LECTURE 13 — GRA 6035

EIVIND ERIKSEN, NOV 18TH 2011

Plan:
1. Finish differential equations / difference equations — [FMEA] 6.4, 11.4
2. Revision of constrained optimization problem; Revision Problems

Last lecture: today
Last problem session: Monday 21st Nov at 17.00–19.45
Exam: $\boxed{\text{Monday Dec 12th}}$

(1) Difference equations, Stability of Differential/difference equations

Linear difference equations with constant coefficients

	homogeneous	inhomogeneous
first order	$y_{t+1} + ay_t = 0$ Char. eqn: $r + a = 0$ $r = -a$ Solution: $y_t = C\cdot(-a)^t$	$y_{t+1} + ay_t = f_t$ Superposition: $y_t = y_t^h + y_t^p$ $y_t^h = C_1\cdot(-a)^t$ y_t^p: "guess solution" and check/adjust param.
Second order	$y_{t+2} + ay_{t+1} + by_t = 0$ Char. eqn: $r^2 + ar + b = 0$ Solution: i) $y_t = C_1\cdot r_1^t + C_2\cdot r_2^t$, $r_1 \neq r_2$ ii) $y_t = C_1\cdot r_1^t + C_2 t\cdot r_1^t$, r_1 iii) $y_t = (\sqrt{b})^t(C_1\cos(\theta t) + C_2\sin(\theta t))$, $\theta = \cos^{-1}\left(-a/2\sqrt{b}\right)$	$y_{t+2} + ay_{t+1} + by_t = f_t$ Superposition: $y_t = y_t^h + y_t^p$ y_t^h: see homogeneous case $(\leftarrow)$ y_t^p: "guess solution" and check/adjust parameters

Example: $y_{t+2} = y_{t+1} + y_t$, $y_0 = 0$, $y_1 = 1$ (Fibonacci)

$y_{t+2} - y_{t+1} - y_t = 0$ ← second order, homogeneous

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{1 - 4\cdot(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2} \Rightarrow r_1 = \frac{1+\sqrt{5}}{2} \neq r_2 = \frac{1-\sqrt{5}}{2}$$

General solution: $y_t = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t$

$y_0 = 0$: $y_0 = C_1 \cdot 1 + C_2 \cdot 1 = 0 \Rightarrow C_1 + C_2 = 0 \Rightarrow C_2 = -C_1$

$y_1 = 1$: $y_1 = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right) - C_1\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$$\Rightarrow C_1\left(\cancel{\tfrac{1}{2}} + \frac{\sqrt{5}}{2} - \cancel{\tfrac{1}{2}} + \frac{\sqrt{5}}{2}\right) = 1$$

$$C_1 \cdot \sqrt{5} = 1 \Rightarrow C_1 = \frac{1}{\sqrt{5}},\ C_2 = -\frac{1}{\sqrt{5}}$$

Solution: $y_t = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^t - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^t$

Check: $t = 6$ gives $y_6 = 8$ (using formula and calculator)

Stability of differential / difference equation:

Linear difference equation: (second order case)

Solution: $y_t = y_t^h + y_t^p = \underbrace{C_1 u_t + C_2 v_t}_{y_t^h} + y_t^p$

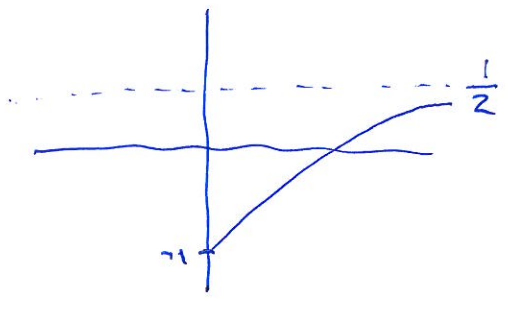
Linear differential equation: (second order case)

Solution: $y(t) = y_h(t) + y_p(t) = \underbrace{C_1 \cdot u(t) + C_2 \cdot v(t)}_{y_h(t)} + y_p(t)$

What is the long term behaviour of the solution?

