LECTURE 12 GRA 6035

EIVIND ERIKSEN, NOV. 11 2011

Plan: Today

1. Linear differential equations of order two - review and new material [FMEA] 6.1-6.4

2. Difference equations [FMEA] 11.1-11.4

Tip: Look at the lecture notes from last year regarding differential equations and difference equation

Plan: Next weeks

Lecture 12: today
Lecture 13: next Friday - revision
Problem Session: Monday 21st.

① Linear second order differential equations

$$\boxed{\ddot{y} + a(t)\cdot\dot{y} + b(t)\cdot y = f(t)}$$

where $y = y(t)$ is the unknown function

$\begin{cases} \text{homogeneous}: & f(t) = 0 \\ \text{inhomogeneous}: & f(t) \neq 0 \end{cases}$

constant coefficients: $\begin{cases} a(t) = a \\ b(t) = b \end{cases}$ are constants

Case I: Homogeneous, constant coefficients

$$\boxed{\ddot{y} + a\dot{y} + by = 0}$$

Characteristic equation:

$$r^2 + ar + b = 0$$

Solutions for r: characteristic roots

$y = e^{rt}$ is solution $\Updownarrow$ r is root in char. eqn.

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

① Two distinct roots $r_1 \neq r_2$ when $a^2 - 4b > 0$

② One double root r_1 when $a^2 - 4b = 0$

③ No (real) roots when $a^2 - 4b < 0$

The general solution is :

1	$r_1 \neq r_2$	$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
2	r_1	$y = C_1 e^{r_1 t} + C_2 \cdot t e^{r_1 t}$
3	$\alpha = -\frac{a}{2}$ $\beta = \frac{\sqrt{4b-a^2}}{2}$	$y = e^{\alpha t} \left(C_1 \cos(\beta t) + C_2 \sin(\beta t) \right)$

Ex:

1. $\ddot{y} - 5\dot{y} + 6y = 0$

$r^2 - 5r + 6 = 0$

$r_1 = 2,\ r_2 = 3 \implies y = C_1 e^{2t} + C_2 e^{3t}$

2. $\ddot{y} - 6\dot{y} + 9y = 0$

$r^2 - 6r + 9 = 0$

$r = 3$ (double root) $\implies y = C_1 e^{3t} + C_2 t \cdot e^{3t}$

3. $\ddot{y} - 2\dot{y} + 2y = 0$

$r^2 - 2r + 2 = 0$

$$r = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2}$$

$$= \frac{2}{2} \pm \frac{\sqrt{-4}}{2}$$

$\alpha = 1,\ \beta = \frac{\sqrt{4}}{2} = 1 \implies y = e^{t} \left(C_1 \cos t + C_2 \sin t \right)$

Case 2: Inhomogeneous, constant coefficients

$$\boxed{\ddot{y} + a\dot{y} + by = f(t)}$$

Superposition principle:

The general solution is

$$\boxed{y = y_h + y_p}$$

where

y_h: is the general solution to the homogeneous diff. eqn.

$$\ddot{y} + a\dot{y} + by = 0$$

y_p: is a particular solution of $\ddot{y} + a\dot{y} + by = f(t)$

Ex: $\ddot{y} - 5\dot{y} + 6y = 18 \leftarrow$ inhomogeneous

$$y = y_h + y_p = \underbrace{C_1 e^{2t} + C_2 e^{3t}}_{y_h} + \underbrace{3}_{y_p} = C_1 e^{2t} + C_2 e^{3t} + 3$$

a) y_h: $\ddot{y} - 5\dot{y} + 6y = 0$

$r^2 - 5r + 6 = 0$

$r = 2, 3$ $\Rightarrow y_h = C_1 e^{2t} + C_2 e^{3t}$

b) y_p: $\ddot{y} - 5\dot{y} + 6y = 18$

Guess: $y =$ constant is a solution

$$\left.\begin{array}{l} y = A \\ \dot{y} = 0 \\ \ddot{y} = 0 \end{array}\right\} \quad 0 - 5\cdot 0 + 6\cdot A = 18$$

$$A = 3$$

$$y_p = 3$$

How to find y_p in general:

$$\ddot{y} + a\dot{y} + by = f(t) \qquad \text{Guess } y_p = ?$$

i) Guess should depend on parameters.

ii) Guess should be of the same kind as $f(t)$.

Ex: $\ddot{y} - 5\dot{y} + 6y = e^t$ $\qquad y = y_h + y_p$

Guess: $y_p = A\cdot e^t$

$$\left.\begin{array}{l} y = Ae^t \\ \dot{y} = Ae^t \\ \ddot{y} = Ae^t \end{array}\right\} \quad A\cancel{e^t} - 5\cdot A\cancel{e^t} + 6A\cancel{e^t} = \cancel{e^t} \quad |: e^t$$

$$A - 5A + 6A = 1$$

$$2A = 1$$

$$A = 1/2$$

$$y_p = \frac{1}{2}e^t$$

Ex: $\ddot{y} - 5\dot{y} + 6y = te^t$

Guess: $y_p = A \cdot te^t + B \cdot e^t$

$f(t) = te^t$

$f'(t) = 1 \cdot e^t + te^t$
$= te^t + e^t$

$f''(t) = te^t + e^t + e^t$
$= te^t + 2e^t$

Choose y_p of the general form fitting f, f', f''.

