LECTURE 5

GKA 6035

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REVIEW LECTURE 4 + ADDITIONAL EXAMPLE

- Eigenvalues
- Eigenvectors
- Diagonalization
- How to compute A^n

A symmetric $\Rightarrow$ A diagonalizable

Ex: Is $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ diagonalizable?

① A is not symmetric.

② Eigenvalues

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = (3-\lambda)\cdot\big((2-\lambda)(1-\lambda)+1\big) - 1\cdot\big((1-\lambda)+1\big) - 1\cdot\big(1-(2-\lambda)\big)$$

$= \cancel{(3-\lambda)(\lambda^2-3\lambda+3)-(2-\lambda)-(-1+\lambda)}$ ← this didn't work

$= (3-\lambda)\cdot\big((2-\lambda)(1-\lambda)\big) + (3-\lambda) + (-1)$

$= (3-\lambda)(2-\lambda)(1-\lambda) + (2-\lambda) = (2-\lambda)\cdot\big((3-\lambda)(1-\lambda)+1\big)$

$= (2-\lambda)(\lambda^2-4\lambda+4) = 0$

$\underline{\lambda=2}$ or: $\lambda^2-4\lambda+4=0$

$\underline{\lambda=2}$

Solution: $\underline{\underline{\lambda=2}}$ (mult. 3)

$\lambda_1 = \underline{2}$ $\lambda_2 = \underline{2}$ $\lambda_3 = \underline{2}$

③ If there are n distinct eigenvalues, then there are n lin. independent eigenvectors. Not the case here, we have to compute:

Eigenvectors for $\lambda=2$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

coefficient matrix with $\lambda=2$

rank = 2

degrees of freedom = 1

z free

$x+z=0$

$-y=0$

$x=-z$

$y=0$

z free

A not diagonalizable since we do not have more than one linearly independent eigenvector.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ eigenvector $\lambda=2$.

Remember:

A Diagonalizable means

$P^{-1}AP = D$, with

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$ diagonal

$$P = \left(\underline{v}_1 \middle| \underline{v}_2 \middle| \cdots \middle| \underline{v}_n \right)$$ invertible

$\Updownarrow$

$\{v_1 \ldots v_n\}$ lin. independent

PLAN : LECTURE 5

1. INTRODUCTION TO OPTIMIZATION (LECTURE 5-8)
2. QUADRATIC FORMS AND DEFINITENESS
3. BORDERED HESSIANS FOR QUADRATIC FORMS

[FMEA] 1.7-1.8

It may be smart to review material about functions, for example

[EMEA] Ch. 4, Ch. 11

for those who need it.

1 INTRO TO OPTIMIZATION

$f(x_1, x_2, \dots, x_n) = f(\underline{x})$ $\qquad$ $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

function in n variables

unconstrained optimization

max/min of $f(\underline{x})$ with no constraints on $\underline{x}$.

constrained optimization

max/min of $f(\underline{x})$ subject to constraints

Ex: max $x_1^2 - 7x_1x_2 + 3x_2^2$

max $\underset{f}{\underset{\shortparallel}{xyz}}$ subject to $\underbrace{x^2+y^2+z^2 \le 1}_{D_f}$

domain of definition

$D_f = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$

② Quadratic forms

A *quadratic form* $Q(\underline{x})$ is a polynomial function where all terms have degree two.

General form in three variables:

$$Q(x_1, x_2, x_3) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{33}x_3^2$$

Ex: $Q(\underline{x}) = x_1^2 + 2x_1x_2 + x_2^2 + x_3^2$

Facts: For any Quadratic form $Q(\underline{x})$

i) $Q(\underline{0}) = Q(0, 0, \dots, 0) = 0$

ii) $\underline{x} = \underline{0}$, $(x_1, \dots, x_n) = (0, \dots, 0)$, is a stationary point

Ex: $Q(\underline{x}) = x_1^2 - 2x_2^2$

$$Q'_{x_1} = \boxed{\begin{array}{l} 2x_1 = 0 \\ -4x_2 = 0 \end{array}} \quad \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}$$