That is, what happens when $t \to \infty$?

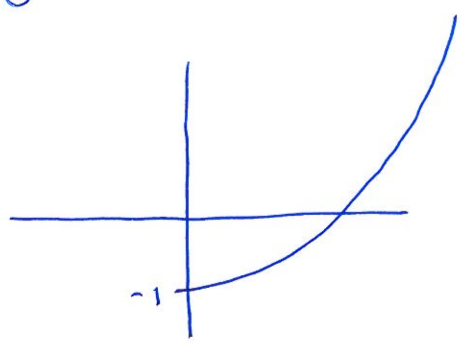
$y(t) = 2e^{-2t} - 3e^{-3t} + \frac{1}{2}$



stable with equilibrium $= \frac{1}{2}$

$\lim_{t \to \infty} y(t) = \frac{1}{2}$

$y(t) = 2e^{t} - 3e^{-t} + \frac{1}{2}$



unstable

$\lim_{t \to \infty} y(t) = \infty$

Note: We use initial conditions to find C_1 and C_2
when initial conditions change, C_1 and C_2 will change

Will this affect <u>long term behaviour</u> or just give short term effects?

Defn: A linear difference / differential equation is <u>globally asymptotically stable</u> if

$$\lim_{t\to\infty} C_1 u(t) + C_2 v(t) = 0 \quad \text{for all } C_1, C_2$$
$$\boxed{C_1 \cdot u_t + C_2 v_t} \longleftarrow \text{difference eqn. case}$$

If this is the case, then changes in initial conditions have <u>no long term effects</u>.

Example: $\ddot{y} + 5\dot{y} + 6y = 3$

$r^2 + 5r + 6 = 0$

$r = -2, -3 \quad \Longrightarrow \quad y = C_1 e^{-2t} + C_2 e^{-3t} + \underbrace{\tfrac{1}{2}}_{y_p(t)}$

$$\lim_{t\to\infty} C_1 e^{-2t} + C_2 e^{-3t} = 0$$
$$\Downarrow$$
<u>globally asympt. stable</u>

(equilibrium $= 1/2$ no matter what the initial conditions / C_1, C_2 are)

Fact:

<u>Differential equation</u>: $\left(\text{All Char. roots } r_i \text{ s.t. } r_i < 0\right) \Longrightarrow$ globally asymptotically stable

<u>Difference equation</u>: $\left(\text{All Char. roots } r_i \text{ s.t. } |r_i| < 1\right) \Longrightarrow$ globally asymptotically stable

Revision Problems

1. Consider the optimization problem

$$\max x^2y^2z^2 \text{ subject to } x^2+y^2+z^2=3$$

a) Write down the Lagrangian $\mathscr{L}$ and the first order conditions for this problem.
b) Find all admissible points that satisfy the first order conditions. Hint: Try to find such points with $x \neq 0, y \neq 0, z \neq 0$ first, these are the most important solutions since $f = 0$ if one of the coordinates are zero.
c) Check that that point $(x,y,z) = (1,1,1)$ is an admissible point that satisfy the first order conditions, and use the bordered Hessian at this point to determine if it is a local maximum points for the constrained optimization problem.
d) Is the set of admissible points closed and bounded? Use this to solve the optimization problem.

2. Consider the optimization problem

$$\max f(x,y,z) = 2z \text{ subject to } x^2+y^2=2,\ x+y+z=1$$

a) Write down the Lagrangian $\mathscr{L}$ and the first order conditions for this problem.
b) Solve the optimization problem. What is the maximum value?
c) Write down the NDCQ for this problem. It NDCQ satisfied for all admissible points (x,y,z)? It is necessary to check NDCQ to solve this optimization problem?
d) Change the last constraint to $x+y+z=b$. Show that the problem has a solution, a maximal value, for each value of b. How does this maximum value change if you increase b?