$y = Ate^t + Be^t$

$\dot{y} = A \cdot (te^t + e^t) + Be^t$
$= Ate^t + (A+B)e^t$

$\ddot{y} = A \cdot (te^t + e^t) + (A+B)e^t$
$= A \cdot te^t + (2A+B)e^t$

$\ddot{y} - 5\dot{y} + 6y = te^t$

$\left[Ate^t + (2A+B)e^t\right] - 5\left[Ate^t + (A+B)e^t\right] + 6\left[Ate^t + Be^t\right] = te^t$

$(A - 5A + 6A)\, te^t + (2A + B - 5A - 5B + 6B) \cdot e^t = 1 \cdot te^t + 0 \cdot e^t$

$2A = 1 \Rightarrow A = 1/2$

$2A + B - 5A - 5B + 6B = 0$

$-3A + 2B = 0 \Rightarrow 2B = 3A = \frac{3}{2} \Rightarrow B = 3/4$

$y_p = \frac{1}{2} \cdot te^t + \frac{3}{4} \cdot e^t$

Conclusion:

- Important to guess "correctly" $\Rightarrow$ Use the computation of f, f', f''.

- Even if we guess "correctly", there are some special case where we don't find solutions. Try to multiply your guess by t.

- Superposition principle can also be used for first order linear diff. eqn.

Ex: $\dot{y} - 5y = e^t$

$$y = y_h + y_p = \underline{\underline{C_1 e^{5t} - \frac{1}{4} e^t}}$$

y_h: $\dot{y} - 5y = 0$

$r - 5$

$r = 5$

y_p: $y = Ae^t$

$A\cancel{e^t} - 5A\cancel{e^t} = \cancel{e^t}$

$-4A = 1$

$A = -1/4$

② Difference equations

Ex: $y_{t+1} = 2y_t$, $y_0 = 1$

$y_0 = 1$
$y_1 = 2 \cdot 1 = 2$ $(t=0)$
$y_2 = 2 \cdot 2 = 4$ $(t=1)$
$y_3 = 8$
$y_4 = 16$
$y_5 = 32$
$\vdots$

$$y_{t+1} = 2^{t+1}$$
$$2y_t = 2 \cdot 2^t = 2^{t+1}$$

$\boxed{y_t = 2^t}$, $t = 0, 1, 2, 3, \ldots$ closed form

Defn: A difference equation is a recurrence relation that relates terms of a sequence with one or more earlier terms.

A solution is a sequence that fits in the recurrance relation.

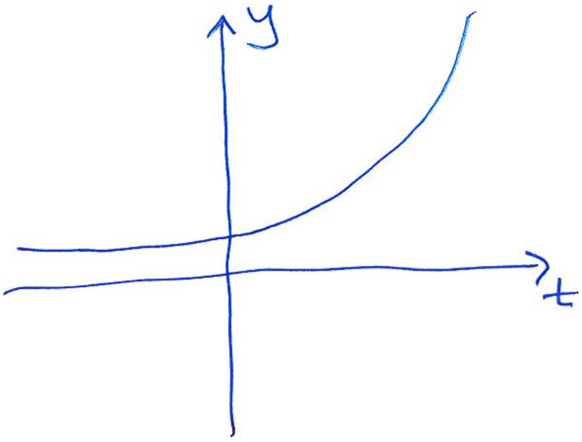
We often want a solution in closed form.

(an expression of y_t that depends on t only)

Differential eqn.

$y' = r \cdot y$

Solution: $y = Ce^{rt}$

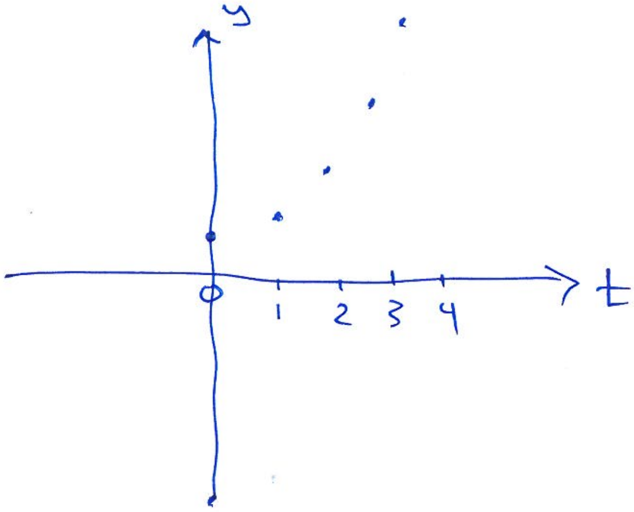


$y' = ry$

Difference eqn.