Q'_{x_2}

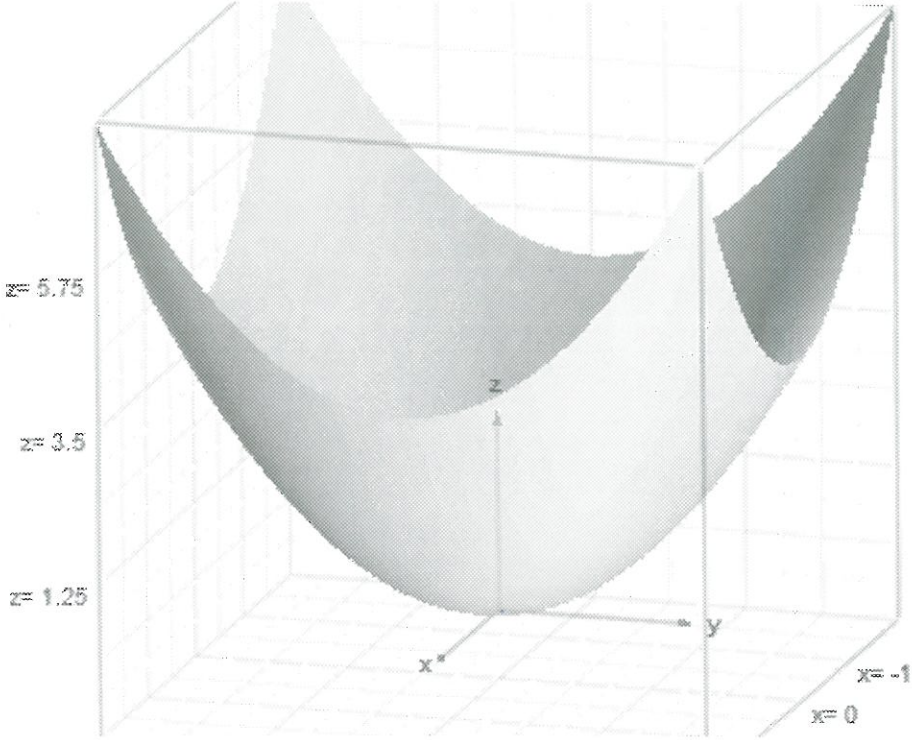
$\underline{x} = (0,0)$ is stationary point

Remember:

Stationary point

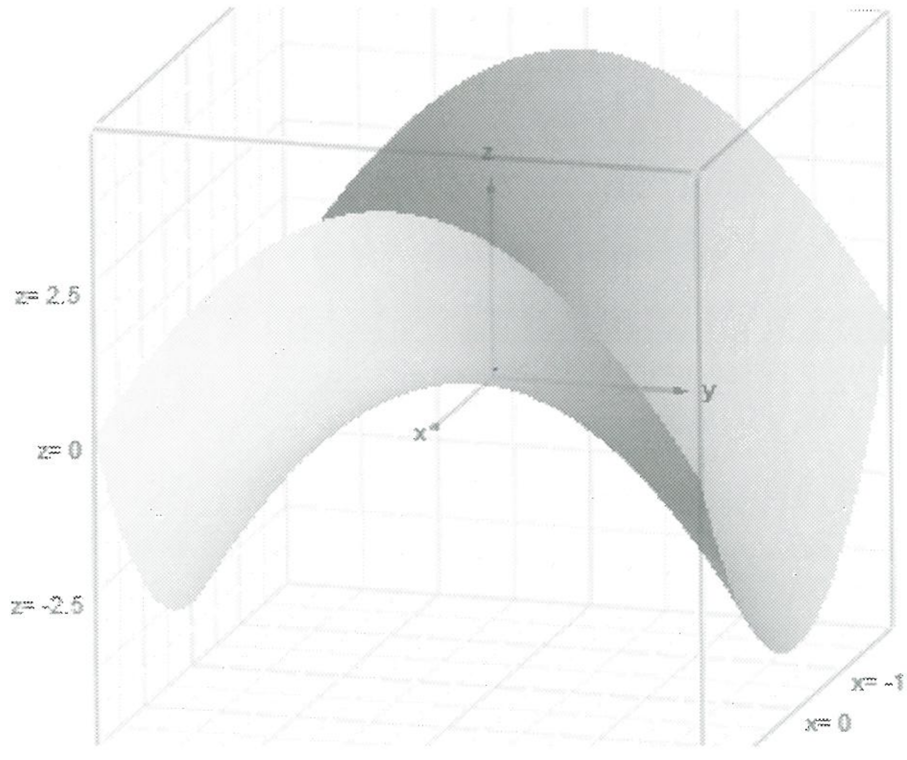
$\Updownarrow$

$Q'_1 = Q'_2 = \dots = Q'_n = 0$



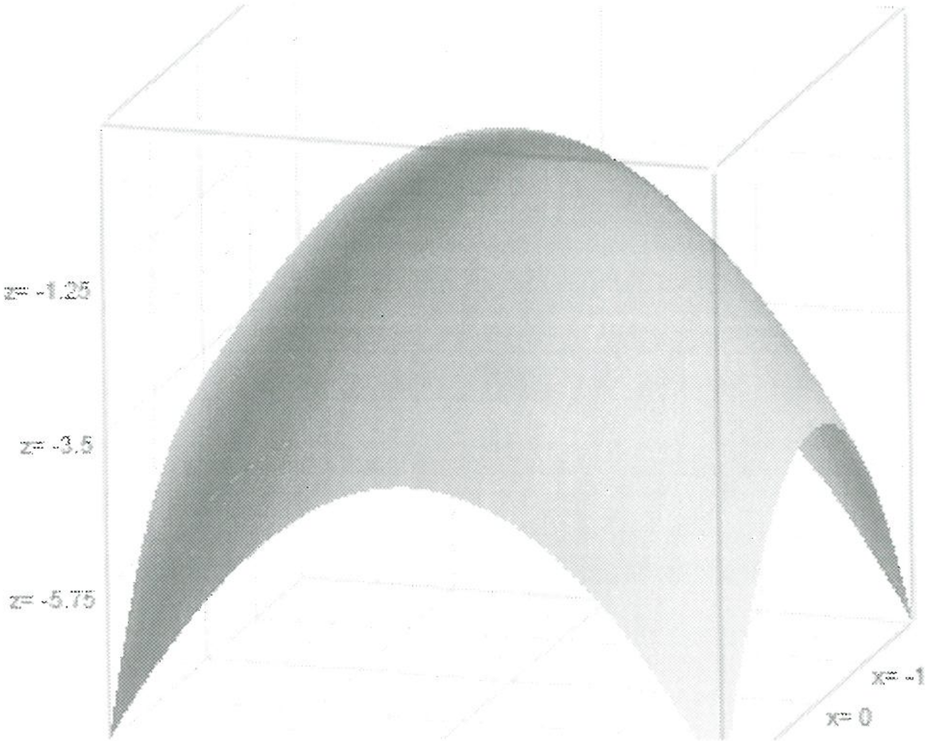
$z = x^2 + y^2$

positive definite



$z = x^2 - y^2$

indefinite



$z = -x^2 - y^2$

negative definite

Defn: $Q(\underline{x})$ quadratic form

We say that Q is

* <u>positive definite</u> if $Q(\underline{x}) > 0$ for all $\underline{x} \neq \underline{0}$
* <u>positive semidefinite</u> if $Q(\underline{x}) \geq 0$ for all $\underline{x}$

* <u>negative definite</u> if $Q(\underline{x}) < 0$ for all $\underline{x} \neq \underline{0}$
* <u>negative semidefinite</u> if $Q(\underline{x}) \leq 0$ for all $\underline{x}$

* <u>indefinite</u> if $Q(\underline{x}) > 0$ for some values of $\underline{x}$
  and $Q(\underline{x}) < 0$ for other values of $\underline{x}$

Note:

positive (semi)definite $\Longleftrightarrow$ $\underline{x} = \underline{0}$ is <u>global minimum</u>
negative (semi)definite $\Longleftrightarrow$ $\underline{x} = \underline{0}$ is <u>global maximum</u>
indefinite $\Longleftrightarrow$ $\underline{x} = \underline{0}$ is <u>saddle point</u>

Matrix form:

$$Q(x_1, x_2, \dots, x_n) = a_{11} x_1^2 + a_{12} x_1 x_2 + \dots$$
$$+ a_{22} x_2^2 + \dots$$
$$\vdots$$
$$+ a_{nn} x_n^2$$

$$= \sum_{\substack{1 \le i \le n \\ 1 \le j \le n \\ i \le j}} a_{ij} x_i x_j$$

$$= (x_1 \; x_2 \; \dots \; x_n) \cdot \begin{pmatrix} a_{11} & a_{12}/2 & \dots \\ \frac{a_{12}}{2} & a_{22} & \\ \vdots & & \ddots \\ & & & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \underline{x}^T \cdot A \cdot \underline{x}$$, where A is a symmetric matrix.