3. Consider the Kuhn-Tucker optimization problem

$$\max f(x,y,z) = 2z \text{ subject to } x^2+y^2 \leq 2,\ x+y+z \leq 1$$

a) Write down the Lagrangian $\mathscr{L}$ and the first order conditions for this problem. Also, write down the complementary slackness conditions.
b) Solve the optimization problem. What is the maximum value?
c) Write down the NDCQ for this problem. It NDCQ satisfied for all admissible points (x,y,z)? It is necessary to check NDCQ to solve this optimization problem?

2 Revision: Constrained optimization problems

a) Bordered Hessian:

$\boxed{\text{Revision Problems, Problem I}}$

We solve the problem and revise the theory at the same time.

Problem I: max $x^2y^2z^2$ subj. to $x^2+y^2+z^2=3$

a) Lagrangian: $\mathcal{L} = x^2y^2z^2 - \lambda\cdot(x^2+y^2+z^2)$

First order conditions:

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 2x\cdot y^2z^2 - \lambda\cdot 2x = 0 \\ \mathcal{L}'_y &= 2y\cdot x^2z^2 - \lambda\cdot 2y = 0 \\ \mathcal{L}'_z &= 2z\cdot x^2y^2 - \lambda\cdot 2z = 0 \end{aligned}}$$

b) Admissible points = points that satisfy the constraints

Constraints:

$\boxed{x^2+y^2+z^2=3}$

Solve FOC (first order conditions) + Constraints

$$\boxed{\begin{aligned} 2x(y^2z^2-\lambda) &= 0 \\ 2y(x^2z^2-\lambda) &= 0 \\ 2z(x^2y^2-\lambda) &= 0 \\ x^2+y^2+z^2 &= 3 \end{aligned}}$$

can divide by x, y, z

Case i): $x\neq 0, y\neq 0, z\neq 0$

$y^2z^2=\lambda$
$x^2z^2=\lambda$
$x^2y^2=\lambda$

$\Rightarrow$ $y^2\cancel{z^2}=x^2\cancel{z^2} \Rightarrow x^2=y^2$

$\cancel{x^2}z^2=\cancel{x^2}y^2 \Rightarrow z^2=y^2$

$\left.\begin{array}{l} x^2=y^2=z^2 \\ x^2+y^2+z^2=3 \end{array}\right\} \Rightarrow x^2=y^2=z^2=1$

$\lambda = y^2z^2=1$

Cand: $(x,y,z;\lambda)=(\pm1,\pm1,\pm1;1)$

$f=x^2y^2z^2=\underline{1}$

Case ii): $x=0$ or $y=0$ or $z=0$

Not so important; if there are such solutions then $f=x^2y^2z^2=0$ is <u>not max</u>

There are a lot of solutions with $f=0$

$x=0, \lambda=0, y^2+z^2=3$
$y=0, \lambda=0, x^2+z^2=3$
$z=0, \lambda=0, x^2+y^2=3$

Theory:

If you have a Lagrange Problem = optimization problem with equality constraints, and you solve

$\boxed{\text{FOC = first order conditions}}$ + $\boxed{\text{Constraints}}$

you get candidates for max / min. You have to do more to check if the solutions are actually max/min.

Methods:

① You can use the Bordered Hessian at a specific point in the list of candidates to find out if the point is a local max/min (details explained later)

② You can use concavity / convexity of the Lagrangian at values of x's from a spesific point in the candidate list to find out if the point is a global max/min (details explained later).

With local / global max/min, we mean local / global max/min among the admissible points.

c) The point $(x,y,z;\lambda)=(1,1,1;1)$ is one of the candidates for max. We use Bordered Hessian to check if it is local max.