$y_{t+1} = (1+r) y_t$

Solution: $y_t = C(1+r)^t$



$$\underbrace{y_{t+1} - y_t}_{\text{change in } y_t} = r y_t$$

change in y_t = measure of change in discrete situation

Ex: Fibonacci sequence

$$y_{t+2} = y_{t+1} + y_t \quad , \quad y_0 = 0 \, , \, y_1 = 1$$

a) Second order difference eqn: the difference between the indices $t, t+1, t+2$ is $\underline{2}$.

b)

t	0	1	2	3	4	5	6	7	8	9	
y_t	0	1	1	2	3	5	8	13	21	34	--

closed formula for $y_t = \; ?$

First order difference equations: Linear

$$y_{t+1} + a(t) \cdot y_t = f(t)$$

linear first order difference equation

homogeneous : $f(t) = 0$
inhomogeneous: $f(t) \neq 0$

constant coefficients: $a(t) = a$ is constant

Case I: Homogeneous, const. coefficients

$$y_{t+1} + a \cdot y_t = 0 \qquad y_{t+1} = (-a) \cdot y_t$$

General solution: $y_t = \underline{\underline{C \cdot (-a)^t}}$

Ex:

$$y_{t+1} = 1.10 \cdot y_t \quad , \quad y_0 = 100$$

$$y_{t+1} - 1.10\, y_t = 0 \Rightarrow y_t = C \cdot 1.10^t$$

Char. eqn:

$$y_{t+1} - 1.10\, y_t = 0 \Rightarrow \text{Char. eqn:} \quad r - 1.10 = 0$$

$$r = 1.10$$

General solution: $y_t = C \cdot r^t$

$y_0 = 100$: $\quad 100 = C \cdot r^0 = C \cdot 1.10^0 = C$

$$\Rightarrow C = 100$$

$$y_t = 100 \cdot 1.10^t$$

Case 2: Inhomogeneous, constant coefficients

$$y_{t+1} + a y_t = f(t)$$

Superposition principle: $y = y_h + y_p$

where

y_h: general solution of $y_{t+1} + a y_t = 0$

y_p: particular solution of $y_{t+1} + a y_t = f(t)$

Ex: $y_{t+1} = 1.05\, y_t - 100$, $y_0 = 5000$

$$y_{t+1} - 1.05\, y_t = -100$$

$$y_t = y_t^h + y_t^p = C \cdot 1.05^t + 2000$$

y_t^h: $y_{t+1} - 1.05\, y_t = 0$

$$r - 1.05 = 0$$
$$r = 1.05 \implies y_t^h = C \cdot 1.05^t$$

y_t^p = a constant

y_t^p: $\left.\begin{array}{l} y_t = A \\ y_{t+1} = A \end{array}\right\}$ $A - 1.05A = -100$

$$-0.05A = -100$$
$$A = 2000$$
$$y_t^p = 2000$$

$y_0 = 5000$: $5000 = C \cdot 1.05^0 + 2000$

$$5000 = C + 2000$$
$$C = 3000$$

Solution: $y_t = 3000 \cdot 1.05^t + 2000$

Note: $C \neq y_0$ in the inhomogeneous case.

Linear second order difference equations.

$$\boxed{y_{t+2} + a(t)\, y_{t+1} + b(t)\, y_t = f(t)}$$

homogeneous: $f(t) = 0$
inhom. $f(t) \neq 0$

constant coeff: $a(t) = a$, $b(t) = b$ are const.

① Homogeneous, constant coeff:

$$\boxed{y_{t+2} + a\, y_{t+1} + b\, y_t = 0}$$

Characteristic equation:

$$r^2 + ar + b = 0$$

Find solution = char. roots.

$y_t = r^t$ is a solution of the difference eqn.
$\Updownarrow$
r is a solution of the char. eqn.

(a) $r_1 \neq r_2$ distinct roots: (if $a^2 - 4b > 0$)

$$y_t = C_1 r_1^t + C_2 \cdot r_2^t$$

(b) r_1 double root: (if $a^2 - 4b = 0$)

$$y_t = C_1 \cdot r_1^t + C_2 \cdot t r_1^t$$

(c) no real solutions: (if $a^2 - 4b < 0$)

$$y_t = (\sqrt{b})^t \cdot \left(C_1 \cos(\theta t) + C_2 \sin(\theta t) \right),$$

where $\theta = \cos^{-1}(-a/2\sqrt{b})$

Ex:

$$y_{t+2} - 3y_{t+1} + 2y_t = 0$$

$$r^2 - 3r + 2 = 0$$

$$r = 1, r = 2 \implies y_t = C_1 \cdot 1^t + C_2 \cdot 2^t$$

$$= C_1 + C_2 \cdot 2^t$$