Ex: $Q(\underline{x}) = x_1^2 + 3x_2^2 = (x_1 \; x_2) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{x}^T A \underline{x}$

$Q(\underline{x}) = x_1^2 + \underline{2x_1x_2} + x_2^2 + x_3^2$

$= (x_1 \; x_2 \; x_3) \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{x}^T A \underline{x}$

Any quadratic form $Q(\underline{x}) = \underline{x}^T A \underline{x}$ for a unique symmetric matrix A — <u>the symmetric matrix</u> of the quadratic form.

Thm: Q quadratic form (n variables), A its symmetric matrix ($n \times n$, $Q(\underline{x}) = \underline{x}^T A \underline{x}$)

Q positive definite $\iff$ all eigenvalues of A are positive ($\lambda_i > 0$)

positive semidefinite $\iff$ — " — non-negative ($\lambda_i \geq 0$)

negative definite $\iff$ $\lambda_i < 0$ (all eigenvalues)

negative semidefinite $\iff$ $\lambda_i \leq 0$ (all eigenvalues)

indefinite $\iff$ there are both positive and negative eigenvalues ($\lambda_i > 0$, $\lambda_j < 0$)

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quadratic form $\rightsquigarrow$ symmetric matrix $\rightsquigarrow$ eigenvalues $\rightsquigarrow$ $\left\{ \text{sign of } \lambda_i \text{ determines definiteness} \right.$

Ex: $Q(\underline{x}) = x_1^2 + 2x_1x_2 + x_2^2 + x_3^2$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \cdot \big((1-\lambda)(1-\lambda) - 1 \big) = 0$$

$$(1-\lambda) \cdot (\lambda^2 - 2\lambda) = 0$$

$$\underline{\lambda = 1}, \ \underline{\lambda = 0}, \ \underline{\lambda = 2}$$

Q is <u>positive semidefinite</u>. (since $\lambda_i \geq 0$ for $i = 1,2,3$)

Principal minors: A symmetric $n \times n$-matrix

A leading principal minor of order k is the minor you get when you keep the first k rows and the first k columns. It is called D_k.

Ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$D_1 = 1$

$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$ row 1, 2 col. 1, 2

$D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$

A principal minor of order k is a minor of order k obtain when you keep the same numbered rows as columns. They are called Δ_k.

Ex: $\Delta_1 = 1, 1, 1$

$\Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ row 2, 3 col. 2, 3

$\Delta_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$

$$\left.\begin{array}{l} \Delta_1 = 1, 1, 1 \\ \Delta_2 = 0, 1, 1 \\ \Delta_3 = 0 \end{array}\right\} \text{Principal minors}$$

$$\left.\begin{array}{l} D_1 = 1 \\ D_2 = 0 \\ D_3 = 0 \end{array}\right\} \text{leading principal minors}$$

Ex: $Q(\underline{x}) = x_1^2 + 2x_2^2 + 7x_3^2$ $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$ $\quad \lambda = 1, \ \lambda = 2, \ \lambda = 7$

$D_1 = 1 \quad D_2 = 2 \quad D_3 = 1 \cdot 2 \cdot 7 = 14$

$Q(\underline{x}) = -x_1^2 - 2x_2^2 - 7x_3^2$ $\quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -7 \end{pmatrix}$ $\quad \lambda_1 = -1, \ \lambda_2 = -2, \ \lambda_3 = -7$

$D_1 = \underline{-1}, \ D_2 = (-1)\cdot(-2) = \underline{2}$

$D_3 = (-1)(-2)(-7) = \underline{-14}$

Result: Q quadratic form, A its symmetric matrix

$\begin{cases} D_1, D_2, \dots, D_n & \text{leading principal m.} \\ \Delta_1, \Delta_2, \dots, \Delta_n & \text{all principal minors} \end{cases}$

Q positive definite $\Longleftrightarrow$ $D_1, D_2, D_3, \dots, D_n > 0$

Q negative definite $\Longleftrightarrow$ D_i has the same sign as $(-1)^i$ for $i = 1, 2, \dots, n$.

