LECTURE 4 GRA 6035

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REVIEW OF LECTURE 3:

- VECTORS
- LINEAR INDEPENDENCE

PLAN FOR LECTURE 4:

① EIGENVALUES AND EIGENVECTORS
② DIAGONALIZATION

} [FMEA] 1.5-1.6

① Eigenvalues and eigenvectors

Definition: A square ($n \times n$) matrix

If we have

$$\boxed{A \cdot \underline{x} = \lambda \cdot \underline{x}}, \quad \underline{x} \neq \underline{0}$$

} A: $n \times n$-matrix
$\underline{x}$: n-vector
λ: a number

then

λ is called an eigenvalue for A

$\underline{x}$ — " — eigenvector for A

Example: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad \underbrace{\underline{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \underline{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\text{2-vectors}}$

$\boxed{A \cdot \underline{x} = \lambda \cdot \underline{x}}$ $\quad$ 2x2-matrix

Are $\underline{x}_1, \underline{x}_2$ eigenvectors?

$$A \cdot \underline{x}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ $\underline{x}_1$ is eigenvector with eigenvalue $\underline{\lambda = 3}$.

$$A \cdot \underline{x_2} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\Rightarrow$ x_2 is an eigenvector with eigenvalue $\underline{\lambda = 1}$

$$A \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \neq \lambda \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\Rightarrow$ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is <u>not</u> an eigenvector

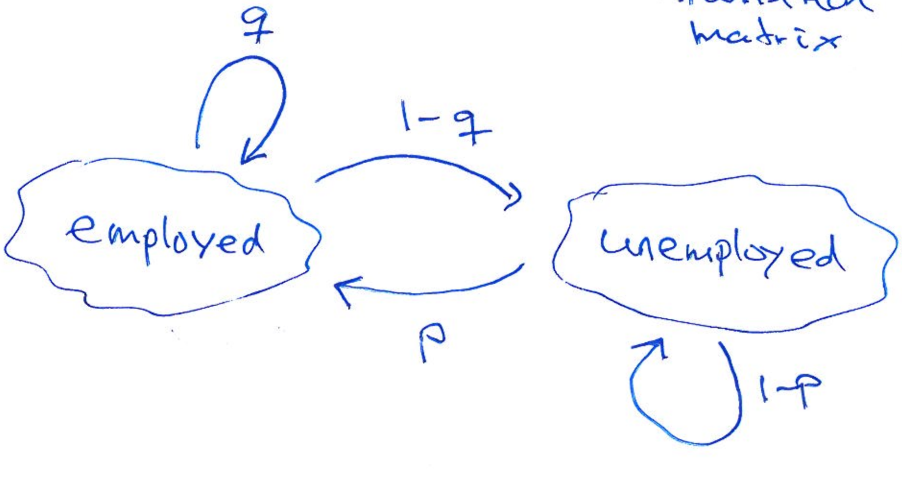
Motivation: Linear dynamical system

$$\underline{v}_0 = \begin{pmatrix} e_0 \\ u_0 \end{pmatrix} \rightsquigarrow \underline{v}_1 = A \cdot \underline{v}_0 \rightsquigarrow \underline{v}_2 = A \cdot \underline{v}_1 = A^2 \underline{v}_0 \rightarrow \dots$$

$$\underline{v}_1 = \begin{pmatrix} e_1 \\ u_1 \end{pmatrix}$$

state vector

transition matrix



$$A = \begin{pmatrix} q & p \\ 1-q & 1-p \end{pmatrix}$$

Ex: $\cancel{x}_0 = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix}$

$$\underline{v}_1 = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix} \cdot \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} = ?$$

$$\underline{v}_2 = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix}^2 \cdot \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} = ?$$

$$\underline{v}_{30} = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix}^{30} \cdot \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} = ?$$

Equilibrium: $\boxed{A \cdot \underline{v} = \underline{v}} = 1 \cdot \underline{v}$ $\quad \left\{ \begin{array}{ll} \lambda = 1 & \text{eigenvalue} \\ \underline{v} & \text{eigenvector} \end{array} \right.$

$$\begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad v = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left. \begin{array}{l} 0.998x + 0.136y = x \\ 0.002x + 0.864y = y \end{array} \right\} \Rightarrow \begin{array}{l} (0.998 - 1)x + 0.136y = 0 \\ 0.002x + (0.864 - 1)y = 0 \end{array}$$

$$-0.002x + 0.136y = 0$$

$$\cancel{0.002x - 0.136y = 0}$$

$$x = \frac{-0.136y}{-0.002} = \cancel{66}\ 68y$$

$$\underline{x = 68y}$$

Normalization: choose $x + y = 1$ to get the answer in percent

$$\boxed{\begin{array}{l} x + y = 68y + y \\ \quad = 69y = 1 \\ \Rightarrow y = \frac{1}{69} \end{array}}$$

long run employment $\rightarrow$ $x = 68 \cdot \frac{1}{69} = \frac{68}{69}$

unemployment $\rightarrow$ $y = \frac{1}{69}$

General method for finding eigenvalues and eigenvectors:

A $n \times n$-matrix

① Find the eigenvalues

$A\underline{x} = \lambda \underline{x}$ equation $\leftarrow$ Find the values of λ such that there is at least one $\underline{x} \neq \underline{0}$ that solves the equation.

$$A\underline{x} = \lambda \underline{x}$$

$$A\underline{x} - \lambda \underline{x} = \underline{0}$$

$$A\underline{x} - \lambda I \cdot \underline{x} = \underline{0}$$

$$(A - \lambda I)\underline{x} = \underline{0}$$

$$(a_{11}-\lambda)\, \cancel{x_1} + a_{12}x_2 + \dots = 0$$
$$a_{21} \quad x_1 + (a_{22}-\lambda)\, x_2 + \dots = 0$$
$$\ddots$$
$$= 0$$

since

$$\lambda \cdot I = \lambda \cdot \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & & & \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 & \dots \\ 0 & \lambda & 0 & \dots \\ \vdots & & & \end{pmatrix}$$

non-trivial solutions $\Big\} \Leftrightarrow |A - \lambda I| = 0$

Conclusion:

The eigenvalues of A are the solutions of the characteristic equation

$$|A - \lambda I| = 0$$

Ex: $A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$ Find the eigenvalues.

Char. eqn:

$$|A - \lambda I| = \left| \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right|$$

$$= \begin{vmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)\cdot(4-\lambda) - 3\cdot 3$$

$$= 4\cdot 4 - 2\cdot 4\cdot \lambda + \lambda^2 - 3\cdot 3$$

$$= \boxed{\lambda^2 - 8\lambda + 7 = 0} \Rightarrow \lambda = 7, \lambda = 1$$

eigenvalues of A

In general: A $n \times n$-matrix $\Rightarrow$ $|A - \lambda I| = 0$ equation of degree n

The case $n=2$:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\det(A) = ad - bc$
$\mathrm{tr}(A) = a + d$ (trace = sum of the entries on the diagonal)

Char. eqn: $\boxed{\lambda^2 - \mathrm{tr}(A)\cdot\lambda + \det(A) = 0}$

Ex: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad tr A = 2 \quad det(A) = 1$

$|A - \lambda I| = 0 \iff \lambda^2 - 2\lambda + 1 = 0$

$\lambda = 1$ (multiplicity 2)

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda_1 = 1 \;,\; \lambda_2 = 1$

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$|A - \lambda I| = 0 \iff \lambda^2 - 0 \cdot \lambda + 1 = 0$

$\lambda^2 + 1 = 0$

no (real) solutions

$\Downarrow$

no eigenvalues

Ex: $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ Find the eigenvalues.

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda) \cdot \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda) \cdot \left((2-\lambda)(2-\lambda) - 1^2 \right) = (1-\lambda)(\lambda^2 - 4\lambda + 3) = 0$$

Concl: $(1-\lambda)(\lambda-1)(\lambda-3) = 0$ $\qquad \lambda = 1$ or $\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 1$ (mult. 2), $\lambda = 3$ $\qquad \lambda_2 = 1, \lambda_3 = 3$

Fact:

If A is an $n \times n$-matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ (not necessarily different), then we have

$$\boxed{\begin{aligned} \det(A) &= \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_n \\ tr(A) &= \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n \end{aligned}}$$

(2) Compute eigenvectors for each eigenvalue

If A has eigenvalue λ, we compute the eigenvectors of A with eigenvalue λ by solving the linear system

$$(A - \lambda I) \cdot \underline{x} = \underline{0}$$

Ex: $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ $\quad \lambda = 3$

Eigenvectors for $\lambda = 3$:

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\uparrow$

$(A - \lambda I)$ with $\lambda = 3$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$\downarrow$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\uparrow$

one free var. z

$$\begin{aligned} -x \quad + z &= 0 \\ -2y \quad &= 0 \end{aligned}$$

$$\begin{cases} x = z \\ y = 0 \\ z \text{ is free} \end{cases}$$

$$\left.\begin{aligned} x &= z \\ y &= 0 \\ z &= z \quad (\text{free}) \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Conclusion:

The eigenvector for A corresponding ~~to~~ to $\lambda = 3$: $\quad z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

In general, when you solve a homogeneous linear system with d degrees of freedom, you get the solutions in standard form as

$$s_1 \cdot \underline{v}_1 + s_2 \cdot \underline{v}_2 + \dots + s_d \cdot \underline{v}_d$$

where $s_1, s_2, \dots, s_d$ are the free variables.

