} - te^{-t})\\ \ddot{x}_p &= A(-e^{-t} - (e^{-t} - te^{-t}))\\ &= Ae^{-t}(t-2)\end{aligned}$$

Putting this into the left hand side of the equation, we obtain

$$\begin{aligned}\ddot{x}_p - x_p &= Ae^{-t}(t-2) - Ate^{-t}\\ &= -2Ae^{-t}\end{aligned}$$

We get a solution for $A = -\frac{1}{2}$. Thus the general solution is

$$x(t) = -\frac{1}{2}te^{-t} + C_1e^{-t} + C_2e^t.$$

Solution.
(continued.) (c) The equation is equivalent to

$$\ddot{x} - 10\dot{x} + 25x = \frac{2}{3}t + \frac{1}{3}.$$

We first solve the homogenous equation for which the characteristic equation is

$$r^2 - 10r + 25 = 0.$$

This has one solution $r = 5$. The general homogenous solution is thus

$$x_h = (C_1 + C_2t)e^{5t}.$$

To find a particular solution, we try

$$x_p = At + B.$$

We have $\dot{x}_p = A$ and $\ddot{x}_p = 0$. Putting this into the equation, we obtain

$$0 - 10A + 25(At + B) = \frac{2}{3}t + \frac{1}{3}$$

We obtain $25A = \frac{2}{3}$ and $-10A + 25B = \frac{1}{3}$. From this we get $A = \frac{2}{75}$ and $-\frac{20}{75} + 25B = \frac{25}{75} \implies B = \frac{45}{25 \cdot 75} = \frac{3}{125}$.
Thus

$$x(t) = \frac{2}{75}t + \frac{3}{125} + (C_1 + C_2t)e^{5t}$$

Problem 10.6.
FMEA 6.3.3 in both editions.

Solution.
(a) We first solve the homogenous equation $\ddot{x} + 2\dot{x} + x = 0$. The characteristic equation is $r^2 + 2r + 1 = 0$ which has the one solution, $r = -1$. We get

$$x_h(t) = (C_1 + C_2 t)e^{-t}.$$

To find a particular solution we try with $x_p = At^2 + Bt + C$. We get $\dot{x}_p = 2At + B$ and $\ddot{x}_p = 2A$. Substituting this into the left hand side of the equation, we get

$$\begin{aligned} &2A + 2(2At + B) + (At^2 + Bt + C) \\ &= 2A + 2B + C + (4A + B)t + At^2 \end{aligned}$$

We get $A = 1$, $(4A + B) = 0$ and $2A + 2B + C = 0$. We obtain $A = 1$, $B = -4$ and $C = -2A - 2B = -2 + 8 = 6$. Thus the general solution is

$$x(t) = t^2 - 4t + 6 + (C_1 + C_2 t)e^{-t}.$$

We get $\dot{x} = 2t - 4 + C_2 e^{-t} + (C_1 + C_2 t)e^{-t}(-1) = 2t - C_1 e^{-t} + C_2 e^{-t} - tC_2 e^{-t} - 4$. From $x(0) = 0$ we get $6 + C_1 = 0 \implies C_1 = -6$. From $\dot{x}(0) = 1$, we get $-C_1 + C_2 - 4 = 1 \implies C_2 = 5 + C_1 = 5 - 6 = -1$. Thus we have

$$x(t) = t^2 - 4t + 6 - (6 + t)e^{-t}.$$

(b) We first solve the homogenous equation $\ddot{x}+4x = 0$. The characteristic equation $r^2+4 = 0$ has no solutions, so we put $\alpha = -\frac{1}{2}0 = 0$ and $\beta = \sqrt{4 - \frac{1}{2}0} = 2$. This gives $x_h = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) = C_1 \cos 2t + C_2 \sin 2t$. To find a solution of $\ddot{x} + 4x = 4t + 1$ we try $x_p = A + Bt$. This gives $\dot{x}_p = B$ and $\ddot{x}_p = 0$. Putting this into the equation, we find that

$$\ddot{x}_p + 4x_p = 0 + 4(A + Bt) = 4A + 4Bt = 4t + 1.$$

This implies that $B = 1$ and $A = \frac{1}{4}$. Thus

$$x(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{4} + t$$

Problem 10.7.
FMEA 6.3.6 in both editions.

Solution.
$x = ue^{rt} \implies \dot{x} = \dot{u}e^{rt} + ure^{rt} = e^{rt}(\dot{u} + ur) \implies \ddot{x} = \ddot{u}e^{rt} + \dot{u}re^{rt} + r(\dot{u}e^{rt} + ure^{rt}) = e^{rt}(\ddot{u} + 2r\dot{u} + ur^2)$. From this we get

$$\begin{aligned}\ddot{x} + a\dot{x} + bx &= e^{rt}[(\ddot{u} + 2r\dot{u} + ur^2) + a(\dot{u} + ur) + bu] \\ &= e^{rt}[\ddot{u} + (2r + a)\dot{u} + (r^2 + ar + b)u]\end{aligned}$$

The characteristic equation is assumed to have one solution $r = \frac{-a}{2}$. Putting $r = \frac{-a}{2}$ into the expression we get

$$\ddot{x} + a\dot{x} + bx = e^{rt}\ddot{u}$$

So $x = ue^{rt}$ is a solution if and only if $e^{rt}\ddot{u} = 0 \Leftrightarrow \ddot{u} = 0$.
The differential equation $\ddot{u} = 0$ has the general solution $u = A + Bt$. Thus $x = (A + Bt)e^{rt}$ is the general solution of $\ddot{x} + a\dot{x} + bx = 0$.

Problem 10.8.
6.3.7 in first edition of FMEA and 6.3.8 in second edition.

Solution.
(a) Substituting $t = e^s$ transforms the equation into $x''(s) + 4x'(s) + 3x'(s) = 0$. The characteristic equation is $r^2 + 4r + 3 = 0$. The solutions are $r = -3, -1$. Thus $x(s) = C_1e^{-3t} + C_2e^{-t}$. Substituting $s = \ln t$ gives $x(t) = C_1t^{-3} + C_2t^{-1}$.
(b) Substituting $t = e^s$ transforms the equation into $x''(s) - 4x'(s) + 3x'(s) = (e^s)^2 = e^{2s}$. First we solve the homogenous equation $x''(s) - 5x'(s) + 3x'(s) = 0$. The characteristic equation is $r^2 - 4r + 3 = 0$, and has the solutions $r = 1$ and $r = 3$. Thus $x_h = C_1e^s + C_2e^{3s}$. To find a particular solution of $x''(s) - 4x'(s) + 3x(s) = (e^s)^2 = e^{2s}$ we try $x_p = Ae^{2s}$. We have $x_p' = 2Ae^{2s}$ and $x_p'' = 4Ae^{2s}$. Substituting this into the equation, gives

$$\begin{aligned}x''(s) - 4x'(s) + 3x(s) &= 4Ae^{2s} - 4 \cdot 2Ae^{2s} + 3 \cdot Ae^{2s} \\ &= -Ae^{2s}\end{aligned}$$

Thus we get $A = -1$, and

$$x(s) = C_1e^s + C_2e^{3s} - e^{2s}$$

Substituting $s = \ln t$ gives

$$x(t) = C_1t + C_2t^3 - t^2.$$

Problem 10.9.
6.3.8 in first edition of FMEA and 6.3.9 in second edition.

Solution.
If $a \neq 0$ we get the general solution

$$x = 100\frac{e^{bt}}{2ab - 3a^2 + b^2} + C_1e^{at} + C_2e^{-3at}$$

provided that $2ab - 3a^2 + b^2 \neq 0$.
When $a = 0$ and $b \neq 0$ we get the general solution

$$x = C_1 + \frac{100}{b^2}e^{bt} + C_2t$$

There are also some other cases to consider, see answers in FMEA.

— Solutions to Exercise Problems —

11. Difference equations

Problem 11.1.
FMEA 11.1.1 in both editions.

Solution.
(a) The difference equation

$$x_{t+1} = ax_t + b$$

has the general solution

$$x_t = a^t \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$$

when $a \neq 1$. For the equation

$$x_{t+1} = 2x_t + 4, \ \ x_0 = 1$$

we obtain

$$\begin{aligned} x_t &= 2^t \left(1 - \frac{4}{1-2} \right) + \frac{4}{1-2} \\ &= 5 \cdot 2^t - 4. \end{aligned}$$

(b) We rewrite the equation as follows:

$$3x_{t+1} = x_t + 2 \iff x_{t+1} = \frac{1}{3}x_t + \frac{2}{3}$$

From the formula and $x_0 = 2$, we get

$$\begin{aligned} x_t &= a^t \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a} \\ &= \left(\frac{1}{3} \right)^t \left(2 - \frac{\frac{2}{3}}{1 - \frac{1}{3}} \right) + \frac{\frac{2}{3}}{1 - \frac{1}{3}} \\ &= \left(\frac{1}{3} \right)^t + 1. \end{aligned}$$

(c) We rewrite the equation as

$$2x_{t+1} + 3x_t + 2 = 0 \iff x_{t+1} = -\frac{3}{2}x_t - 1.$$

We get ($x_0 = -1$),

$$\begin{aligned} x_t &= a^t \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a} \\ &= \left(-\frac{3}{2} \right)^t \left(-1 - \frac{-1}{1 - (-\frac{3}{2})} \right) + \frac{-1}{1 - (-\frac{3}{2})} \\ &= -\frac{3}{5} \left(-\frac{3}{2} \right)^t - \frac{2}{5} \end{aligned}$$

(d) We rewrite as

$$x_{t+1} - x_t + 3 = 0 \iff x_{t+1} = x_t - 3.$$

(We cannot use the formula applied above.) In this case we get

$$x_t = x_0 + tb = 3 - 3t.$$

Problem 11.2.
FMEA 11.2.1 in both editions.

Solution.
We have to solve the difference equation

$$a_{t+1} = (1 + 0.2)a_t + 100 - 50,\ a_0 = 1000$$

or equivalently

$$a_{t+1} = 1.2a_t + 50,\ a_0 = 1000$$

The solution is

$$\begin{aligned} a_t &= (1.2)^t(1000 - \frac{50}{1-1.2}) + \frac{50}{1-1.2} \\ &= (1.2)^t 1250 - 250 \end{aligned}$$

Problem 11.3.
FMEA 11.3.1 in both editions.

Solution.
(a) We substitute $x_t = A + B2^t$ into $x_{t+2} - 3x_{t+1} + 2x_t$ and obtain

$$\begin{aligned} (A + B2^{t+2}) - 3(A + B2^{t+1}) + 2(A + B2^t) &= 2B2^t - 3B2^{t+1} + B2^{t+2} \\ &= 2^t B(2 - 3 \cdot 2 + 2^2) \\ &= 0 \end{aligned}$$

(b) We substitute $x_t = A3^t + B4^t$ into $x_{t+2} - 7x_{t+1} + 12x_t$ and obtain

$$\begin{aligned} &(A3^{t+2} + B4^{t+2}) - 7(A3^{t+1} + B4^{t+1}) + 12(A3^t + B4^t) \\ &= 12A3^t + 12B4^t - 7A3^{t+1} + A3^{t+2} - 7B4^{t+1} + B4^{t+2} \\ &= 3^t(12A - 7A \cdot 3 + A \cdot 3^2) + 4^t(12B - 7B \cdot 4 + B \cdot 4^2) \\ &= 0 \end{aligned}$$

Problem 11.4.
FMEA 11.3.2 in both editions.

Solution.
First we prove that $x_t = A + Bt$ is a solution by substituting it into the equation. We get

$$\begin{aligned} x_{t+2} - 2x_{t+1} + x_t &= (A + B(t + 2)) - 2(A + B(t + 1)) + (A + Bt) \\ &= A - 2A + A + Bt + 2B - 2Bt - 2B + Bt \\ &= 0 \end{aligned}$$

To prove that his is the general solution, we must show that any solution may be written as

$$x_t = A + Bt$$

for some A and B. Assume that x'_t is any solution of the difference equation. Since for any solution x_t, we have $x_{t+2} = 2x_{t+1} - x_t$, we see if for some choice of A and B, $x_0 = x'_0$ and $x_1 = x'_1$ then $x_t = x'_t$ for all t. We get

$$x'_0 = x_0 = A \text{ and } x'_1 = x_1 = A + B.$$

Thus $B = x'_1 - x'_0$ and $A = x'_0$ gives $x_t = x'_t$.

Problem 11.5.
FMEA 11.4.1ab in both editions.

Solution.
(a) The characteristic equation is $r^2 - 6r + 8 = 0$ and has solution: $r_1 = 2$ and $r_2 = 4$. Thus we get general solution

$$x_t = A2^t + B4^t.$$

(b) The characteristic equation is $r^2 - 8r + 16 = 0$ and has one solution $r = 4$. Thus we get the general solution

$$x_t = (A + Bt)4^t$$

Problem 11.6.
FMEA 11.4.5 in both editions.