(i.e. $(-1)^i D_i > 0$ for $i = 1, 2, \dots, n$)

Q positive semidefinite $\Longleftrightarrow$ $\Delta_i \geq 0$ for all principal minors

Q negative semidefinite $\Longleftrightarrow$ $(-1)^i \Delta_i \geq 0$ — " —

($\Delta_1 \leq 0, \Delta_2 \geq 0, \Delta_3 \leq 0, \dots$)

Q indefinite $\Longleftrightarrow$ all cases that are neither positive semidefinite nor negative semidefinite

Ex:

a) $Q(x_1,x_2,x_3) = x_1^2 + 2x_1x_2 + x_2^2 + x_3^2$ $\quad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$D_1 = 1, \; D_2 = 0, \; D_3 = 0$ leading principal

$\Delta_1 = 1, 1, 1 \quad \Delta_2 = 0, 1, 1 \quad \Delta_3 = 0$ principal

$\left.\begin{array}{l}\Delta_i \geq 0 \text{ for all} \\ \text{principal minors}\end{array}\right\} \Rightarrow$ positive semidefinite

b) $Q(x_1,x_2,x_3) = x_1^2 + 2x_1x_2 + x_2^2 - x_3^2$ $\quad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$D_1 = 1, \; D_2 = 0, \; D_3 = 0$

$\Delta_1 = 1, 1, -1 \quad \Delta_2 = 0, -1, -1, \quad \Delta_3 = 0$

Q <u>not</u> positive semidefinite

Q <u>not</u> negative semidefinite

Q is <u>indefinit</u>

c) $Q(\underline{x}) = x^2 + 2yz$ $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$D_1 = 1 \quad D_2 = 0, \quad D_3 = 1 \cdot (-1) = -1$

$\Delta_1 = \quad \Delta_2 \quad \Delta_3$

Doesn't match pos/neg. semidefinite pattern

$\Downarrow$

<u>indefinite</u>

(3) Quadratic forms with linear constraints

Ex: Max/min $\underbrace{Q(\underline{x}) = x^2 + 6xy - y^2}_{\substack{\text{quadratic form} \\ \text{in } n \text{ variables} \\ (n=2)}}$ subject to $\underbrace{2x - y = 0}_{\substack{m \text{ linear} \\ \text{constraints} \\ (m=1)}}$

Example of constraint optimalization.

When A is the symmetric matrix of the quadratic form Q and B is the coefficient matrix of the constraints,

$$A = \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 \end{pmatrix}$$

we form the bordered Hessian matrix

$$H = \left(\begin{array}{c|c} 0 & B \\ \hline B^T & A \end{array}\right)$$

Result: Assume that B has rank m.
Consider the last $n-m$ leading principal minors of H,

$$D_{2m+1}, D_{2m+2}, \dots, D_{m+n} = |H|$$

last principal minor is $|H|$

All these leading principal minors have same sign as $(-1)^m$ $\iff$ $Q(\underline{x})$ restricted to $\{\underline{x} : B\underline{x} = \underline{0}\}$ is positive definite and $\underline{x} = \underline{0}$ is global minimum

These leading principal minors alternate in sign and $D_{m+n} = |H|$ has the same sign as $(-1)^n$ $\iff$ $Q(\underline{x})$ restricted to $\{\underline{x} : B\underline{x} = \underline{0}\}$ is negative definite and $\underline{x} = \underline{0}$ is global maximum.

In the example: Max/min $Q(\underline{x}) = x^2 + 6xy - y^2$ subject to $2x - y = 0$

$A = \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -1 \end{pmatrix}$

$n = 2$ (two var's) $m = 1$ (one constraint)

B has rk $1 = m$ ok

$n - m = 2 - 1 = 1$

$\Rightarrow$ one leading principal minor to check, $D_3 = |H|$ (the last one)

$$|H| = \left| \begin{array}{c|c} 0 & B \\ \hline B^T & A \end{array} \right| = \begin{vmatrix} 0 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & -1 \end{vmatrix}$$

$$= -2\left(2\cdot(-1) - 3(-1)\right) + (-1)\left(2\cdot 3 - 1\cdot(-1)\right) = -2 - 7 = \underline{-9}$$

positive definite $\Leftrightarrow$ $|H|$ has same sign as $(-1)^m = (-1)^1 = -1$ ← Yes, this condition is satisfied!

negative definite $\Leftrightarrow$ $|H|$ has same sign as $(-1)^n = (-1)^2 = +1$ ← No, this condition is not satisfied

Conclusion: $Q(\underline{x})$ restricted to $\{2x - y = 0\}$ is positive definite

$(x, y) = (0, 0)$ is global minimum.

Alternative solution:

$2x - y = 0 \Rightarrow y = 2x$

$Q(x, y) = x^2 + 6xy - y^2$

$= x^2 + 6x(2x) - (2x)^2 = 9x^2$

↑ positive definite

$x = 0$ global min. $x = 0 \Rightarrow y = 2x = 0$

$\Rightarrow (x, y) = (0, 0)$