LECTURE 2

GRA 6035

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REVIEW OF LECTURE I:

- LINEAR SYSTEMS AND THEIR SOLUTIONS
- GAUSSIAN ELIMINATION
- THE RANK OF A MATRIX

} [LS6E] Ch. 1-3

Rank of a matrix: rk A = # pivots in an echelon form of A
= # pivot positions in A
(# = number of)

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$\operatorname{rk} A = \underline{\underline{3}}$

Note: i) If $A = 0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, then $\operatorname{rk} A = 0$ (no pivots)

This is the only type of a matrix with rank = 0.

ii) If A is $m \times n$-matrix (m rows, n columns), then $\operatorname{rk} A \leq m$ and $\operatorname{rk} A \leq n$

Ex: $A = \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} \Rightarrow \operatorname{rk} A = 0, 1, 2$ (depending on the numbers *)

2x3-matrix

$$\hat{A} = (\boxed{A} \mid \underline{b})$$

Proposition: We consider a linear system in n var's $x_1, x_2, \ldots, x_n$ with coefficient matrix A and augmented matrix $\hat{A}$:

i) If $\operatorname{rk} A = \operatorname{rk} \hat{A}$, then the system is consistent (has solutions)
If $\operatorname{rk} A \neq \operatorname{rk} \hat{A}$, then the system is inconsistent (no solutions)

ii) Assume that the system is consistent.
If $\operatorname{rk} A = n$, the system has a unique solution.
If $\operatorname{rk} A < n$, the system has infinitely many solutions and $n - \operatorname{rk} A$ degrees of freedom.

PLAN FOR LECTURE 2:

1. MATRIX ALGEBRA
2. DETERMINANTS
3. MINORS AND RANK

[FMEA] 1.1, 1.3, 1.4, 1.9

1. MATRIX ALGEBRA:

An **$m\times n$-matrix** is a rectangular table of numbers with m rows and n columns.

Ex: $A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \end{pmatrix}$ is a 2×3-matrix

$a_{11} = 2 \quad a_{12} = 3 \quad a_{13} = 5$

$a_{21} = 1 \quad a_{22} = 4 \quad a_{23} = 7$

↑ row 2, column 1

Matrix Operations:

i) Addition / subtraction:
- position by position
- defined if the matrices have the same size

Ex:
$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is not defined

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$

ii) Scalar multiplication
- position by position
- scalar = number

Ex:
$2\cdot\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$

iii) Multiplication.
- $A\cdot B$ is defined if the sizes are compatible

$$\underset{m\times n}{A}\,;\ \underset{n\times p}{B} \longrightarrow \underset{m\times p}{AB}$$

- the entries in the product are computed as

$$\underbrace{(a_{i1}\ a_{i2}\ \dots\ a_{in})}_{\text{row } i \text{ of } A}\cdot \underbrace{\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}}_{\text{column } j \text{ of } B} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}}_{\text{entry in position } (i,j) \text{ in } AB}$$

Ex:
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 10 \end{pmatrix}$$

$1\cdot 1 + 2\cdot 0 = 1$

$2\times 2 \quad 2\times 2$

Identity matrix

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- $A\cdot I = I\cdot A = A$

iv) Transpose

- A^T is obtained by making the rows in A the column in A^T

Ex:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Inverse of $2 = 2^{-1} = \frac{1}{2}$ $\left\{ 2 \cdot \frac{1}{2} = \frac{1}{2} \cdot 2 = 1 \right.$

v) Inverse

- An inverse of A is a matrix A^{-1} such that $A \cdot A^{-1} = A^{-1} \cdot A = I$
- A matrix has at most one inverse
- If A has an inverse, A is called <u>invertible</u>

Ex:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{1} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}}$$

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^{-1}$ does not exist so $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ <u>not</u> invertible

- When $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 2x2:

If $ad-bc=0$, then A is not invertible
If $ad-bc \neq 0$, then

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

<u>Note</u>: When A is square ($n \times n$-matrix), we can compute powers

$$A^2 = A \cdot A$$
$$A^3 = A \cdot A \cdot A$$
$$\vdots$$
$$A^n = \underbrace{A \cdot A \cdot A \cdot \ldots \cdot A}_{n \text{ copies}}$$

MATRIX LAWS:

$\begin{cases} A, B, C & \text{- matrices} \\ r & \text{- a number} \end{cases}$

Whenever the operations are defined, we have:

$$A+B = B+A$$
$$(A+B)+C = A+(B+C)$$
$$A+0 = A$$
$$A+(-A) = 0$$
$$r\cdot(A+B) = rA + rB$$

$$(AB)C = A(BC)$$
$$A\cdot(B+C) = AB+AC$$
$$(A+B)\cdot C = AC+BC$$
$$(rA)\cdot B = A\cdot(rB) = r\cdot(AB)$$
$$A\cdot I = IA = A$$

BUT: $AB \neq BA$

$$(A+B)^T = A^T + B^T$$
$$(rA)^T = rA^T$$
$$(AB)^T = B^T A^T$$
$$(A^T)^T = A$$

$$\left.\begin{aligned} A\cdot A^{-1} &= A^{-1}\cdot A = I \\ (rA)^{-1} &= r^{-1}\cdot A^{-1} \\ (AB)^{-1} &= B^{-1}A^{-1} \\ (A^T)^{-1} &= (A^{-1})^T \\ (A^{-1})^{-1} &= A \end{aligned}\right\}$$

when A, B are invertible matrices and $r \neq 0$

Special matrices:

$$0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

zero matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

identity matrix

Diagonal matrix:

$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$$

Upper / lower triangular matrix:

$$U = \begin{pmatrix} d_1 & * & * & \cdots & * \\ 0 & d_2 & * & \cdots & * \\ 0 & 0 & d_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$$

upper

$$L = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ * & d_2 & 0 & \cdots & 0 \\ * & * & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & d_n \end{pmatrix}$$

lower

Ex: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ upper triangular

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ — " —

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ diagonal

Note: A square echelon form is upper triangular

Symmetric:

$A = A^T$

Ex: $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 7 \\ -1 & 7 & 4 \end{pmatrix}$

Vector:

$\underline{v}$ $m \times 1$-matrix (column vector)

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

matrix with one column

Partioned matrices:

Ex:

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right) \cdot \left(\begin{array}{cc|cc} 0 & 0 & 2 & 4 \\ 0 & 0 & 7 & 5 \\ \hline 7 & 5 & 0 & 0 \end{array}\right)$$

$$A = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 4 \\ 7 & 5 \end{pmatrix} \qquad C = \begin{pmatrix} 7 & 5 \end{pmatrix}$$

$$= \left(I \mid A \right) \cdot \left(\begin{array}{c|c} O & B \\ \hline C & O \end{array}\right)$$

$$= \left(I \cdot O + A \cdot C \mid IB + A \cdot O \right)$$

$$= \left(AC \mid B \right) = \underline{\underline{\left(\begin{array}{cc|cc} 21 & 15 & 2 & 4 \\ 35 & 25 & 7 & 5 \end{array}\right)}}$$

$$A \cdot C = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 7 & 5 \end{pmatrix} = \begin{pmatrix} 21 & 15 \\ 35 & 25 \end{pmatrix}$$

2 Determinants

If A is a square $(n \times n)$ matrix, then we can compute the determinant

$$\det(A) = |A|$$

The result is a number.

The case $n=2$:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The case $n=1$:

$$|a| = a$$

$$\det((a)) = a$$

The general case:

1. Cofactor expansion

A $n \times n$-matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Determinant using cofactor expansion along the first row:

$$\boxed{|A| = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + \cdots + a_{1n} \cdot C_{1n}.}$$

Defn. of cofactors

The cofactor in position (i,j) is

$$C_{ij} = \underset{\text{sign}}{(-1)^{i+j}} \cdot \underset{\text{minor}}{M_{ij}}$$

where

M_{ij} = determinant of the matrix you get when you delete row i and column j.

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ $\qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$$|A| = 1\cdot C_{11} + 1\cdot C_{12} + 1\cdot C_{13}$$

$$= 1\cdot(+1)\cdot M_{11} + 1\cdot(-1)\cdot M_{12} + 1\cdot(+1)\cdot M_{13}$$

$$= +1\cdot\begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= +(-4-2) - (4-1) + (2+1)$$

$$= -6 - 3 + 3 = \underline{\underline{-6}}$$

② Determinant using row operations:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow{-1,\,-1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{\frac{1}{2}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$1\cdot C_{11} + 0\cdot C_{21} + 0\cdot C_{31}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = +1\cdot\begin{vmatrix} -2 & 0 \\ 1 & 3 \end{vmatrix} - 0\cdot * + 0\cdot * = -6$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1\cdot(-2)\cdot 3 = \underline{-6}$$

Facts:

a) The determinant of a triangular matrix is the product of the diagonal entries.

b) i) When you interchange two rows, the determinant changes with a factor -1.

ii) When you add a multiple of one row to another row, the determinant does not change.

iii) When you multiply a row with $c \neq 0$, the determinant changes with a factor c.