Solution.
The characteristic equation is

$$r^2 - 4(ab + 1)r + 4a^2b^2 = 0$$

This has the solutions $r_1 = 2ab - 2\sqrt{2ab+1} + 2$ and $r_2 = 2ab + 2\sqrt{2ab+1} + 2$. The general solution is

$$D_n = Ar_1^n + Br_2^n$$

assuming $2ab + 1 > 0$.

— Solutions to Exercise Problems —

12. More on difference equations

Problem 12.1.
Find the general solution of the difference equation $3x_{t+2} - 12x_t = 4$.

Solution.
We have that

$$3x_{t+2} - 12x_t = 4 \Longleftrightarrow x_{t+2} - 4x_t = \frac{4}{3}$$

The characteristic equation is

$$3r^2 - 12 = 0 \Longleftrightarrow r^2 - 4 = 0 \Longleftrightarrow r = \pm 2$$

The general homogenous solution is

$$x_t^{(h)} = A \cdot (-2)^t + B \cdot 2^t$$

We search for a particular solution on the form

$$x_t^{(p)} = c$$

We get that

$$c - 4c = \frac{4}{3} \Longleftrightarrow -3c = \frac{4}{3} \Longleftrightarrow c = -\frac{4}{9}$$

Thus we

$$x_t = A \cdot (-2)^t + B \cdot 2^t - \frac{4}{9}$$

Problem 12.2.
FMEA 11.4.2a in both editions.

Solution.
The characteristic equation is

$$r^2 + 2r + 1 = 0$$

The solution is $r = -1$.The general homogenous solution is

$$x_t^{(h)} = (A + tB) \cdot (-1)^t$$

We search for a particular solution of the form

$$x_t^{(p)} = c \cdot 2^t$$

Substituting into the equation, we get

$$c \cdot 2^{t+2} + 2c \cdot 2^{t+1} + c \cdot 2^t = c2^t(4 + 4 + 1) = 9c \cdot 2^t \implies c = 1$$

Thus we get

$$x_t = (A + tB) \cdot (-1)^t + 2^t$$

Problem 12.3.
FMEA 11.4.7b in both editions.

Solution.
The characteristic equation is

$$r^2 - r - 1 = 0$$

has solutions $r_1 = \frac{1}{2}\sqrt{5} + \frac{1}{2} = 1.618$ and $r_2 = \frac{1}{2} - \frac{1}{2}\sqrt{5} = -0.618\,03$. Since $r_1 > 1$, the solution

$$x_t = Ar_1^t + Br_2^2 \to \infty \text{ as } t \to \infty$$

Thus the difference equation is not stable.

Part 3

Exam Problems with Solutions

CHAPTER 4

Exam Problems

This chapter contains exam problems in GRA6035 Mathematics. The exam problems consist of midterm exams and final exams for the years 2007–2010. The midterm exams are multiple choice exams and counts for 20% of the grade, and the final exams are ordinary written exams, and count for 80% of the grade. Solutions to the exam problems are given in Chapter 5.

NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Multiple-choice examination in:	**GRA 60352 Mathematics (Mid-term exam (20%))**
Examination date:	03.10.07, 14:00 – 15:00
Permitted examination aids:	Bilingual dictionary and advanced calculator as a specific calculator defined in the student handbook
Answer sheets:	Answer sheet for multiple choice examinations
Total number of pages:	5
Number of attachments:	1 (example of how to use the answer sheet)

PLEASE READ THE FOLLOWING BEFORE YOU BEGIN!

- Students must themselves assure that the examination papers are complete.
- Students must provide the following information on the answer sheet:
  - Examination code
  - Personal initials
  - Student registration number

  The student registration number must be recorded with both the appropriate numbers and by putting an "X" by the corresponding number in the columns below.
- Pens with green ink and pencils cannot be used in filling in answer sheets. Answer sheets must not be used for writing rough drafts.
- **All answers must be recorded with an "X" under the letter you believe corresponds with the correct answer.**
- **Cancel an "X" by filling in the box completely (boxes that are completely filled in will not be registered). "X" in two boxes for one question will be registered as a wrong answer.**
- The attached example shows you how the answer sheet would be filled in if A were the correct answer for question 1, B correct for question 2, C correct for question 3 and D correct for question 4. An "X" under E indicates that you choose not to answer question 5.
- **Your answers are to be recorded on the answer sheet. Answers written on the examination papers and not on the answer sheets will not be graded.**
- There is only *one* right answer for each question. Because the questions are weighted equally, it can be to your advantage to answer the simplest questions first.
- Wrong answers are given -1 point, unanswered questions get 0 points (indicated by an "X" next to E") and correct answers are given 3 points.
- You can keep the examination papers.

Question 1

Compute the matrix product

$$\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -1 & 5 \end{pmatrix}.$$

What is the answer?

A. $\begin{pmatrix} 6 & 6 \\ 5 & 5 \end{pmatrix}$

B. $\begin{pmatrix} 6 & 0 \\ 8 & 5 \end{pmatrix}$

C. $\begin{pmatrix} 6 & 0 \\ -2 & 5 \end{pmatrix}$

D. $\begin{pmatrix} 6 & 0 \\ 5 & 5 \end{pmatrix}$

E. I prefer not to answer.

Question 2

Let M be the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 2 \\ -1 & 7 & -4 \end{pmatrix}.$$

What is the rank of M?

A. 1
B. 2
C. 3
D. 4

E. I prefer not to answer.

Question 3

Consider the following system of linear equations

$$\begin{aligned} 2x_2 - 4x_3 + x_4 &= 3 \\ 5x_1 - 5x_2 + 2x_3 + 2x_4 &= -2 \end{aligned}$$

Does the system have solutions?

A. The system has solutions and two degrees of freedom.
B. The system has no solutions.
C. The system has solutions and one degree of freedom.
D. The system has exactly one solution.

E. I prefer not to answer.

Question 4

Consider the two vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

Are the vectors, u and v, linearly independent?

A. No, only three vectors in $\mathbb{R}^3$ can be linearly independent.
B. Yes.
C. No, since $\mathbf{u}$ is a multiple of $\mathbf{v}$.
D. No, since $\mathbf{u}$ is not a multiple of $\mathbf{v}$.

E. I prefer not to answer.

Question 5

Consider the function

$$Q(x_1, x_2, x_3) = 2x_1^2 - x_2^2 + x_3^2.$$

Is this a quadratic form?

A. No, this is not a quadratic form.
B. Yes, it is a quadratic form and it is positive definite.
C. Yes, it is a quadratic form and it is negative definite.
D. Yes, it is a quadratic form, but it is neither positive nor negative definite.

E. I prefer not to answer.

Question 6

Let M be the matrix

$$M = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}.$$

What is $\mathrm{adj}(M)$**?**

A. $\begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$

B. $\begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$

C. $\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$

D. $\begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix}$

E. I prefer not to answer.

Question 7

Consider the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}$$

Is M diagonalizable?

A. No.
B. Yes, because M is symmetric.
C. Yes, because M has three distinct eigenvalues.
D. Yes, because $|M| = 0$.

E. I prefer not to answer.

Question 8

Let

$$M = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 3 & -1 \\ 2 & 0 & 3 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

Are u and v eigenvectors for M?

A. Yes, **u** and **v** are eigenvectors for M.
B. No, only **v** is an eigenvector for M.
C. No, only **u** is an eigenvector for M.
D. No, neither **u** nor **v** is an eigenvector for M.

E. I prefer not to answer.

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Written examination in:	GRA 60353 Mathematics (Final exam (80%))
Examination date:	10.12.07, 09:00 – 12:00
Permitted examination aids:	Bilingual dictionary and advanced calculator as a specific calculator defined in the student handbook
Answer sheets:	Squares
Total number of pages:	2

Question 1

Let A and B be two matrices defined by

$$A = \begin{pmatrix} -9 & 0 & 5 \\ -4 & 1 & 2 \\ -2 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & t & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(a) Compute the determinant $|A|$ and the determinant $|B|$. Compute AB.

(b) What is the rank of A? Determine the values of t for which B has rank 2. Determine the values of t for which the rank of AB is 2.

Two firms numbered 1 and 2 share the market for a certain commodity. In course of the next year, the following changes occur:

$$\begin{cases} \text{Firm 1 keeps 25 \% of its customers, while losing 75 \% to Firm 2.} \\ \text{Firm 2 keeps 50 \% of its customers, while losing 50 \% to Firm 1.} \end{cases}$$

We can represent market shares of the two firms by means of *a market share vector,* defined as a column vector whose components are all nonnegative and sum to 1. Define the *transition matrix* T and the initial share vector $\mathbf{s}$ by

$$T = \begin{pmatrix} 0.25 & 0.50 \\ 0.75 & 0.50 \end{pmatrix} \text{ and } \mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}.$$

(c) Show that T has an eigenvector with eigenvalue 1, and find such an eigenvector $\mathbf{v}$ which is also a market share vector. How will the marked shares develop when $\mathbf{s} = \mathbf{v}$?

Let M be any two by two matrix such that it has an eigenvector $\mathbf{v}_1$ with eigenvalue 1 and an eigenvector $\mathbf{v}_2$ with eigenvalue 2.

(d) Show directly from the definition of linear independence and the definition of an eigenvector that $\mathbf{v}_1$ and $\mathbf{v}_2$ must be linearly independent.

Question 2

Consider the function

$$f(x_1, x_2, x_3) = x_3^3 + x_1^2 + x_2^2 + x_3^2 + 2x_2x_3 - 2x_1 + 12x_2$$

(a) Find f_1', f_2' and f_3'. Show that $(1, -8, 2)$ and $(1, -4, -2)$ are the only stationary points of f.

(b) Classify the stationary points.

Consider the function g defined by

$$g(x, y) = x^2 + 4xy + 4y^2 + e^y - y$$

(c) Show that the function g is convex.

(d) Does g have a global minimum or maximum value? If this is the case, then find this value.

Question 3

(a) Find the solution of

$$\dot{x} = (t-2)x^2$$

that satisfies $x(0) = 1$.

(b) Find the general solution of the second-order differential equation

$$\ddot{x} - 5\dot{x} + 6x = e^{7t}.$$

(c) Find the general solution of the first-order differential equation

$$\dot{x} + 2tx = te^{-t^2+t}.$$

(d) Find the solution of

$$3x^2e^{x^3+3t}\dot{x} + 3e^{x^3+3t} - 2e^{2t} = 0$$

with $x(1) = -1$.

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Multiple-choice examination in:	**GRA 60352 Mathematics (Mid-term exam (20%))**
Examination date:	01.10.08, 14:00 – 15:00
Permitted examination aids:	Bilingual dictionary and advanced calculator as a specific calculator defined in the student handbook
Answer sheets:	Answer sheet for multiple choice examinations
Total number of pages:	5
Number of attachments:	1 (example of how to use the answer sheet)

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This exam has 8 questions.

Question 1

Compute the matrix product

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

What is the answer?

A. $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

B. $\begin{pmatrix} 4 & 2 \\ -2 & -1 \end{pmatrix}$

C. $\begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$

D. $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

E. I prefer not to answer.

Question 2

Let M be the matrix

$$M = \begin{pmatrix} 1 & -3 & 5 & 0 \\ 3 & 0 & 0 & 0 \\ -2 & -6 & -10 & 0 \end{pmatrix}.$$

What is the rank of M?

A. 1

B. 2

C. 3

D. 4

E. I prefer not to answer.

Question 3

Compute the determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & -2 & -2 & -2 \\ -3 & 2 & 2 & 2 \\ -3 & 3 & 8 & 9 \end{vmatrix}$$

if possible. **The answer is:**

A. It is not possible to compute the determinant of a 4×4 matrix.

B. 0

C. -2

D. 2

E. I prefer not to answer.

Question 4

Consider the following system of linear equations

$$\begin{array}{rrrrcl} x_1 & +x_2 & +x_3 & +x_4 & = & 4 \\ 2x_1 & +2x_2 & +3x_3 & +3x_4 & = & 8 \\ & & x_3 & +x_4 & = & 0 \end{array}$$

Does the system have solutions?

A. The system has solutions and two degrees of freedom.

B. The system has no solutions.

C. The system has solutions and one degree of freedom.

D. The system has exactly one solution.

E. I prefer not to answer.

Question 5

Which statement is *not* true?

A. The rank of an $m \times n$ matrix is less or equal to the minimum of m and n.

B. Four vectors in $\mathbb{R}^3$ are always linearly dependent.

C. If three vectors in $\mathbb{R}^4$ are linearly independent, then at least one of the vectors is a linear combination of the remaining two vectors.

D. If A is any matrix, then the rank of A is equal to the rank of the transposed matrix A^T.

E. I prefer not to answer.

Question 6

Consider the function

$$Q(x_1, x_2, x_3) = x_1^2 - x_2^2 + x_3^2 + 2x_1x_2$$

Is this a quadratic form?