Method: $\begin{cases} n = \#\text{variables} = 3 & (x,y,z) \\ m = \#\text{constraints} = 1 & (x^2+y^2+z^2=3) \end{cases}$

i) Let $C=(g'_x, g'_y, g'_z) = (2x \;\; 2y \;\; 2z)$ be the matrix which contain all partial derivatives of all constraint functions

Let $\mathcal{L}''$ be the Hessian of the Lagrangian $L = x^2y^2z^2 - \lambda(x^2+y^2+z^2)$

$$\mathcal{L}'' = \begin{pmatrix} 2y^2z^2-2\lambda & 4xyz^2 & 4xzy^2 \\ 4xyz^2 & 2x^2z^2-2\lambda & 4yzx^2 \\ 4xzy^2 & 4yzx^2 & 2x^2y^2-2\lambda \end{pmatrix}$$

The bordered Hessian at $(1,1,1;1)$ is the matrix

$$B = \left(\begin{array}{c|c} 0 & C \\ \hline C^T & \mathcal{L}'' \end{array}\right)(1,1,1;1) = \left(\begin{array}{c|ccc} 0 & 2 & 2 & 2 \\ \hline 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ 2 & 4 & 4 & 0 \end{array}\right)$$

ii) Compute the last $n-m=3-1=2$ leading principal minors of B: D_3, D_4

$$D_3 = \begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 4 \\ 2 & 4 & 0 \end{vmatrix} = -2(0-8)+2\cdot(8-0) = 16+16 = 32 > 0$$

$$D_4 = \begin{vmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ 2 & 4 & 4 & 0 \end{vmatrix} \underset{\substack{\uparrow \\ \text{row operations}}}{=} \begin{vmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 0 & 4 & -4 & 0 \\ 0 & 4 & 0 & -4 \end{vmatrix} = -2 \cdot \begin{vmatrix} 2 & 2 & 2 \\ 4 & -4 & 0 \\ 4 & 0 & -4 \end{vmatrix}$$

$$= -2\left(2\cdot(0+16) + (-4)\cdot(-8-8)\right) = -2\cdot 96 = -192 < 0$$

$D_3 > 0$, $D_4 < 0$

Theory:

Compute the last n-m leading principal minors D_i at a point $(\underline{x}; \underline{\lambda})$ that satisfies $\boxed{\text{FOC}}$ + $\boxed{\text{Constraints}}$. Then

$$\left.\begin{array}{l}\text{sign of } D_i = \text{sign of } (-1)^m \text{ for all } i \\ \text{corresponding to n-m last } D_i\end{array}\right\} \Rightarrow \underline{x} \text{ local min}$$

$$\left.\begin{array}{l}\text{sign of } D_i \text{ alternating, with} \\ \text{last sign} = \text{sign of } (-1)^n\end{array}\right\} \Rightarrow \underline{x} \text{ local max}$$

Note: The last n-m $D_i \neq 0$, otherwise we cannot use this test
that is, if one $D_i = 0$, the test is inconclusive

For the case $\begin{cases} n=3 \\ m=1 \end{cases}$:

$D_3 < 0,\ D_4 < 0 \Rightarrow$ local min
$D_3 > 0,\ D_4 < 0 \Rightarrow$ local max

(ii) Use the test above to conclude:

Since $D_3 > 0,\ D_4 < 0 \Rightarrow (x,y,z) = (1,1,1)$ is local max

d) Admissible points: All (x,y,z) with $x^2+y^2+z^2=3$

Equality constraint $\Rightarrow$ closed set

$$\left.\begin{array}{l}\text{Must have } x^2 \leq 3,\ y^2 \leq 3,\ z^2 \leq 3 \\ -\sqrt{3} \leq x \leq \sqrt{3},\ -\sqrt{3} \leq y \leq \sqrt{3},\ -\sqrt{3} \leq z \leq \sqrt{3}\end{array}\right\} \Rightarrow \text{bounded set}$$

The set is closed and bounded $\Rightarrow$ there is a max

extreme value theorem

Theory: When we know that there is a max, the max must be at one of these points:

* A solution to $\boxed{\text{FOC}}$ + $\boxed{\text{Constraints}}$
* A point where $\boxed{\text{NDCQ does not hold}}$ that is admissible

The same applies for min.