Matrix Laws for determinants:

$\begin{cases} A, B & \text{matrices} \\ r & \text{number} \end{cases}$

Whenever the operations are defined, we have:

$|AB| = |A| \cdot |B|$

$|A^T| = |A|$

$|A^{-1}| = 1/|A|$ when A is invertible

$|rA| = r^n \cdot A$ when A is $n \times n$-matrix

Applications:

a) Inverse matrices:

A $n \times n$-matrix

Fact:

If $|A| \neq 0$, then A is invertible (A^{-1} exists)

If $|A| = 0$, then A is not invertible.

Moreover, if $|A| \neq 0$

$$A^{-1} = \frac{1}{|A|} \cdot \underbrace{\underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}}_{\text{cofactor matrix}}{}^{T}}_{\text{adjoint matrix}}$$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$

$|A| = -6 \neq 0 \Rightarrow A^{-1}$ exists

$$A^{-1} = \frac{1}{-6} \cdot \begin{array}{c} \begin{array}{ccc} -6 & 3 & 3 \\ \shortparallel & \shortparallel & \shortparallel \end{array} \\ \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T \end{array}$$

$$= -\frac{1}{6} \begin{pmatrix} -6 & * & * \\ 3 & * & * \\ 3 & * & * \end{pmatrix}$$

b) Linear systems

$x + y + z = 1$

$x - y + z = 4$

$x + 2y + 4z = 7$

$\longleftrightarrow$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$$

3x3 linear system

matrix form

$$A \cdot \underline{x} = \underline{b}$$

$$A^{-1} \cdot (A\underline{x}) = A^{-1}(\underline{b})$$

$$(A^{-1}A)\underline{x} = A^{-1}\underline{b}$$

$$I\underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = A^{-1}\underline{b}$$

Note: $|A| = -6 \neq 0$

$\Rightarrow A^{-1}$ exists

(and we can compute it)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{6} \cdot \begin{pmatrix} 6 & * & * \\ 3 & * & * \\ 3 & * & * \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ -1.5 \\ 2.5 \end{pmatrix}$$

(3) ## Minors and ranks

Ex: $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & -1 & 2 & 1 \\ 1 & 7 & 7 & 3 \end{pmatrix}$ 3×4-matrix

A <u>minor</u> of A of order k is the determinant of a submatrix of order k of A obtained by deleting some rows and some columns.

Ex:

Minors of order 3: $\overset{=0}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 1 & 7 & 7 \end{vmatrix}}$, $\overset{\neq 0}{\begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 1 \\ 1 & 7 & 3 \end{vmatrix}}$, $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 2 & 1 \\ 1 & 7 & 3 \end{vmatrix}$, $\begin{vmatrix} 2 & 3 & 5 \\ -1 & 2 & 1 \\ 7 & 7 & 3 \end{vmatrix}$

deleted col. 4 — deleted col. 3 — " col. 2 — " col. 1

Minors of order 2: $\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}, \ldots$

deleted row 3
col 3,4

Minors of order 1: $1, 2, 3, 5, 2, -1, 2, 1,$
$1, 7, 7, 3$

Rank and minors:

Proposition:

$\operatorname{rk} A =$ maximal order of a non-zero minor

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & -1 & 2 & 1 \\ 1 & 7 & 7 & 3 \end{pmatrix}$$

3×4

Maximal order of minors: 3

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 1 & 7 & 7 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 1 \\ 1 & 7 & 3 \end{vmatrix} \neq 0$$

deleted col. 3

This means:

$$\underline{\underline{\operatorname{Rk} A = 3}}$$

Ex:

$$\begin{aligned} x + 2y + 3z + 5w &= 0 \\ 2x - y + 2z + w &= 0 \\ x + 7y + 7z + 3w &= 0 \end{aligned}$$

$x=0, y=0, z=0, w=0$ is a solution.

$3 = \operatorname{rk} A$

$\Downarrow$

$4 - 3 = 1$ degree of freedom

$n - \operatorname{rk} A$

In fact, z is a free variable since we deleted col. $3 = z$

Prop:

If A is a square $n \times n$-matrix, then

$$\operatorname{rk} A = n \iff \det(A) \neq 0.$$