A. No, this is not a quadratic form.

B. Yes, it is a quadratic form and it is positive definite.

C. Yes, it is a quadratic form and it is negative definite.

D. Yes, it is a quadratic form, but it is neither positive nor negative definite.

E. I prefer not to answer.

Question 7

Consider the matrices

$$M = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \text{ and } N = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

Are M and N diagonalizable?

A. Both M and N are diagonalizable.

B. M is diagonalizable, but N is not diagonalizable.

C. M is not diagonalizable, but N is diagonalizable.

D. Neither of the matrices are diagonalizable.

E. I prefer not to answer.

Question 8

Which function is neither convex nor concave?

A. $f(x, y) = x^2 + y^2$

B. $f(x, y) = -x^2 - y^2$

C. $f(x, y) = -x^2 + y^2$

D. $f(x, y) = x^2 + y^2 - x$

E. I prefer not to answer.

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Written examination in:	GRA 60353 Mathematics (Final exam (80%))
Examination date:	10.12.08, 09:00 – 12:00
Permitted examination aids:	Bilingual dictionary and advanced calculator as a specific calculator defined in the student handbook
Answer sheets:	Squares
Total number of pages:	2

Question 1

Let B and C be two matrices defined by

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \text{ and } C = \begin{pmatrix} c & -5 \\ 5 & 1 \end{pmatrix}$$

(a) Compute BC and CB. For which values of c do we have that $BC = CB$?

Let

$$A = \begin{pmatrix} -2 & 6 \\ 3 & 1 \end{pmatrix}$$

(b) Write down the characteristic equation and find the eigenvalues of A.

(c) Find the eigenvectors of A. Is A diagonalizable? If so, find a matrix P such that $D = P^{-1}AP$ is a diagonal matrix and find D.

(d) Let E be a square matrix and assume that λ is an eigenvalue of E. Show that λ^2 is an eigenvalue of E^2.

Question 2

Let

$$h(x, y, z) = y^4 + x^2 + 2x + y^2 + yz - 1$$

(a) Find the Hessian matrix $\mathbf{h}''(x, y, z)$.

(b) Show that h has a unique stationary point and classify this point as a local maximum, local minimum or a saddle point.

Consider the problem

$$\max f(x, y, z) = 2z \text{ subject to } \begin{cases} g_1(x, y, z) = x^2 + y^2 = 2 \\ g_2(x, y, z) = x + y + z = 1 \end{cases}$$

(c) Write down the first order conditions using the Lagrangian

$$\mathcal{L}(x, y, z) = f(x, y, z) - \lambda_1 g_1(x, y, z) - \lambda_2 g_2(x, y, z)$$

and show that one obtains $\lambda_2 = 2$, $x = -\frac{1}{\lambda_1}$ and $y = -\frac{1}{\lambda_1}$

(d) Substitute $x = -\frac{1}{\lambda_1}$ and $y = -\frac{1}{\lambda_1}$ into one of the constraints to obtain $\lambda_1 = \pm 1$ and for each of the different sets of values for the multipliers, find out if $\mathcal{L}$ is convex or concave. Solve the maximization problem.

Question 3

(a) Find the general solution of the following differential equation

$$\dot{x} + \frac{1}{t}x = 2 \qquad (t > 0)$$

(b) Find the general solution of the linear second order differential equation

$$\ddot{x} + 3\dot{x} + 2x = 2t + 5$$

(c) Solve the difference equation

$$x_{t+2} = 4x_{t+1} - 4x_t, \qquad x_0 = 0, x_1 = 2$$

(d) Consider the differential equation

$$\dot{x} + 2x^2 = 0, \; x(0) = 1$$

Explain why this is not a *linear* differential equation, and solve it as a separable differential equation.

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Multiple-choice examination in:	**GRA 60352 Mathematics (Mid-term exam (20%))**
Examination date:	29.04.09, 16:00 – 17:00
Permitted examination aids:	Bilingual dictionary and advanced calculator as a specific calculator defined in the student handbook
Answer sheets:	Answer sheet for multiple choice examinations
Total number of pages:	5
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This exam has 8 questions.

Question 1

Let M be the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 \\ -6 & 2 & 5 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}.$$

What is the rank of M?

A. 1

B. 2

C. 3

D. 4

E. I prefer not to answer.

Question 2

Let M be the matrix

$$M = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.$$

Compute the eigenvalues of M. **Which statement is correct?**

A. The matrix M has no real eigenvalues.

B. The eigenvalues are -1 and 4.

C. The eigenvalues are 2 and 1.

D. The matrix M has only one real eigenvalue.

E. I prefer not to answer.

Question 3

Compute the determinant

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & -2 & -2 \\ -3 & 0 & 2 & 2 \\ -3 & 0 & 1 & 2 \end{vmatrix}$$

if possible. **The answer is:**

A. It is not possible to compute the determinant of a 4×4 matrix.

B. 0

C. -4

D. 2

E. I prefer not to answer.

Question 4

Which function is both convex and concave?

A. $f(x,y) = x^2 + y^2$

B. $f(x,y) = -x^2 - y^2$

C. $f(x,y) = -x^2 + y^2$

D. $f(x,y) = x + y$

E. I prefer not to answer.

Question 5

Let

$$A = \begin{pmatrix} 1 & -3 \\ 3 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & -2 \\ 2 & 1 \end{pmatrix}$$

Compute $(AB)^{-1}A$. **The answer is:**

A. $\begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$

B. $\begin{pmatrix} 1 & -3 \\ 3 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 0 & -2 \\ 2 & 1 \end{pmatrix}$

D. $\begin{pmatrix} -\frac{1}{6} & \frac{1}{2} \\ -\frac{23}{36} & \frac{5}{12} \end{pmatrix}$

E. I prefer not to answer.

Question 6

Let

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 1 & 4 \\ -2 & -6 & -10 \end{pmatrix}$$

and let B be any 3×3 matrix. **Which statement is correct?**

A. The matrix product AB is not defined.

B. The columns of the matrix AB are linearly independent.

C. The columns of the matrix AB are linearly dependent.

D. It is not possible to decide if the columns of the matrix AB are linearly independent without knowing B.

E. I prefer not to answer.

Question 7

Let

$$M = \begin{pmatrix} 4 & 3 & -5 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Which statement is correct?

A. $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of M.

B. Only $\mathbf{v}$ is an eigenvector of M.

C. $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of M.

D. Only $\mathbf{w}$ is an eigenvector of M.

E. I prefer not to answer.

Question 8

Let $f(x_1, x_2, x_3)$ and $g(x_1, x_2, x_3)$ be positive definite quadratic forms.

Which statement is correct?

A. The function $f(x_1, x_2, x_3) + g(x_1, x_2, x_3)$ need not be a quadratic form.

B. The function $f(x_1, x_2, x_3) + g(x_1, x_2, x_3)$ is a positive definite quadratic form.

C. The function $f(x_1, x_2, x_3) + g(x_1, x_2, x_3)$ is a negative semidefinite quadratic form.

D. The function $f(x_1, x_2, x_3) + g(x_1, x_2, x_3)$ is a quadratic form, but it need not be definite.

E. I prefer not to answer.

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Written examination in:	GRA 60353 Mathematics (Final exam (80%))
Examination date:	04.05.09, 13:00 – 16:00
Permitted examination aids:	Bilingual dictionary. BI-approved exam calculator: TEXAS INSTRUMENTS BA II Plus™
Answer sheets:	Squares
Total number of pages:	2

Question 1

Consider the following system of linear equations

$$\begin{array}{rrrcl} x_1 & & -30x_3 & = & 0 \\ -2x_1 & -10x_2 & +70x_3 & = & 0 \\ -x_1 & -10x_2 & +40x_3 & = & 0 \end{array}$$

(a) Write down the coefficient matrix of the system. Solve the system, and state the number of degrees of freedom. What is the rank of the coefficient matrix?

Consider the quadratic form

$$Q(x_1, x_2, x_3) = ax_1^2 + x_2^2 + ax_3^2 + 2(a-2)x_1x_3$$

(b) Find a matrix A such that $Q = \mathbf{x}^T A\mathbf{x}$. For which values of a is the quadratic form positive definite?

Define

$$T = \begin{pmatrix} 1 & 4 \\ c & 1 \end{pmatrix}.$$

(c) For which values of c does T have two distinct eigenvalues?

(d) Find a value of c such that T is not diagonalizable.

Question 2

Consider the function f defined on the subset $S = \{(x, y, z) : z > 0\}$ of $\mathbb{R}^3$ by

$$f(x, y, z) = x^2 - 2x + y^2 + z^3 - 3z.$$

(a) Show that the subset S is convex. Find the stationary points of f.

(b) Find the Hessian matrix of f. Is f concave or convex? Does f have a global extreme point? Justify your answer.

Consider the problem

$$\max f(x, y) = \ln(x + 1) + \ln(y + 1) \text{ subject to } \begin{cases} y \leq 5 \\ x + y \leq 2 \end{cases}$$

(c) Write down the necessary Kuhn-Tucker conditions.

(d) Solve the problem.

Question 3

(a) Find the general solution of the following differential equation

$$\dot{x} + tx = 3t \qquad (t > 0)$$

(b) Find the general solution of the linear second order differential equation

$$\ddot{x} + 5\dot{x} + 6x = e^t$$

(c) Solve the following difference equation:

$$x_{t+1} - 3x_t - 2 = 0, \quad x_0 = 0$$

(d) Consider the following system of difference equations:

$$x_{t+1} = x_t + 3y_t$$
$$y_{t+1} = 2x_t$$

with $x_0 = 1$ and $y_0 = 0$. Derive a second order difference equation for x_t and solve this equation and the system.

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Multiple-choice examination in:	**GRA 60352 Mathematics (Mid-term exam (20%))**
Examination date:	28.09.09, 14:00 – 15:00
Permitted examination aids:	Bilingual dictionary. BI-approved exam calculator: TEXAS INSTRUMENTS BA II Plus™
Answer sheets:	Answer sheet for multiple choice examinations
Total number of pages:	5
Number of attachments:	1 (example of how to use the answer sheet)

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This exam has 8 questions.

Question 1

Consider the matrix

$$A = \begin{pmatrix} -8 & 10 & -4 \\ -6 & 8 & -4 \\ 0 & 0 & -2 \end{pmatrix}.$$

Compute the cofactor A_{33} . **What is the answer?**

A. $\begin{pmatrix} -8 & -4 \\ -6 & -4 \end{pmatrix}$

B. -4

C. 4

D. None of the above.

E. I prefer not to answer.

Question 2

Let M be the matrix

$$M = \begin{pmatrix} 1 & 0 & 5 & 9 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 6 & 8 \\ 3 & 0 & 7 & 7 \end{pmatrix}.$$

What is the rank of M?

A. 1

B. 2

C. 3

D. 4

E. I prefer not to answer.

Question 3

Consider the matrix

$$A = \begin{pmatrix} -8 & 10 & -4 \\ -6 & 8 & -4 \\ 0 & 0 & -2 \end{pmatrix}.$$

Compute the eigenvalues of A. **Which statement is true?**

A. The matrix A has no eigenvalues.

B. The matrix A has exactly one eigenvalue.

C. The matrix A has exactly two distinct eigenvalues.

D. The matrix A has three distinct eigenvalues.

E. I prefer not to answer.

Question 4

Consider the following system of linear equations

$$\begin{array}{rrrcl} x_1 & +5x_2 & +9x_3 & = & 1 \\ 2x_1 & +6x_2 & +8x_3 & = & 2 \\ 3x_1 & +7x_2 & +7x_3 & = & 4 \end{array}$$

Does the system have solutions?

A. The system has no solutions.

B. The system has solutions and one degree of freedom.

C. The system has solutions and two degrees of freedom.

D. The system has exactly one solution.

E. I prefer not to answer.

Question 5

Consider the function

$$f(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + x_3^4 + 2x_1x_2$$

Compute the Hessian matrix $\mathbf{f}''(\mathbf{x})$ of f. **What is the correct answer?**

A. The Hessian matrix $\mathbf{f}''(\mathbf{x})$ of f is positive semidefinite for all $\mathbf{x}$.

B. The Hessian matrix $\mathbf{f}''(\mathbf{x})$ of f can be both positive and negative semidefinite.

C. The Hessian matrix $\mathbf{f}''(\mathbf{x})$ of f is negative semidefinite for all $\mathbf{x}$.

D. The Hessian matrix $\mathbf{f}''(\mathbf{x})$ of f can be indefinite.

E. I prefer not to answer.

Question 6

Consider the matrix

$$A = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & -10 \\ 0 & 0 & -2 \end{pmatrix}.$$

This matrix has the eigenvectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

where $\mathbf{u}$ and $\mathbf{v}$ have the eigenvalue $\lambda = 8$ and $\mathbf{w}$ has the eigenvalue $\lambda = -2$. **Which statement is correct?**

A. The matrix A does not have three distinct eigenvalues. Hence it is not diagonalizable.

B. The matrix A does not have three linearly independent eigenvectors, and it is not diagonalizable.

C. The matrix A is diagonalizable.

D. The matrix A is not invertible.

E. I prefer not to answer.

Question 7

Assume that A, B and C are $n \times n$ matrices. Simplify the following matrix expression:

$$C(A+B)(A-B) - CA^2 + (CB + CA)B$$

The answer can be written as:

A. 0

B. ABC

C. $-CBA$

D. CBA

E. I prefer not to answer.

Question 8

Consider the function

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^2$$

Is this a quadratic form?