$\boxed{\text{FOC}}$ + $\boxed{\text{Constraints}}$ gave $(\pm 1, \pm 1; \pm 1; 1)$ and some points with $\boxed{f=0}$
$\boxed{f=1}$

NDCQ (non-degenerate constraint qualification) is

$$\operatorname{rk}\begin{pmatrix} 2x & 2y & 2z \end{pmatrix} = 1$$

The only point where this does not hold, is $(x,y,z)=(0,0,0)$ and this is not admissible since $x^2+y^2+z^2 = 0 \neq 3$.

$\boxed{\text{NDCQ does not hold}}$ + $\boxed{\text{Constraints hold}}$ give no points
$\qquad \qquad \qquad \qquad \qquad \qquad \qquad$ ‖
$\qquad \qquad \qquad \qquad \qquad \qquad$ admissible

Conclusion:

Candidates: $\underset{f=1}{(\pm 1; \pm 1, \pm 1; 1)}$, $\underset{f=0}{\text{some points}}$ $\Rightarrow$ max: $f=1$ for $(x,y,z)=(\pm 1, \pm 1, \pm 1)$

b) <u>Lagrange Problem</u> : $\boxed{\text{Revision Problems , Problem 2}}$

<u>Problem 2</u>: max $2z$ subject to $\begin{cases} x^2+y^2=2 \\ x+y+z=1 \end{cases}$

a) Lagrangian: $\mathcal{L} = 2z - \lambda_1(x^2+y^2) - \lambda_2(x+y+z)$

FOC:
$$\boxed{\begin{aligned} \mathcal{L}'_x &= -\lambda_1 \cdot 2x - \lambda_2 = 0 \\ \mathcal{L}'_y &= -\lambda_1 \cdot 2y - \lambda_2 = 0 \\ \mathcal{L}'_z &= 2 - \lambda_2 \qquad\; = 0 \end{aligned}}$$

b) $\boxed{\text{FOC}} + \boxed{\text{Constraints}}$:

$$\boxed{\begin{aligned} -\lambda_1 \cdot 2x - \lambda_2 &= 0 \\ -\lambda_1 \cdot 2y - \lambda_2 &= 0 \\ 2 \qquad - \lambda_2 &= 0 \\ x^2+y^2 &= 2 \\ x+y+z &= 1 \end{aligned}} \Rightarrow \lambda_2 = 2$$

$\Rightarrow -\lambda_1 \cdot 2x = 2 \Rightarrow x = -1/\lambda_1 \;,\; \lambda_1 \neq 0$

$-\lambda_1 \cdot 2y = 2 \Rightarrow y = -1/\lambda_1 \;,\; \lambda_1 \neq 0$

$\Downarrow$

$x = y$

$x^2+y^2=2 \Rightarrow x^2=y^2=1 \Rightarrow x=y=\pm 1$

$x=y=1 \Rightarrow \lambda_1 = -1 \Rightarrow z=-1$

$x=y=-1 \Rightarrow \lambda_1 = +1 \Rightarrow z = 3$

<u>Solutions</u>: $(x,y,z;\lambda_1,\lambda_2) = (1,1,-1;-1,2)$

$(-1,-1,3;\; 1,2)$

<u>Values</u>: $f(1,1,-1) = -2$

$f(-1,-1,3) = 6$

Candidates for max: $f = 6$ at $(-1,-1,3;1,2)$

<u>Theory: Convex/concave $\mathcal{L}$</u>

Given $(x,y,z;\lambda_1,\lambda_2)$ that solves $\boxed{\text{FOC}} + \boxed{\text{Constraints}}$. Consider

$\mathcal{L}(x,y,z)$ as a function of (x,y,z),
with λ_1 and λ_2 fixed from the points we look at

Then we have:

$\mathcal{L}$ convex $\Rightarrow$ (x,y,z) is <u>global min</u>
$\mathcal{L}$ concave $\Rightarrow$ (x,y,z) is <u>global max</u>