Fact:

If you use an echelon form of the coefficient matrix to solve the system, then

$$\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_d\}$$

are linearly independent.

Eigenvectors for $\lambda = 1$:

$$\begin{pmatrix} 2-1 & 0 & 1 \\ 0 & 1-1 & 0 \\ 1 & 0 & 2-1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{matrix} \\ -1 \\ \end{matrix}$$

$$\Downarrow$$

$$x + \cancel{0} + z = 0 \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

y, z free

$$\left.\begin{aligned} x &= -z \\ y &= y \\ z &= z \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix}$$

$$= y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent

Fact:

If λ is an eigenvector of multiplicity m, the equations determining the eigenvector for λ has

- at least one degree of freedom
- at most m degrees of freedom

Ex: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\lambda = 1$ eigenvalue
multiplicity $= 2$

Eigenvectors for $\lambda = 1$:

$$\begin{pmatrix} 1-1 & 1 \\ 0 & 1-1 \end{pmatrix} \underline{x} = \underline{0} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow y = 0$$

x free
one degree of freedom

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \cdot \underline{\underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}} \leftarrow \begin{cases} \text{one degree of freedom} \\ \text{multiplicity } 2 \end{cases}$$

② Diagonalization:

$$A = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix} \qquad A^{100} = ? \qquad A^n = ?$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix} \qquad D^{100} = \begin{pmatrix} 1^{100} & 0 \\ 0 & 0.4^{100} \end{pmatrix}$$

$$D^n = \begin{pmatrix} 1^n & 0 \\ 0 & 0.4^n \end{pmatrix}$$

Definition:

A matrix A is diagonalizable if there exists an invertible matrix P such that

$$P^{-1}AP = D$$

is diagonal.

Motivation:

$$P^{-1}AP = D \iff A = PDP^{-1}$$

$$\Downarrow$$

~~For~~ $$A^n = A \cdot A \cdot A \cdot \ldots \cdot A$$

$$= PD\cancel{P^{-1}} \cdot \cancel{P}D\cancel{P^{-1}} \cdot \cancel{P}DP^{-1} \ldots PDP^{-1}$$

$$= P \cdot D^n \cdot P^{-1}$$

How to find a diagonalization, if possible:

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A ($n \times n$-matrix), and $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ are eigenvector for A ($\underline{v}_i$ is eigenvector with eigenvalue λ_i), then

$$A \cdot \underline{v}_1 = \lambda_1 \underline{v}_1 \qquad A\underline{v}_2 = \lambda_2 \underline{v}_2 \qquad \cdots \qquad A\underline{v}_n = \lambda_n \underline{v}_n$$

Form the matrixes:

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix} \qquad P = \left(\underline{v}_1 \,\middle|\, \underline{v}_2 \,\middle|\, \ldots \,\middle|\, \underline{v}_n \right)$$

eigenvalues on the diagonal

eigenvectors as columns

Then:

$$A \cdot P = P \cdot D \qquad \left\{ \begin{aligned} & A \cdot (\underline{v}_1 | \underline{v}_2 \ldots | \underline{v}_n) = (\lambda_1 \underline{v}_1 | \lambda_2 \underline{v}_2 | \ldots) \\ & = P \cdot D \end{aligned} \right.$$

If moreover $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ are linearly independent, then P is invertible, and then

$$AP = PD \Longrightarrow P^{-1}AP = D$$

Facts:

① A is diagonalizable $\Longleftrightarrow$ A has n linearly independent eigenvectors

$\Longleftrightarrow$ ⓐ all eigenvalues are real (n eigenvalue counted with multiplicity)

n eigenvalues counted with multiplicity

and

ⓑ the number of degrees of freedom for each eigenvalue equals the multiplicity.

must check that the number of degrees of freedom $= m$ for each eigenvalue λ of multiplicity $m \geq 2$

Ex: $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Diagonalizable: * $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 3$ (three eigenvalues, counted with multiplicity) ✓
* The eigenvectors for $\lambda = 1$ are given by two free variables ✓

$\lambda_1 = 1$ (mult 2)
$\lambda = 3$ (mult 1)

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $P = \left(\underline{v_1} \middle| \underline{v_2} \middle| \underline{v_3} \right)$

② Fact: A symmetric matrix is always diagonalizable.