A. No, this is not a quadratic form.

B. Yes, it is a quadratic form and it is positive definite.

C. Yes, it is a quadratic form and it is positive semidefinite.

D. Yes, it is a quadratic form, but it is indefinite.

E. I prefer not to answer.

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Written examination in:	GRA 60353 Mathematics (Final exam (80%))
Examination date:	10.12.09, 9:00 – 12:00
Permitted examination aids:	Bilingual dictionary. BI-approved exam calculator: TEXAS INSTRUMENTS BA II Plus™
Answer sheets:	Squares
Total number of pages:	2

Question 1

(a) For which values of t are the three vectors

$$\begin{pmatrix} 1 \\ t \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -t \\ -1 \\ -1 \end{pmatrix}$$

linearly dependent?

(b) Find c_1, c_2 such that

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

(c) Determine if

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

is diagonalizable.

Question 2

Consider the function f defined by

$$f(x_1, x_2, x_3) = x_1 + x_2^2 + x_3^3 - x_1 x_2 - 3x_3$$

(a) Find the stationary points of f.

(b) Classify each stationary point of f as a local maximum point, a local minimum point or a saddle point.

Consider the function

$$f(x, y; p) = -px + e^{x+y^2}$$

which we view as a function in two variables x and y with an undetermined parameter p.

(c) Determine if f is convex or concave as a function in x and y.

(d) The function f has a global minimum point. Find this point $(x^*(p), y^*(p))$ and determine the corresponding minimal value function $f^*(p)$.

(e) Find the derivative of the minimal value function with respect to p.

Question 3

(a) Find the general solution of the differential equation

$$\dot{x} + at\,x = 2t$$

with parameter $a \neq 0$. What is the general solution if $a = 0$?

(b) Solve the following initial value problem:

$$\ddot{x} + 2\dot{x} + x = 4e^t, \quad x(0) = 1, \quad \dot{x}(0) = 2$$

(c) Solve the difference equation

$$x_{t+1} - x_t = rx_t + s, \quad x_0 = 100s$$

with parameters $r, s > 0$.

(d) Solve the following difference equation:

$$x_{t+2} + 2x_{t+1} + x_t = 4t + 4, \quad x_0 = 1, \quad x_1 = 1$$

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Multiple-choice examination in:	**GRA 60352 Mathematics (Mid-term exam (20%))**
Examination date:	29.04.10, 9:00 – 10:00
Permitted examination aids:	Bilingual dictionary. BI-approved exam calculator: TEXAS INSTRUMENTS BA II Plus^{TM}
Answer sheets:	Answer sheet for multiple choice examinations
Total number of pages:	5
Number of attachments:	1 (example of how to use the answer sheet)

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This exam has 8 questions.

Question 1

Let M be the matrix

$$M = \begin{pmatrix} 1 & 2 & 9 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 3 & 5 & 8 \\ 3 & 0 & 7 & 0 \end{pmatrix}.$$

What is the rank of M?

A. 1

B. 2

C. 3

D. 4

E. I prefer not to answer.

Question 2

Consider the matrix

$$A = \begin{pmatrix} -8 & 10 & -4 & 1 \\ 0 & -6 & 8 & -4 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Compute the eigenvalues of A. **Which statement is true?**

A. The matrix A has exactly one eigenvalue.

B. The matrix A has exactly two distinct eigenvalue.

C. The matrix A has exactly three distinct eigenvalues.

D. The matrix A has four distinct eigenvalues.

E. I prefer not to answer.

Question 3

Let A be a 5×4-matrix of rank four, and consider the following system of linear equations

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

Does the system have solutions?

A. The system has no solutions.

B. The system has exactly one solution.

C. The system has solutions and one degree of freedom.

D. The number of solutions depends on the entries in the matrix A.

E. I prefer not to answer.

Question 4

Consider the matrix

$$A = \begin{pmatrix} -8 & 10 & -4 \\ -6 & 8 & -4 \\ 0 & 0 & -2 \end{pmatrix}$$

and the three vectors

$$\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ \frac{5}{2} \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -\frac{3}{2} \end{pmatrix} \quad \mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Which statment is true?

A. The vectors $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for the matrix A, but $\mathbf{w}$ is not an eigenvector for A.

B. None of the vectors are eigenvectors for the matrix A.

C. All of the three vectors are eigenvectors for the matrix A.

D. Only $\mathbf{w}$ is an eigenvector for the matrix A.

E. I prefer not to answer.

Question 5

Consider the function

$$f(x_1, x_2, x_3) = x_1^2 + 2x_2^3 + x_3^3 + 2x_1x_2$$

Compute the Hessian matrix $\mathbf{f}''(\mathbf{x})$ of f. **What is the correct answer?**

A. The Hessian matrix $\mathbf{f}''(\mathbf{x})$ of f is positive semidefinite for all $\mathbf{x}$.

B. The Hessian matrix $\mathbf{f}''(\mathbf{x})$ of f can be both positive and negative semidefinite.

C. The Hessian matrix $\mathbf{f}''(\mathbf{x})$ of f is negative semidefinite for all $\mathbf{x}$.

D. The Hessian matrix $\mathbf{f}''(\mathbf{x})$ of f can be indefinite.

E. I prefer not to answer.

Question 6

Which statement is *not* true?

A. If λ is an eigenvalue of an invertibel matrix A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

B. If $|A| = 0$ then $\lambda = 0$ is an eigenvalue of A.

C. If A is an invertible $n \times n$ matrix, then A has n distinct (different) eigenvalues.

D. If λ is an eigenvalue of a matrix A, then λ^3 is an eigenvalue of A^3.

E. I prefer not to answer.

Question 7

Assume that A and B are $n \times n$ matrices. Simplify the following matrix expression:

$$(A+2B)(A-2B)-(2A+B)^2$$

The answer can be written as:

A. 0

B. $-3A^2-4AB-5B^2$

C. $-3A^2+4AB-5B^2$

D. $-3A^2-6AB+2BA-5B^2$

E. I prefer not to answer.

Question 8

Consider the matrices

$$M=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } N=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Are M and N diagonalizable?

A. Both M and N are diagonalizable.

B. M is diagonalizable, but N is not diagonalizable.

C. M is not diagonalizable, but N is diagonalizable.

D. Neither of the matrices are diagonalizable.

E. I prefer not to answer.

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Written examination in:	**GRA 60353 Mathematics (Final exam (80 %))**
Examination date:	04.05.10, 9:00 – 12:00
Permitted examination aids:	Bilingual dictionary. BI-approved exam calculator: TEXAS INSTRUMENTS BA II Plus™
Answer sheets:	Squares
Total number of pages:	2

Question 1

Let

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 3 & x & 3 \\ 2 & 3 & 2 \end{pmatrix}$$

(a) Discuss the rank of A for different values of x.

(b) Explain that A is a symmetric matrix. Show that A is indefinte for all values of x.

(c) Two firms numbered 1 and 2 share the market for a certain commodity. In the course of each week, the following changes occur:

$$\begin{cases} \text{Firm 1 keeps } \frac{1}{3} \text{ of its customers, while losing } \frac{2}{3} \text{ to Firm 2.} \\ \text{Firm 2 keeps } \frac{1}{4} \text{ of its customers, while losing } \frac{3}{4} \text{ to Firm 1.} \end{cases}$$

Show that in the long run, Firm 1 will have $\frac{9}{17}$ of the customers.

(d) Let E be a square matrix and assume that 2 is an eigenvalue of E. Show that 10 is an eigenvalue of the matrix $E^3 + E$.

Question 2

Let

$$h(x, y, z) = 2x^3 - 9x^2 + 12x + y^2 - yz + z^2$$

(a) Find the Hessian matrix $\mathbf{h}''(x, y, z)$.

(b) Find and classify the stationary points of h.

Consider the problem

$$\max f(x, y, z) = 2z^3 - 9y^2 + 12x \text{ subject to } \begin{cases} g_1(x, y, z) = z - x = 0 \\ g_2(x, y, z) = y - z = 0 \end{cases}$$

(c) Write down the first order conditions using the Lagrangian

$$\mathcal{L}(x, y, z) = f(x, y, z) - \lambda_1 g_1(x, y, z) - \lambda_2 g_2(x, y, z)$$

and show that one obtains $\lambda_1 = -12$, $\lambda_2 = -18y$.

(d) Find all solutions of the first order conditions. Is $\mathcal{L}$ convex as a function of x, y and z? Is $\mathcal{L}$ concave as a function of x, y and z?

Question 3

(a) Find the general solution of the linear second order differential equation

$$\ddot{x} - 7\dot{x} + 12x = e^{-t}$$

(b) Solve the following difference equation:

$$x_{t+1} - 1.08x_t + 30 = 0, \quad x_0 = 800$$

(c) Solve the following difference equation:

$$x_{t+2} - 11x_{t+1} + 30x_t = 0, \quad x_0 = 0, \quad x_1 = 1$$

(d) Find the general solution of the following differential equation

$$\dot{x} + tx = t^3 \qquad (t > 0)$$

CHAPTER 5

Solutions to Exam Problems

This chapter contains solutions to exam problems in GRA6035 Mathematics. The exam problems are multiple choice midterm exams and ordinary written final exams, and are given in Chapter 4.

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Solutions:	**GRA 60352 Mathematics (Mid-term exam (20%))**
Examination date:	03.10.07, 14:00 – 15:00
Total number of pages:	5

Question 1 : D
Question 2 : C
Question 3 : A
Question 4 : B
Question 5 : D
Question 6 : A
Question 7 : C
Question 8 : C

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Solutions in:	GRA 60353 Mathematics (Final exam (80%))
Examination date:	10.12.07, 09:00 – 12:00
Permitted examination aids:	Bilingual dictionary and advanced calculator as a specific calculator defined in the student handbook
Answer sheets:	Squares
Total number of pages:	4

Question 1

Let A and B be two matrices defined by

$$A = \begin{pmatrix} -9 & 0 & 5 \\ -4 & 1 & 2 \\ -2 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & t & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(a) Compute the determinant $|A|$ and the determinant $|B|$. Compute AB.

Solution.

$$|A| = \begin{vmatrix} -9 & 0 & 5 \\ -4 & 1 & 2 \\ -2 & 0 & 1 \end{vmatrix} = 1, |B| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & t & 0 \\ 0 & 0 & 3 \end{vmatrix} = 3t$$

$$AB = \begin{pmatrix} -9 & 0 & 5 \\ -4 & 1 & 2 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & t & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -9 & 0 & 6 \\ -4 & t & 2 \\ -2 & 0 & 1 \end{pmatrix}$$

(b) What is the rank of A? Determine the values of t for which B has rank 2. Determine the values of t for which the rank of AB is 2.

Solution.
$|A| \neq 0 \implies \operatorname{rank} A = 3$.
$|B| = 3t$, so $t = 0 \iff \operatorname{rank} B < 3$.
$t = 0 \implies B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ which has rank 2 since $\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3$, so $\operatorname{rank} B = 2 \iff t = 0$.
Since A is invertible, $\operatorname{rank} AB = \operatorname{rank} B$, so $\operatorname{rank} AB = 2 \iff t = 0$.

Two firms numbered 1 and 2 share the market for a certain commodity. In course of the next year, the following changes occur:

$$\begin{cases} \text{Firm 1 keeps 25 \% of its customers, while losing 75 \% to Firm 2.} \\ \text{Firm 2 keeps 50 \% of its customers, while losing 50 \% to Firm 1.} \end{cases}$$

We can represent market shares of the two firms by means of *a market share vector,* defined as a column vector whose components are all nonnegative and sum to 1. Define the *transition matrix* T and the initial share vector $\mathbf{s}$ by

$$T = \begin{pmatrix} 0.25 & 0.50 \\ 0.75 & 0.50 \end{pmatrix} \text{ and } \mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}.$$

(c) Show that T has an eigenvector with eigenvalue 1, and find such an eigenvector $\mathbf{v}$ which is also a market share vector. How will the marked shares develop when $\mathbf{s} = \mathbf{v}$?