(solves Lagrange Problem with min/max)

We use this theory at the candidate $(-1,-1,3;1,2)$:

$$\mathcal{L} = 2z - \lambda_1(x^2+y^2) - \lambda_2(x+y+z) = +2z - 1\cdot(x^2+y^2) - 2(x+y+z)$$
$$= \cancel{2z} - x^2 - y^2 - 2x - 2y - \cancel{2z} = \underline{-x^2 - y^2 - 2x - 2y}$$

This function is $\underline{\text{concave}}$ since $\mathcal{L}'' = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

This means that $\underline{(-1,-1,3) \text{ gives max}}$

$\underline{\text{max value } f=6}$

c) $\underline{\text{NDCQ}}$: $\boxed{\operatorname{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 1 & 1 \end{pmatrix} = 2}$ ← the condition we should check

$\underline{\text{Check NDCQ}}$: $\operatorname{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{cases} 1, & x=y=0 \\ 2, & \text{otherwise} \end{cases}$

$x=y=0$ is not possible at admissible point, so $\operatorname{rk}=2$

$\underline{\text{NDCQ satisfied for all admissible points}}$

For the solution method we used in b), it was $\underline{\text{not}}$ necessary to check NDCQ.

d) $\underline{x+y+z=b}$: $\boxed{\text{FOC}}$ + $\boxed{\text{Constraints}}$ give almost same solution as in b) (with $b=3$):

$\lambda_2 = 2 \Rightarrow x = y = -1/\lambda_1$

$x^2+y^2 = 2 \Rightarrow x = y = \pm 1$

$z = b-1-1 = b-2$ if $x=y=1$

$z = b+1+1 = b+2$ if $x=y=-1$

$\Downarrow$

$\underline{\text{Candidates}}$:

$(x,y,z;\lambda_1,\lambda_2) = (1,1,b-2;-1,2)$

$(-1,-1,b+2;1,2)$

$f = 2(b-2)$ in $(1,1,b-2)$

$f = 2(b+2)$ in $(-1,-1,b+2)$

$\underline{\text{Best candidate}}$: $(-1,-1,b+2)$, $f = 2b+4$

In $(-1,-1,b+2;1,2)$, we still have

$\mathcal{L} = -x^2 - y^2 - 2x - 2y$ is concave

so $\underline{(x,y,z) = (-1,-1,b+2) \text{ is max}}$

$\underline{f = 2b+4}$ is max value

If b increases, then max value $f = 2b+4$ increases $\left(\frac{\partial f}{\partial b} = 2 = \lambda_2 \right)$.

c) Kuhn-Tucker Problem: Revision Problems, Problem 3

Problem 3: max $2z$ subject to $\begin{cases} x^2+y^2 \leq 2 \\ x+y+z \leq 1 \end{cases}$

Kuhn-Tucker problem in std form
(max problem $g_i(\underline{x}) \leq b_i$)

Note: If the problem is not std form, change it before you start

min f subj. to ... $\rightarrow$ max $-f$ subj. to ...

$g_i(\underline{x}) \geq b_i$ $\rightarrow$ $-g_i(\underline{x}) \leq -b_i$

a) L and FOC the same as in Problem 2.

Complementary slackness conditions:

$$\boxed{\begin{array}{l} \lambda_1 \geq 0 \text{ and } (x^2+y^2<2 \Rightarrow \lambda_1 = 0) \\ \lambda_2 \geq 0 \text{ and } (x+y+z<1 \Rightarrow \lambda_2 = 0) \end{array}}$$

b) Solve $\boxed{\text{FOC}}$ + $\boxed{\text{Constraints}}$ + $\boxed{\text{Complementary Slackness Cond.}}$ = CSC

Divide in 4 cases:

$x^2+y^2=2$ $x+y+z=1$	$x^2+y^2=2$ $x+y+z<1$	$x^2+y^2<2$ $x+y+z=1$	$x^2+y^2<2$ $x+y+z<1$	Constraints
$-\lambda_1 \cdot 2x - \lambda_2 = 0$ $-\lambda_1 \cdot 2y - \lambda_2 = 0$ $2 - \lambda_2 = 0$	$\leftarrow$ same	$\leftarrow$ same	$\leftarrow$ same	FOC
$\lambda_1 \geq 0$ $\lambda_2 \geq 0$	$\lambda_1 \geq 0$ $\lambda_2 = 0$	$\lambda_1 = 0$ $\lambda_2 \geq 0$	$\lambda_1 = 0$ $\lambda_2 = 0$	CSC
From Problem 2: $(1,1,-1;-1,2)$ $(-1,-1,3;1,2)$ First is no soln. since $\lambda_1 = -1 < 0$	no soln: $\lambda_2 = 2$ and $\lambda_2 = 0$	$\lambda_1 = 0$ $\lambda_2 = -\lambda_1 \cdot 2x = 0$ $\lambda_2 = 2 \Rightarrow$ impossible no soln	no soln: $\lambda_2 = 2$ and $\lambda_2 = 0$	

In conclusion, we find one candidate for max by solving

$\boxed{FOC}$ + $\boxed{\text{Constraints}}$ + $\boxed{CSC}$

The point $(x,y,z;\lambda_1,\lambda_2) = (-1,-1,3;1,2)$
with $f = 6$

Theory: Kuhn-Tucker problems

Solutions of $\boxed{FOC}$ + $\boxed{\text{Constraints}}$ + $\boxed{CSC}$ give candidates for max. We must do more to check that a point is actually max.

Method 1: Check that $L(x,y,z)$ is concave when we fix λ_1,λ_2 from a specific candidate point. In that case, the point is max.

Method 2: i) Make sure that there is a max. ← Extreme value theorem

ii) If so, max must be one of the following points

* Solutions of $\boxed{FOC}$ + $\boxed{\text{Constraints}}$ + $\boxed{CSC}$
* Admissible points that do no satisfy NDCQ

Make this list and compute f at each point.

The candidate $(-1,-1,3;1,2)$ gives concave L just as in Problem 2b).
Therefore $(-1,-1,3)$ is max

$f=6$ is max value.

c) NDCQ for Kuhn-Tucker problems: Check all the cases

$\left.\begin{matrix} x^2+y^2 = 2 \\ x+y+z=1 \end{matrix}\right\}$ NDCQ: $\operatorname{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 1 & 1 \end{pmatrix} = 2$ ok for admissible points (same as Problem 2).

$\left.\begin{matrix} x^2+y^2=2 \\ x+y+z<1 \end{matrix}\right\}$ NDCQ: $\operatorname{rk}\begin{pmatrix} 2x & 2y & 0 \end{pmatrix} = 1$ ok for admissible points ($x=y=0$ not adm.)

$\left.\begin{array}{l} x^2+y^2<2 \\ x+y+z=1 \end{array}\right\}$ NDCQ: $\operatorname{rk}(1\ 1\ 1)=1$ ok.

$\left.\begin{array}{l} x^2+y^2<2 \\ x+y+z<1 \end{array}\right\}$ NDCQ: no condition to check ok.

Conclusion: NDCQ satisfied for all admissible points
Not necessary to check this to use Method I in b).

Theory: NDCQ for Kuhn-Tucker problems

Constraints:

$$\boxed{\begin{array}{l} g_1(\underline{x})\le b_1 \\ g_2(\underline{x})\le b_2 \\ \vdots \\ g_m(\underline{x})\le b_m \end{array}}$$

Divide into all the different cases.
Include the row

$$\left(\frac{\partial g_i}{\partial x_1}\quad \frac{\partial g_i}{\partial x_2}\quad \cdots \quad \frac{\partial g_i}{\partial x_n}\right)$$

in the matrix when $g_i(\underline{x})=b_i$ (binding constraints)

NDCQ: rk (this matrix) = #rows (the matrix)