Solution.
$|A - I| = \begin{vmatrix} 0.25 - 1 & 0.50 \\ 0.75 & 0.50 - 1 \end{vmatrix} = \begin{vmatrix} -0.75 & 0.5 \\ 0.75 & -0.5 \end{vmatrix} = 0.$
$A - I = \begin{bmatrix} -0.75 & 0.5 \\ 0.75 & -0.5 \end{bmatrix}$, so $(A - I)\mathbf{x} = 0 \iff -0.75x_1 + 0.5x_2 = 0 \iff -0.75x_1 = -0.5x_2 \implies x_1 = \frac{-0.5}{-0.75}x_2 = \frac{2}{3}x_2$.
$\mathbf{x} = \begin{pmatrix} \frac{2}{3}x_2 \\ x_2 \end{pmatrix}$, or $\mathbf{x} = \begin{pmatrix} 2t \\ 3t \end{pmatrix}$ where t is any number. $2t + 3t = 5t = 1 \implies t = 1/5 \implies$
$\mathbf{v} = \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$.
$\mathbf{s} = \mathbf{v} \implies$ market shares will stay constant. Firm 1 has 40 % and Firm 2 has 60 %.

Let M be any two by two matrix such that it has an eigenvector $\mathbf{v}_1$ with eigenvalue 1 and an eigenvector $\mathbf{v}_2$ with eigenvalue 2.

(d) Show directly from the definition of linear independence and the definition of an eigenvector that $\mathbf{v}_1$ and $\mathbf{v}_2$ must be linearly independent.

Solution.
$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0 \implies A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2 2\mathbf{v}_2 = 0$
Subtracting the first equation form the second, we obtain $c_2\mathbf{v}_2 = 0 \implies c_2 = 0$. Then we have $c_1\mathbf{v}_1 = 0 \implies c_1 = 0$. This shows that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Question 2

Consider the function

$$f(x_1, x_2, x_3) = x_3^3 + x_1^2 + x_2^2 + x_3^2 + 2x_2x_3 - 2x_1 + 12x_2$$

(a) Find f_1', f_2' and f_3'. Show that $(1, -8, 2)$ and $(1, -4, -2)$ are the only stationary points of f.

Solution.
$f_1' = 2x_1 - 2.$
$f_2' = 2x_2 + 2x_3 + 12$
$f_3' = 3x_3^2 + 2x_3 + 2x_2$
$2x_1 - 2 = 0 \implies x_1 = 1$
$2x_2 + 2x_3 + 12 = 0 \implies 2x_2 + 2x_3 = -12$
$3x_3^2 + 2x_3 + 2x_2 = 0 \implies 3x_3^2 - 12 = 0 \implies x_2 = \pm 2$
$2x_2 + 2x_3 = -12 \implies x_2 = -6 - x_3 = -6 \mp 2$
Stationary points:

$$(1, -8, 2) \text{ and } (1, -4, -2).$$

(b) Classify the stationary points.

Solution.
Hessian matrix

$$\mathbf{f}'' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 6x_3+2 \end{pmatrix}.$$

$$\mathbf{f}''(1,-8,2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 14 \end{pmatrix}$$

$D_1 = 2 > 0,\ D_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0,\ D_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 14 \end{vmatrix} = 48 > 0 \implies (1,-8,2)$ is a local minimum.

$$\mathbf{f}''(1,-4,-2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & -10 \end{pmatrix}$$

$D_1 = 2 > 0,\ D_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0,\ D_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & -10 \end{vmatrix} = -48 < 0 \implies (1,-4,-2)$ is a saddle point.

Consider the function g defined by

$$g(x,y) = x^2 + 4xy + 4y^2 + e^y - y$$

(c) Show that the function g is convex.

Solution.
$g_1' = 2x + 4y$
$g_2' = 4x + 8y + e^y - 1$
Hessian matrix

$$\begin{pmatrix} 2 & 4 \\ 4 & 8+e^y \end{pmatrix}.$$

$D_1 = 2,\ D_2 = \begin{vmatrix} 2 & 4 \\ 4 & 8+e^y \end{vmatrix} = 2e^y > 0 \implies$ Hessian matrix is positive definite $\implies g$ is convex.

(d) Does g have a global minimum or maximum value? If this is the case, then find this value.

Solution.
$g_1' = 2x + 4y = 0 \implies y = -\frac{1}{2}x$
$g_2' = 4x+8y+e^y-1 = 0 \implies 4x+8(-\frac{1}{2}x)+e^{-\frac{1}{2}x}+1 = e^{-\frac{1}{2}x}-1 = 0 \implies x = 0 \implies y = 0.$
$(0,0)$ is the only stationary point. g convex $\implies (0,0)$ is a global minimum.
$g(0,0) = 1.$

Question 3

(a) Find the solution of

$$\dot{x} = (t-2)x^2$$

that satisfies $x(0) = 1$.

Solution.
$\dot{x} = (t-2)x^2 \implies \frac{1}{x^2}\dot{x} = t-2 \implies \int \frac{1}{x^2}dx = \int(t-2)dt \implies -\frac{1}{x} = \frac{1}{2}t^2 - 2t + C \implies x = \frac{-2}{t^2-4t+2C}$
$x(0) = \frac{-2}{2C} = \frac{-1}{C} = 1 \implies C = -1 \implies x(t) = \frac{-2}{t^2-4t-2}$

(b) Find the general solution of the second-order differential equation

$$\ddot{x} - 5\dot{x} + 6x = e^{7t}.$$

Solution.
$\ddot{x} - 5\dot{x} + 6x = 0, r^2 - 5r + 6 = 0 \implies r = 3, r = 2 \implies x_h(t) = Ae^{2t} + Be^{3t}$.
$x_p = Ce^{7t} \implies \dot{x}_p = 7Ce^{7t} \implies \ddot{x}_p = 49Ce^{7t}$
$\ddot{x}_p - 5\dot{x}_p + 6x_p = Ce^{7t}(49 - 5\cdot 7 + 6) = 20Ce^{7t} = 1 \implies C = \frac{1}{20}$
$x(t) = Ae^{2t} + Be^{3t} + \frac{1}{20}e^{7t}$.

(c) Find the general solution of the first-order differential equation

$$\dot{x} + 2tx = te^{-t^2+t}.$$

Solution.
Integrating factor $e^{t^2} \implies xe^{t^2} = \int te^{-t^2+t}e^{t^2}dt = \int te^t dt = te^t - e^t + C \implies x(t) = (te^t - e^t + C)e^{-t^2}$

(d) Find the solution of

$$3x^2e^{x^3+3t}\dot{x} + 3e^{x^3+3t} - 2e^{2t} = 0$$

with $x(1) = -1$.

Solution.
$\frac{\partial}{\partial t}(3x^2e^{x^3+3t}) = 9x^2e^{3t+x^3}$
$\frac{\partial}{\partial x}(3e^{x^3+3t} - 2e^{2t}) = 9x^2e^{3t+x^3}$
Exact.
$h'_x = 3x^2e^{x^3+3t} \implies h = e^{x^3+3t} + \alpha(t) \implies h'_t = 3e^{x^3+3t} + \alpha'(t)$
$h'_t = 3e^{x^3+3t} + \alpha'(t) = 3e^{x^3+3t} - 2e^{2t} \implies \alpha'(t) = -2e^{2t} \implies \alpha(t) = -e^{2t} + C \implies h = e^{x^3+3t} - e^{2t} + C$
$h = e^{x^3+3t} - e^{2t} = K$
$x(1) = -1 \implies e^{(-1)^3+3} - e^2 = K \implies K = 0 \implies e^{x^3+3t} - e^{2t} = 0 \implies e^{x^3+3t} = e^{2t} \implies x^3 + 3t = 2t \implies x^3 = -t \implies x(t) = \sqrt[3]{-t}$.

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Solutions:	**GRA 60352 Mathematics (Mid-term exam (20%))**
Examination date:	01.10.08, 14:00 – 15:00
Permitted examination aids:	Bilingual dictionary and advanced calculator as a specific calculator defined in the student handbook
Answer sheets:	Answer sheet for multiple choice examinations
Total number of pages:	5
Number of attachments:	1 (example of how to use the answer sheet)

PLEASE READ THE FOLLOWING BEFORE YOU BEGIN!

- Students must themselves assure that the examination papers are complete.
- Students must provide the following information on the answer sheet:
  - Examination code
  - Personal initials
  - Student registration number

  The student registration number must be recorded with both the appropriate numbers and by putting an "X" by the corresponding number in the columns below.
- Pens with green ink and pencils cannot be used in filling in answer sheets. Answer sheets must not be used for writing rough drafts.
- **All answers must be recorded with an "X" under the letter you believe corresponds with the correct answer.**
- **Cancel an "X" by filling in the box completely (boxes that are completely filled in will not be registered). "X" in two boxes for one question will be registered as a wrong answer.**
- The attached example shows you how the answer sheet would be filled in if A were the correct answer for question 1, B correct for question 2, C correct for question 3 and D correct for question 4. An "X" under E indicates that you choose not to answer question 5.
- **Your answers are to be recorded on the answer sheet. Answers written on the examination papers and not on the answer sheets will not be graded.**
- There is only *one* right answer for each question. Because the questions are weighted equally, it can be to your advantage to answer the simplest questions first.
- Wrong answers are given -1 point, unanswered questions get 0 points (indicated by an "X" next to E") and correct answers are given 3 points.
- You can keep the examination papers.

This exam has 8 questions.

Question 1

Compute the matrix product

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

What is the answer?

A. $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

B. $\begin{pmatrix} 4 & 2 \\ -2 & -1 \end{pmatrix}$

C. $\begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$

$\boxed{\text{D.}}$ $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

E. I prefer not to answer.

Solution.

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \to \boxed{\text{D}}$$

Question 2

Let M be the matrix

$$M = \begin{pmatrix} 1 & -3 & 5 & 0 \\ 3 & 0 & 0 & 0 \\ -2 & -6 & -10 & 0 \end{pmatrix}.$$

What is the rank of M?

A. 1

B. 2

$\boxed{\text{C.}}$ 3

D. 4

E. I prefer not to answer.

Solution.

$$\begin{vmatrix} 1 & -3 & 5 \\ 3 & 0 & 0 \\ -2 & -6 & -10 \end{vmatrix} = -180 \neq 0 \implies r(M) = 3 \to \boxed{\text{C}}$$

Question 3

Compute the determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & -2 & -2 & -2 \\ -3 & 2 & 2 & 2 \\ -3 & 3 & 8 & 9 \end{vmatrix}$$

if possible. **The answer is:**

A. It is not possible to compute the determinant of a 4×4 matrix.

$\boxed{\text{B.}}$ 0

C. -2

D. 2

E. I prefer not to answer.

Solution.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & -2 & -2 & -2 \\ -3 & 2 & 2 & 2 \\ -3 & 3 & 8 & 9 \end{vmatrix} = 1 \cdot \begin{vmatrix} -2 & -2 & -2 \\ 2 & 2 & 2 \\ 3 & 8 & 9 \end{vmatrix} = 0 \to \boxed{\text{B}}$$

Question 4

Consider the following system of linear equations

$$\begin{array}{rrrrcr} x_1 & +x_2 & +x_3 & +x_4 & = & 4 \\ 2x_1 & +2x_2 & +3x_3 & +3x_4 & = & 8 \\ & & x_3 & +x_4 & = & 0 \end{array}$$

Does the system have solutions?

$\boxed{\text{A.}}$ The system has solutions and two degrees of freedom.

B. The system has no solutions.

C. The system has solutions and one degree of freedom.

D. The system has exactly one solution.

E. I prefer not to answer.

Solution.
The system is equivalent to

$$\begin{array}{ccccc} x_1 & +x_2 & & & = & 4 \\ & & x_3 & +x_4 & = & 0 \end{array}$$

This gives

$$\begin{aligned} x_1 &= 4 - x_2 \\ x_2 &= \text{free} \\ x_3 &= -x_4 \\ x_4 &= \text{free} \end{aligned}$$

$\rightarrow \boxed{\text{A}}$

Question 5

Which statement is *not* true?

A. The rank of an $m \times n$ matrix is less or equal to the minimum of m and n.

B. Four vectors in $\mathbb{R}^3$ are always linearly dependent.

$\boxed{\text{C.}}$ If three vectors in $\mathbb{R}^4$ are linearly independent, then at least one of the vectors is a linear combination of the remaining two vectors.

D. If A is any matrix, then the rank of A is equal to the rank of the transposed matrix A^T.

E. I prefer not to answer.

Solution.

A. The rank is equal to the order of the largest minor $\neq 0$, this is $\leq \min(m, n)$ $\rightarrow$true

B. The rank of a 3×4 matrix is $\leq \min(3, 4) = 3 < 4$ $\rightarrow$true

C. $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent$\rightarrow$false

D. $|M| = |M^T|$ for any square submatrix of A $\rightarrow$true

$\rightarrow \boxed{\text{C}}$

Question 6

Consider the function

$$Q(x_1, x_2, x_3) = x_1^2 - x_2^2 + x_3^2 + 2x_1x_2$$

Is this a quadratic form?

A. No, this is not a quadratic form.

B. Yes, it is a quadratic form and it is positive definite.

C. Yes, it is a quadratic form and it is negative definite.

$\boxed{\text{D.}}$ Yes, it is a quadratic form, but it is neither positive nor negative definite.

E. I prefer not to answer.

Solution.
$Q(0,1,0) = -1$, $Q(1,0,0) = 1 \to \boxed{\text{D}}$

Question 7

Consider the matrices

$$M = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \text{ and } N = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

Are M and N diagonalizable?

$\boxed{\text{A.}}$ Both M and N are diagonalizable.

B. M is diagonalizable, but N is not diagonalizable.

C. M is not diagonalizable, but N is diagonalizable.

D. Neither of the matrices are diagonalizable.

E. I prefer not to answer.

Solution.
M is symmetric $\implies$ M is diagonalizable. N has eigenvalues $1, 2$ and 3. Since these are distinct, N is diagonalizable $\rightarrow$ $\boxed{\text{A}}$

Question 8

Which function is neither convex nor concave?

A. $f(x, y) = x^2 + y^2$

B. $f(x, y) = -x^2 - y^2$

$\boxed{\text{C.}}$ $f(x, y) = -x^2 + y^2$

D. $f(x, y) = x^2 + y^2 - x$

E. I prefer not to answer.

Solution.

	$f(x,y)$	Hessian	
A	$x^2 + y^2$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	convex
B	$-x^2 - y^2$	$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$	concave
C	$-x^2 + y^2$	$\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$	not convex or concave
D	$x^2 + y^2 - x$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	convex

$\rightarrow$ $\boxed{\text{C}}$

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Solutions in:	GRA 60353 Mathematics (Final exam (80%))
Examination date:	10.12.08, 09:00 – 12:00
Permitted examination aids:	Bilingual dictionary and advanced calculator as a specific calculator defined in the student handbook
Answer sheets:	Squares
Total number of pages:	4

Question 1

Let B and C be two matrices defined by

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \text{ and } C = \begin{pmatrix} c & -5 \\ 5 & 1 \end{pmatrix}$$

(a) Compute BC and CB. For which values of c do we have that $BC = CB$?

Solution.

$$BC = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c & -5 \\ 5 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} c-5 & -6 \\ c+10 & -3 \end{pmatrix}}}$$

$$CB = \begin{pmatrix} c & -5 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} c-5 & -c-10 \\ 6 & -3 \end{pmatrix}}}$$

$CB = BC \Longleftrightarrow \underline{\underline{c = -4}}$

Let

$$A = \begin{pmatrix} -2 & 6 \\ 3 & 1 \end{pmatrix}$$

(b) Write down the characteristic equation and find the eigenvalues of A.

Solution.

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 6 \\ 3 & 1-\lambda \end{vmatrix} = \underline{\underline{\lambda^2 + \lambda - 20 = 0}} \Longleftrightarrow \underline{\underline{\lambda = 4 \text{ or } \lambda = -5}}$$

(c) Find the eigenvectors of A. Is A diagonalizable? If so, find a matrix P such that $D = P^{-1}AP$ is a diagonal matrix and find D.

Solution.

Since the 2×2 matrix A has two distinct eigenvalues, $\underline{\underline{A \text{ is diagonalizable}}}$.

$\lambda = 4 \implies \begin{pmatrix} -2-4 & 6 \\ 3 & 1-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \iff -x_1 + x_2 = 0 \iff$

$\underline{\underline{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ where } x_2 \text{ is free}}}$

$\lambda = -5 \implies \begin{pmatrix} -2-(-5) & 6 \\ 3 & 1-(-5) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \iff x_1 + 2x_2 = 0 \iff$

$\underline{\underline{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ where } x_2 \text{ is free}}}$

$\underline{\underline{P = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \implies D = P^{-1}AP = \begin{pmatrix} 4 & 0 \\ 0 & -5 \end{pmatrix}}}$ (other answers possible)

(d) Let E be a square matrix and assume that λ is an eigenvalue of E. Show that λ^2 is an eigenvalue of E^2.

Solution.

$E\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq 0 \implies E^2\mathbf{x} = E(E\mathbf{x}) = E(\lambda\mathbf{x}) = \lambda E\mathbf{x} = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x} \implies \lambda^2$ is an eigenvaue of E^2

Question 2

Let

$$h(x, y, z) = y^4 + x^2 + 2x + y^2 + yz - 1$$

(a) Find the Hessian matrix $\mathbf{h}''(x, y, z)$.

Solution.

$$\underline{\underline{\mathbf{h}''(x, y, z) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 12y^2 + 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}}}$$

(b) Show that h has a unique stationary point and classify this point as a local maximum, local minimum or a saddle point.

Solution.

$h'_x = 2x + 2 = 0$, $h'_y = 4y^3 + 2y + z = 0$, $h'_z = y = 0 \implies \underline{\underline{(x, y, z) = (-1, 0, 0)}}$

$\mathbf{h}''(-1, 0, 0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. $D_1 = 2$, $D_2 = 4$, $D_3 = -2 \implies \underline{\underline{\text{saddle point}}}$

Consider the problem

$$\max f(x,y,z) = 2z \text{ subject to } \begin{cases} g_1(x,y,z) = x^2 + y^2 = 2 \\ g_2(x,y,z) = x + y + z = 1 \end{cases}$$

(c) Write down the first order conditions using the Lagrangian

$$\mathcal{L}(x,y,z) = f(x,y,z) - \lambda_1 g_1(x,y,z) - \lambda_2 g_2(x,y,z)$$

and show that one obtains $\lambda_2 = 2$, $x = -\frac{1}{\lambda_1}$ and $y = -\frac{1}{\lambda_1}$

Solution.

$\mathcal{L} = 2z - \lambda_1(x^2 + y^2) - \lambda_2(x + y + z)$
$\mathcal{L}'_x = \underline{\underline{-2\lambda_1 x - \lambda_2 = 0}} \iff x = -\frac{\lambda_2}{2\lambda_1}$
$\mathcal{L}'_y = \underline{\underline{-2\lambda_2 y - \lambda_2 = 0}} \iff y = -\frac{\lambda_2}{2\lambda_1}$
$\mathcal{L}'_z = \underline{\underline{2 - \lambda_2 = 0}} \iff \underline{\underline{\lambda_2 = 2}}$
$\lambda_2 = 2 \implies x = y = -\frac{\lambda_2}{2\lambda_1} = \underline{\underline{-\frac{1}{\lambda_1}}}$

(d) Substitute $x = -\frac{1}{\lambda_1}$ and $y = -\frac{1}{\lambda_1}$ into one of the constraints to obtain $\lambda_1 = \pm 1$ and for each of the different sets of values for the multipliers, find out if $\mathcal{L}$ is convex or concave. Solve the maximization problem.

Solution.
$x = -\frac{1}{\lambda_1}$ and $y = -\frac{1}{\lambda_1} \implies (-\frac{1}{\lambda_1})^2 + (-\frac{1}{\lambda_1})^2 = 2 \iff \frac{1}{\lambda_1^2} = 1 \iff \underline{\underline{\lambda_1 = \pm 1}}$
$\lambda_1 = -1, \lambda_2 = 2 \implies \mathcal{L} = 2z + (x^2 + y^2) - 2(x + y + z)$ is $\underline{\underline{\text{convex}}}$
$\lambda_1 = 1, \lambda_2 = 2 \implies \mathcal{L} = 2z - (x^2 + y^2) - 2(x + y + z)$ is $\underline{\underline{\text{concave}}}$
$\mathcal{L}$ is concave for $\lambda_1 = 1, \lambda_2 = 2$, so this gives a maximum. We get $x = y = -\frac{1}{1} = -1 \implies$ $z = 1 - x - y = 1 + 2 = 3$
Thus $\underline{\underline{f(-1,-1,3) = 6 \text{ is the maximum value}}}$

Question 3

(a) Find the general solution of the following differential equation

$$\dot{x} + \frac{1}{t}x = 2 \qquad (t > 0)$$

Solution.
Integrating factor is $e^{\int \frac{1}{t}dt} = e^{\ln t} = t$: $\dot{x}t + x = 2t \iff \frac{d}{dt}(xt) = 2t \iff xt = t^2 + C \iff \underline{\underline{x = t + Ct^{-1}}}$

(b) Find the general solution of the linear second order differential equation

$$\ddot{x} + 3\dot{x} + 2x = 2t + 5$$

Solution.

$r^2 + 3r + 2 = 0 \iff r = -1 \vee r = -2 \implies x_h = A \cdot e^{-t} + B \cdot e^{-2t}$
$x_p = Ct + D \implies \dot{x}_p = C \implies \ddot{x}_p = 0$
$\ddot{x}_p + 3\dot{x}_p + 2x_p = 0 + 3C + 2(Ct + D) = 3C + 2D + 2Ct \implies 2C = 2$ and $3C + 2D = 5 \implies$
$C = 1$ and $D = 1 \implies x_p = t + 1$
We get $\underline{\underline{x = A \cdot e^{-t} + B \cdot e^{-2t} + t + 1}}$

(c) Solve the difference equation

$$x_{t+2} = 4x_{t+1} - 4x_t, \quad x_0 = 0, x_1 = 2$$

Solution.

$x_{t+2} = 4x_{t+1} - 4x_t \iff x_{t+2} - 4x_{t+1} + 4x_t = 0$
$r^2 - 4r + 4 = 0 \iff r = 2$
$x_t = (A + Bt) \cdot 2^t \implies x_0 = A, x_1 = (A + B) \cdot 2 \implies A = 0$ and $B = 1$
$\underline{\underline{x_t = t \cdot 2^t}}$

(d) Consider the differential equation

$$\dot{x} + 2x^2 = 0, \ x(0) = 1$$

Explain why this is not a *linear* differential equation, and solve it as a separable differential equation.

Solution.

Because of the term x^2 the equation is not linear. $\dot{x} + 2x^2 = 0 \iff \frac{dx}{dt} = -2x^2 \iff$
$-\frac{1}{x^2}\frac{dx}{dt} = 2 \iff \int -\frac{1}{x^2} dx = \int 2dt \iff \frac{1}{x} = 2t + C \iff x = \frac{1}{2t+C}$
$x(0) = \frac{1}{C} = 1 \implies C = 1 \implies \underline{\underline{x(t) = \frac{1}{2t+1}}}$

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Solutions:	**GRA 60352 Mathematics (Mid-term exam (20%))**
Examination date:	29.04.09, 16:00 – 17:00
Permitted examination aids:	Bilingual dictionary and advanced calculator as a specific calculator defined in the student handbook
Answer sheets:	Answer sheet for multiple choice examinations
Total number of pages:	4

This exam has 8 questions.

Question 1

Let M be the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 \\ -6 & 2 & 5 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}.$$

What is the rank of M?

A. 1

B. 2

C. 3

D. 4

E. I prefer not to answer.

Solution.
$\operatorname{rank} M = 3 \implies \boxed{\text{C}}$

Question 2

Let M be the matrix

$$M = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.$$

Compute the eigenvalues of M. **Which statement is correct?**

A. The matrix M has no real eigenvalues.

B. The eigenvalues are -1 and 4.

C. The eigenvalues are 2 and 1.

D. The matrix M has only one real eigenvalue.

E. I prefer not to answer.

Solution.
The eigenvalues are $-1, 4 \implies \boxed{\text{B}}$

Question 3

Compute the determinant

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & -2 & -2 \\ -3 & 0 & 2 & 2 \\ -3 & 0 & 1 & 2 \end{vmatrix}$$

if possible. **The answer is:**

A. It is not possible to compute the determinant of a 4×4 matrix.

B. 0

C. -4

D. 2

E. I prefer not to answer.

Solution.
$\boxed{\text{C}}$

Question 4

Which function is both convex and concave?

A. $f(x, y) = x^2 + y^2$

B. $f(x, y) = -x^2 - y^2$

C. $f(x, y) = -x^2 + y^2$

D. $f(x, y) = x + y$

E. I prefer not to answer.

Solution.
$\boxed{\text{D}}$

Question 5

Let

$$A = \begin{pmatrix} 1 & -3 \\ 3 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & -2 \\ 2 & 1 \end{pmatrix}$$

Compute $(AB)^{-1}A$. **The answer is:**

A. $\begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$

B. $\begin{pmatrix} 1 & -3 \\ 3 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 0 & -2 \\ 2 & 1 \end{pmatrix}$

D. $\begin{pmatrix} -\frac{1}{6} & \frac{1}{2} \\ -\frac{23}{36} & \frac{5}{12} \end{pmatrix}$

E. I prefer not to answer.

Solution.
$(AB)^{-1}A = B^{-1}A^{-1}A = B^{-1} = \begin{pmatrix} 0 & -2 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \Longrightarrow \boxed{\text{A}}$

Question 6

Let

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 1 & 4 \\ -2 & -6 & -10 \end{pmatrix}$$

and let B be any 3×3 matrix. **Which statement is correct?**

A. The matrix product AB is not defined.

B. The columns of the matrix AB are linearly independent.

C. The columns of the matrix AB are linearly dependent.

D. It is not possible to decide if the columns of the matrix AB are linearly independent without knowing B.

E. I prefer not to answer.

Solution.
$|A| = 0 \implies |AB| = |A||B| = 0 \cdot |B| = 0 \Longrightarrow$ columns of AB are linearly dependent $\Longrightarrow$ $\boxed{\text{C}}$

Question 7

Let

$$M = \begin{pmatrix} 4 & 3 & -5 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Which statement is correct?

A. $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of M.

B. Only $\mathbf{v}$ is an eigenvector of M.

C. $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of M.

D. Only $\mathbf{w}$ is an eigenvector of M.

E. I prefer not to answer.

Solution.

$$M\mathbf{u} = \begin{pmatrix} 4 & 3 & -5 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \mathbf{u} \text{ is an eigenvector with } \lambda = 0.$$

$$M\mathbf{v} = \begin{pmatrix} 4 & 3 & -5 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \implies \mathbf{v} \text{ is an eigenvector with } \lambda = 2.$$

$$M\mathbf{w} = \begin{pmatrix} 4 & 3 & -5 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \implies \mathbf{w} \text{ is not an eigenvector.}$$

$\boxed{\text{C}}$

Question 8

Let $f(x_1, x_2, x_3)$ and $g(x_1, x_2, x_3)$ be positive definite quadratic forms.

Which statement is correct?

A. The function $f(x_1, x_2, x_3) + g(x_1, x_2, x_3)$ need not be a quadratic form.

B. The function $f(x_1, x_2, x_3) + g(x_1, x_2, x_3)$ is a positive definite quadratic form.

C. The function $f(x_1, x_2, x_3)+g(x_1, x_2, x_3)$ is a negative semidefinite quadratic form.

D. The function $f(x_1, x_2, x_3) + g(x_1, x_2, x_3)$ is a quadratic form, but it need not be definite.

E. I prefer not to answer.

Solution.
$f(x_1, x_2, x_3)$ and $g(x_1, x_2, x_3)$ be positive definite quadratic forms $\Longrightarrow f(x_1, x_2, x_3) > 0$ and $g(x_1, x_2, x_3) > 0$ for all $\mathbf{x} \neq \mathbf{0}$. Thus $f(x_1, x_2, x_3) + g(x_1, x_2, x_3) > 0$ for all $\mathbf{x} \neq \mathbf{0} \implies \boxed{\text{B}}$

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Solutions:	GRA 60353 Mathematics (Final exam (80%))
Examination date:	04.05.09, 13:00 – 16:00
Total number of pages:	4

Question 1

Consider the following system of linear equations

$$\begin{array}{rrrcl} x_1 & & -30x_3 & = & 0 \\ -2x_1 & -10x_2 & +70x_3 & = & 0 \\ -x_1 & -10x_2 & +40x_3 & = & 0 \end{array}$$

(a) Write down the coefficient matrix of the system. Solve the system, and state the number of degrees of freedom. What is the rank of the coefficient matrix?

Solution.
Coefficient matrix:

$$A = \begin{pmatrix} 1 & 0 & -30 \\ -2 & -10 & 70 \\ -1 & -10 & 40 \end{pmatrix}$$

The last equation is the sum of the two first equations, hence the third equation is superflous. Adding two times the first equation to the second, we obtain

$$\begin{array}{rrrcl} x_1 & & -30x_3 & = & 0 \\ & -10x_2 & +10x_3 & = & 0 \end{array}$$

From this we get

$$x_1 = 30x_3$$
$$x_2 = x_3$$
$$x_3 = \text{free}$$

Number of degrees of freedom: $\underline{\underline{1}}$ The rank of coefficient matrix: $3 - 1 = \underline{\underline{2}}$

Consider the quadratic form

$$Q(x_1, x_2, x_3) = ax_1^2 + x_2^2 + ax_3^2 + 2(a-2)x_1x_3$$

(b) Find a matrix A such that $Q = \mathbf{x}^T A\mathbf{x}$. For which values of a is the quadratic form positive definite?

Solution.

$$A = \begin{pmatrix} a & 0 & a-2 \\ 0 & 1 & 0 \\ a-2 & 0 & a \end{pmatrix}$$

$D_1 = a$, $D_2 = \begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} = a$, $D_3 = \begin{vmatrix} a & 0 & a-2 \\ 0 & 1 & 0 \\ a-2 & 0 & a \end{vmatrix} = 4a-4$

Positive definite $\Leftrightarrow D_1 > 0,\ D_2 > 0,\ D_3 > 0 \Leftrightarrow 4a-4 > 0, a > 0 \Leftrightarrow \underline{\underline{a > 1}}$

Define

$$T = \begin{pmatrix} 1 & 4 \\ c & 1 \end{pmatrix}.$$

(c) For which values of c does T have two distinct eigenvalues?

Solution.

$\begin{vmatrix} 1-\lambda & 4 \\ c & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 4c + 1 = 0 \Longleftrightarrow \lambda = 1 \pm 2\sqrt{c}$

Two distinct eigenvalues $\Longleftrightarrow \underline{\underline{c > 0}}$

(d) Find a value of c such that T is not diagonalizable.

Solution.

Several answers are possible: $c < 0 \implies$ no real eigenvalues, hence T is not diagonalizable. If $c = 0$ one also finds that the matrix is not diagonalizable. (It is not possible to find two linearly independent eigenvectors in this case.)

Question 2

Consider the function f defined on the subset $S = \{(x,y,z) : z > 0\}$ of $\mathbb{R}^3$ by

$$f(x,y,z) = x^2 - 2x + y^2 + z^3 - 3z.$$

(a) Show that the subset S is convex. Find the stationary points of f.

Solution.

Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in S. A point on the line between P_1 and P_2 is given as

$$\begin{aligned} (x,y,z) &= s(x_1,y_1,z_1) + (1-s)(x_2,y_2,z_2) \\ &= (sx_1 + (1-s)x_2, sy_1 + (1-s)y_2, z_1 + (1-s)z_2) \end{aligned}$$

where $0 \le s \le 1$. Since $z_1 > 0$ and $z_2 > 0$ it follows that $z = z_1 + (1-s)z_2 > 0$. Hence (x,y,z) is in S. This shows that S is convex.

$f'_x = 2x - 2 = 0, f'_y = 2y = 0, f'_z = 3z^2 - 3 = 0 \Longleftrightarrow (x,y,z) = (1,0,\pm 1)$. Since $(1,0,-1)$ is not in S, the only stationary point is $\underline{\underline{(1,0,1)}}$.

(b) Find the Hessian matrix of f. Is f concave or convex? Does f have a global extreme point? Justify your answer.

Solution.

The Hessian matrix is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6z \end{pmatrix}$$

We have $D_1 = 2, D_2 = 4$ and $D_3 = 24z > 0$ since (x,y,z) is in S. <u>Thus f is convex.</u> Since f is convex, <u>$(1,0,1)$ is a global minimum.</u>

Consider the problem

$$\max f(x,y) = \ln(x+1) + \ln(y+1) \text{ subject to } \begin{cases} y \leq 5 \\ x+y \leq 2 \end{cases}$$

(c) Write down the necessary Kuhn-Tucker conditions.

Solution.
$\mathcal{L} = \ln(x+1) + \ln(y+1) - \lambda_1(y-5) - \lambda_2(x+y-2)$
$\mathcal{L}'_x = \frac{1}{x+1} - \lambda_2 = 0$
$\mathcal{L}'_y = \frac{1}{y+1} - \lambda_1 - \lambda_2 = 0$
$\lambda_1 \geq 0 \quad (\lambda_1 = 0 \text{ if } y < 5)$
$\lambda_2 \geq 0 \ (\lambda_2 = 0 \text{ if } x+y < 2)$

(d) Solve the problem.

Solution.
Several strategies are possible. Since we must have $x > -1$ and $y > -1$, the constraint $y \leq 5$ is inessential. Nevertheless, we solve the problem with two constraints:
$\underline{y = 5, x+y = 2}$: $y = 5 \implies x = 2-5 = -3 \implies$
$f(x,y)$ is not defined, hence this case gives no solutions.
$\underline{y < 5, x+y = 2}$:
$\underline{y = 2-x, \lambda_1 = 0}$,

$$\left.\begin{array}{lll} \mathcal{L}'_x & = \frac{1}{x+1} - \lambda_2 & = 0 \\ \mathcal{L}'_y & = \frac{1}{y+1} - \lambda_1 - \lambda_2 & = 0 \end{array}\right\} \implies \frac{1}{x+1} = \frac{1}{y+1} \implies x = y \implies x + x = 2 \implies$$

$x = 1 \implies (1,1)$ is a possible solution.
$\underline{y = 5, x+y < 2}$:
$\lambda_2 = 0 \implies \frac{1}{x+1} = 0$ which is imposible, hence no solutions
$\underline{y < 5, x+y < 2}$:
$\lambda_2 = 0 \implies \frac{1}{z+1} = 0$ which is imposible, hence no solutions
Maximum: $f(1,1) = \ln(1+1) + \ln(1+1) = \underline{\underline{2\ln 2}}$

Question 3

Find the general solution of the following differential equation

(a)

$$\dot{x} + tx = 3t \qquad (t > 0)$$

Solution.
Integrating factor: $e^{\int t dt} = e^{\frac{1}{2}t^2}$
$\frac{d}{dt}(xe^{\frac{1}{2}t^2}) = 3te^{\frac{1}{2}t^2} \implies xe^{\frac{1}{2}t^2} = \int 3te^{\frac{1}{2}t^2} dt = 3e^{\frac{1}{2}t^2} + C \implies \underline{\underline{x = 3 + Ce^{-\frac{1}{2}t^2}}}$

(b) Find the general solution of the linear second order differential equation

$$\ddot{x} + 5\dot{x} + 6x = e^t$$

Solution.
$r^2 + 5r + 6 = 0$, solution is: $-2, -3 \implies x_h = Ae^{-2t} + Be^{-3t}$
$x_p = Ce^t \implies \dot{x}_p = Ce^t \implies \ddot{x}_p = Ce^t \implies Ce^t + 5Ce^t + 6Ce^t = 12Ce^t = e^t \implies C = \frac{1}{12}$
$\underline{\underline{x = Ae^{-2t} + Be^{-3t} + \frac{1}{12}e^t}}$

(c) Solve the following difference equation:

$$x_{t+1} - 3x_t - 2 = 0, \quad x_0 = 0$$

Solution.
$x_{t+1} - 3x_t - 2 = 0 \Longleftrightarrow x_{t+1} = 3x_t + 2 \Longleftrightarrow x_t = 3^t(x_0 - \frac{2}{1-3}) + \frac{2}{1-3} = 3^t\,(0+1) - 1 \implies$
$\underline{\underline{x_t = 3^t - 1}}$

(d) Consider the following system of difference equations:

$$\begin{aligned} x_{t+1} &= x_t + 3y_t \\ y_{t+1} &= 2x_t \end{aligned}$$

with $x_0 = 1$ and $y_0 = 0$. Derive a second order difference equation for x_t and solve this equation and the system.

Solution.
$x_{t+1} = x_t + 3y_t \implies x_{t+2} = x_{t+1} + 3y_{t+1} = x_{t+1} + 3(2x_t) = x_{t+1} + 6x_t$ Thus we obtain $\underline{\underline{x_{t+2} - x_{t+1} - 6x_t = 0}}$. Characteristic equation is $r^2 - r - 6 = 0$, solution is: $3, -2 \implies x_t = A \cdot 3^t + B \cdot (-2)^t$. From $x_0 = 1$, $y_0 = 0$, $x_1 = 1 + 3 \cdot 0 = 1$. Thus $A + B = 1, 3A - 2B = 1$. $A = \frac{3}{5}, B = \frac{2}{5} \implies \underline{\underline{x_t = \frac{3}{5} \cdot 3^t + \frac{2}{5} \cdot (-2)^t}}$. $y_t = 2x_{t-1} = 2(\frac{3}{5} \cdot 3^{t-1} + \frac{2}{5} \cdot (-2)^{t-1}) \implies$
$\underline{\underline{y_t = \frac{4}{5}\,(-2)^{t-1} + \frac{6}{5}3^{t-1}}}$

BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Solutions:	**GRA 60352 Mathematics (Mid-term exam (20%))**
Examination date:	28.09.09, 14:00 – 15:00
Total number of pages:	4

Question 1 : B
Question 2 : B
Question 3 : C
Question 4 : A
Question 5 : A
Question 6 : C
Question 7 : D
Question 8 : C

NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Solutions:	GRA 60353 Mathematics (Final exam (80%))
Examination date:	10.12.09, 9:00 – 12:00
Permitted examination aids:	Bilingual dictionary. BI-approved exam calculator: TEXAS INSTRUMENTS BA II Plus™
Total number of pages:	2

Question 1

(a)
$$\begin{vmatrix} 1 & 2 & -t \\ t & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix} = 3t - 3t^2 = -3t(t-1) = 0 \implies \underline{\underline{t = 0 \text{ or } t = 1}}$$

(b) We get a system with three equations:
$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_2 &= -1 \\ c_1 + 3c_2 &= -1 \end{aligned}$$
which has the unigue solution $\underline{\underline{c_1 = 2 \text{ and } c_2 = -1}}$.

(c) There is one eigenvalue $\lambda = 1$. To find all possible eigenvectors we solve
$$(A - I)\mathbf{x} = 0 \iff \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$
$$\iff \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ where } t \text{ is a free variable.}$$
Since it is not possible to find two linearly independent eigenvectors, the matrix $\underline{\underline{A \text{ is not diagonalizable}}}$.

Question 2

(a)
$$\left.\begin{aligned} f_1' &= 1 - x_2 = 0 \\ f_2' &= 2x_2 - x_1 = 0 \\ f_3' &= 3x_3^2 - 3 = 0 \end{aligned}\right\} \implies (x_1, x_2, x_3) = \underline{\underline{(2, 1, \pm 1)}}.$$

(b) $\mathbf{f}'' = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6x_3 \end{pmatrix}$. Since $\Delta_2 = \begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix} = -1$, the Hessian matrix is always indefinite. Thus $\underline{\underline{(2, 1, \pm 1) \text{ are both saddlepoints}}}$.

(c) $\mathbf{f}'' = \begin{pmatrix} e^{y^2+x} & 2ye^{y^2+x} \\ 2ye^{y^2+x} & 2e^{y^2+x} + 4y^2e^{y^2+x} \end{pmatrix}$, $D_1 = e^{y^2+x} > 0$, $D_2 = 2e^{2(y^2+x)} > 0$
$\underline{\underline{\text{The function is convex in } x \text{ and } y}}$.

(d)
$$\left.\begin{aligned} f_x' &= -p + e^{x+y^2} = 0 \\ f_y' &= 2ye^{x+y^2} = 0 \end{aligned}\right\} \text{ and } p > 0 \implies \underline{\underline{y^*(p) = 0 \text{ and } x^*(p) = \ln p}} \implies$$
$\underline{\underline{f^*(p) = -p\ln p + p}}$ (no solutions if $p \leq 0$)

(e) $\frac{df^*}{dp} = -x^*(p) = \underline{\underline{-\ln p}}$

Question 3

(a) The integrating factor is $e^{\frac{1}{2}at^2}$ and we get

$$\begin{aligned}\frac{d}{dt}(xe^{\frac{1}{2}at^2}) &= 2te^{\frac{1}{2}at^2} \\ xe^{\frac{1}{2}at^2} &= \int 2te^{\frac{1}{2}at^2}\,dt = \frac{2}{a}e^{\frac{1}{2}at^2} + C \text{ if } a \neq 0\end{aligned}$$

Thus

$$\underline{\underline{x(t) = \frac{2}{a} + Ce^{-\frac{1}{2}at^2}}}$$

If $a = 0$ we get $\dot{x} = 2t \implies \underline{\underline{x(t) = t^2 + C}}$

(b) The homegenous solution: $x_h = (C_1 + C_2 t)e^{-t}$. The particular solution: $x_p = Ae^t \implies A + 2A + A = 4 \implies A = 1 \implies x_p = e^t$. The general solution: $x(t) = (C_1 + C_2 t)e^{-t} + e^t$. Initial conditions: $x(0) = C_1 + 1 = 1 \implies C_1 = 0$. $\dot{x} = C_2 e^{-t} - (C_1 + C_2 t)e^{-t} + e^t \implies \dot{x}(t) = C_2 - C_1 + 1 = C_2 + 1 = 2 \implies C_2 = 1$. Solution

$$\underline{\underline{x(t) = e^t + te^{-t}}}$$

(c)

$$\begin{aligned}x_t &= (r+1)x_t + s \implies \\ x_t &= (r+1)^t\left(x_0 - \frac{s}{1-(r+1)}\right) + \frac{s}{1-(r+1)} \\ &= \underline{\underline{\frac{(r+1)^t(100r+1) - 1}{r}s}}\end{aligned}$$

(d) The homogenous solution: $x_t = (C_1 + C_2 t)(-1)^t$. Particular solution: $x_t^{(p)} = At + B \implies A(t+2) + B + 2(A(t+1) + B) + At + B = 4A + 4B + 4At = 4t + 4 \implies A = 1$ and $B = 0$. General solution: $x_t = (C_1 + C_2 t)(-1)^t + t$. Initial conditions: $x_0 = C_1 = 1$, $x_1 = (C_1 + C_2)(-1) + 1 = 1 \implies C_2 = -1$. Solution:

$$\underline{\underline{x_t = (1-t)(-1)^t + t}}$$

NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Solutions:	**GRA 60352 Mathematics (Mid-term exam (20%))**
Examination date:	28.09.09, 09:00 – 10:00
Total number of pages:	5

Question 1 : D
Question 2 : D
Question 3 : D
Question 4 : C
Question 5 : D
Question 6 : C
Question 7 : B
Question 8 : D

NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Written examination in:	**GRA 60353 Mathematics (Final exam (80 %))**
Examination date:	04.05.10, 9:00 – 12:00
Permitted examination aids:	Bilingual dictionary. BI-approved exam calculator: TEXAS INSTRUMENTS BA II Plus^{TM}
Answer sheets:	Squares
Total number of pages:	2

Question 1

Let

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 3 & x & 3 \\ 2 & 3 & 2 \end{pmatrix}$$

(a) Discuss the rank of A for different values of x.

Solution.
The rank is 3 if

$$\begin{vmatrix} 1 & 3 & 2 \\ 3 & x & 3 \\ 2 & 3 & 2 \end{vmatrix} = 9 - 2x \neq 0 \iff x \neq \frac{9}{2}$$

If $x = \frac{9}{2}$ the rank is 2.

(b) Explain that A is a symmetric matrix. Show that A is indefinte for all values of x.

Solution.
The matrix is symmetric since $A^T = A$.
$D_1 = 1$
$D_2 = \begin{vmatrix} 1 & 3 \\ 3 & x \end{vmatrix} = x - 9$
$D_3 = 9 - 2x$
For A to be (semi)definite, D_1 and D_3 (or be zero). Thus $9 - 2x \geq 0 \iff x \leq \frac{9}{2}$. Under this condition, $D_2 < 0$. This shows that A is indefinte.

(c) Two firms numbered 1 and 2 share the market for a certain commodity. In the course of each week, the following changes occur:

$$\begin{cases} \text{Firm 1 keeps } \frac{1}{3} \text{ of its customers, while losing } \frac{2}{3} \text{ to Firm 2.} \\ \text{Firm 2 keeps } \frac{1}{4} \text{ of its customers, while losing } \frac{3}{4} \text{ to Firm 1.} \end{cases}$$

Show that in the long run, Firm 1 will have $\frac{9}{17}$ of the customers.

Solution.
Let x_s and y_s be the number of customers of firm 1 and firm 2 respectivly after s weeks. Then
$$\begin{pmatrix} x_{s+1} \\ y_{s+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{3}{4} \\ \frac{2}{3} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x_s \\ y_s \end{pmatrix}$$
Consider
$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} = \begin{pmatrix} \frac{9}{17} \\ \frac{8}{17} \end{pmatrix}$$
Then
$$\begin{pmatrix} \frac{1}{3} & \frac{3}{4} \\ \frac{2}{3} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{9}{17} \\ \frac{8}{17} \end{pmatrix} = \begin{pmatrix} \frac{9}{17} \\ \frac{8}{17} \end{pmatrix}$$
It follows that $\begin{pmatrix} x_s \\ y_s \end{pmatrix}$ will approach $\begin{pmatrix} \frac{9}{17} \\ \frac{8}{17} \end{pmatrix}$ as s increases.

(d) Let E be a square matrix and assume that 2 is an eigenvalue of E. Show that 10 is an eigenvalue of the matrix $E^3 + E$.

Solution.
Let $\mathbf{v}$ be an eigenvector for E with eigenvalue 2. Then
$$(E^3 + E)\mathbf{v} = E^3\mathbf{v} + E\mathbf{v} = 2^3\mathbf{v} + 2\mathbf{v} = 10\mathbf{v}$$
From this follows that 10 is an eigenvalue for the matrix $E^3 + E$.

Question 2

Let
$$h(x, y, z) = 2x^3 - 9x^2 + 12x + y^2 - yz + z^2$$

(a) Find the Hessian matrix $\mathbf{h}''(x, y, z)$.

Solution.
$$\mathbf{h}''(x, y, z) = \begin{pmatrix} 12x - 18 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

(b) Find and classify the stationary points of h.

Solution.
$h'_x = 6x^2 - 18x + 12 = 0$.
$h'_y = 2y - z = 0$
$h'_z = 2z - y = 0$
gives $(1, 0, 0)$ and $(2, 0, 0)$. We get that $\mathbf{h}''(1, 0, 0)$ that is indefinite, hence $(1, 0, 0)$ is a saddle point. We get that $\mathbf{h}''(2, 0, 0)$ is positive definite, hence $(2, 0, 0)$ is a local minimum point.

Consider the problem
$$\max f(x, y, z) = 2z^3 - 9y^2 + 12x \text{ subject to } \begin{cases} g_1(x, y, z) = z - x = 0 \\ g_2(x, y, z) = y - z = 0 \end{cases}$$

(c) Write down the first order conditions using the Lagrangian
$$\mathcal{L}(x, y, z) = f(x, y, z) - \lambda_1 g_1(x, y, z) - \lambda_2 g_2(x, y, z)$$
and show that one obtains $\lambda_1 = -12$, $\lambda_2 = -18y$.

Solution.
$\mathcal{L}'_x = 12 - \lambda_1(-1) - \lambda_2 \cdot 0 = 12 + \lambda_1 = 0 \implies \lambda_1 = -12$
$\mathcal{L}'_y = -18y - \lambda_1 \cdot 0 - \lambda_2 \cdot 1 = -18y - \lambda_2 = 0 \implies \lambda_2 = -18y$
$\mathcal{L}'_z = 6z^2 - \lambda_1 \cdot 1 - \lambda_2 \cdot (-1) = 6z^2 - \lambda_1 + \lambda_2 = 0$

(d) Find all solutions of the first order conditions. Is $\mathcal{L}$ convex as a function of x, y and z? Is $\mathcal{L}$ concave as a function of x, y and z?

Solution.
$6z^2 - \lambda_1 + \lambda_2 = 0 \implies 6z^2 + 12 - 18y = 6z^2 - 18z + 12 = 0 \implies z = 1$ or $z = 2$.We get

$$(1,1,1) \longleftrightarrow \lambda_1 = -12, \lambda_2 = -18$$
$$(2,2,2) \longleftrightarrow \lambda_1 = -12, \lambda_2 = -24$$

Hessian matrix of $\mathcal{L}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & 12z \end{pmatrix}$$

Thus $\mathcal{L}$ is neither convex nor concave as a function of x, y and z.

Question 3

(a) Find the general solution of the linear second order differential equation

$$\ddot{x} - 7\dot{x} + 12x = e^{-t}$$

Solution.
Characteristic equation

$$r^2 - 7r + 12 = 0$$

gives general homogenous solution: $x_h = Ae^{3t}+Be^{4t}$. Finding particular solution $x_p = Ce^{-t}$ gives $C = \frac{1}{20}$. Thus

$$x = Ae^{3t} + Be^{4t} + \frac{1}{20}e^{-t}$$

(b) Solve the following difference equation:

$$x_{t+1} - 1.08x_t + 30 = 0, \quad x_0 = 800$$

Solution.

$$\begin{aligned} x_t &= (1.08)^t(800 - \frac{-30}{-0.08}) + \frac{-30}{-0.08} \\ &= (1.08)^t \cdot 425 + 375 \end{aligned}$$

(c) Solve the following difference equation:

$$x_{t+2} - 11x_{t+1} + 30x_t = 0, \quad x_0 = 0, \quad x_1 = 1$$

Solution.
$x_t = 6^t - 5^t$

(d) Find the general solution of the following differential equation

$$\dot{x} + tx = t^3 \qquad (t > 0)$$

Solution.
We get
$$xe^{\frac{1}{2}t^2} = \int t^3 e^{\frac{1}{2}t^2} dt = e^{\frac{1}{2}t^2}\left(t^2 - 2\right) + C$$
Thus
$$x = t^2 - 2 + Ce^{-\frac{1}{2}t^2}$$

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