HELLO

04.11.11

ln - e GYMNASTICS

$$\ln x = \cancel{\frac{1}{3}\ln(1+t^3)}$$

$$e^{\ln x} = e^{\frac{1}{3}\ln(1+t^3)}$$

$$\overset{?}{=} (1+t^3)^{1/3}$$

FLASHBACK: $\boxed{(e^a)^b = e^{ab}}$

$$e^{\frac{1}{3}\ln(1+t^3)} = \left(e^{\ln(1+t^3)}\right)^{\frac{1}{3}} = (1+t^3)^{\frac{1}{3}}$$

FLASHBACK:

$$(x^x)'_x = \left[\left(e^{\overset{x}{\overset{\shortparallel}{\ln x}}}\right)^x\right]'_x$$

$$= \left(e^{x\ln x}\right)'_x$$

$$= \underline{e^{x\ln x}\left(\ln x + x\cdot\frac{1}{x}\right)}$$

LAST EPISODE:

WE SOLVED

$$\dot{x} + \boxed{2}x = 3$$

IS A RELIC OF A DIFFERENTIATION!

MORE SPECIFICALLY IT IS THE PRODUCT OF A CHAIN RULE DIFFERENTIATION:

$$\left[e^{2x}\right]'_x = \boxed{2}e^{2x}$$

WE NEED THIS

$$e^{2x}\dot{x} + 2e^{2t}\cdot \cancel{x} = 3e^{2t}$$

PRODUCT RULE OF DIFFERENTIATION IN REVERSE

$$\parallel$$

THIS IS t!!

$$\int (e^{2t}x)'_t \, dt = \int 3e^{2t} \, dt$$

$$e^{2t}x = 3\cdot\frac{1}{2}e^{2t} + C$$

$$x(t) = \frac{3}{2}\frac{e^{2t}}{e^{2t}} + \frac{C}{e^{2t}}$$

$$x(t) = \frac{3}{2} + Ce^{-2t}$$

THIS LECTURE WILL TAKE THIS

IDEA TO EXTREMES!!

LET US SOLVE ALL EQUATIONS OF THE TYPE

$$\dot{x} + ax = b \qquad a, b \text{ CONSTANTS.}$$

THIS IS THE RELIC OF $\left[e^{at}\right]'_t = a e^{at}$

SO LET US MULTIPLY WITH e^{at}:

$$e^{at}\dot{x} + ae^{at}x = be^{at}$$

YES

$$\int (e^{at}x)'_t \, dt = \int b e^{at} \, dt$$

$$e^{at}x = b \cdot \frac{1}{a} e^{at} + C$$

$$x(t) = \frac{b}{a} + Ce^{-at}$$

THEORETIC CONSIDERATIONS:

STABLE / UNSTABLE

WE SEE THAT :

$$\lim_{t \to \infty} x(t) = \frac{b}{a} \qquad \text{WHEN } a > 0$$

$$\left(\lim_{t \to \infty} C e^{-at} = 0 \;! \right)$$

WE SAY IN THIS CASE THAT THE SOLUTION

IS **STABLE.**

IF $a < 0 \Rightarrow \lim_{t \to \infty} C e^{at} = \pm\infty$,

WE SAY THAT THE SOLUTION

IS **UNSTABLE.**

BACK ON TRACK:

$$\dot{x} + 2x = 3$$

MULTIPLIED WITH e^{2t}.

e^{2t} IS CALLED THE **INTEGRATING FACTOR**.

(WHY? JUST BECAUSE MULTIPLYING WITH e^{2t} SOLVES THE PROBLEM, SINCE WE CAN INTEGRATE THE LEFT HAND SID: $\left(e^{2t}\dot{x} + 2e^{2t}x\right)$

$$= \left(e^{2t}x\right)'_t$$

)

HOW FAR CAN WE TAKE THIS IDEA?

ALL THE WAY:

WE ARE ABLE SOLVE

$$\dot{x} + a(t)\,x = b(t) \quad !!!$$

LET US LOOK AT THAT NOW!

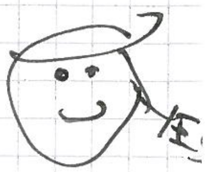
LET US TRY TO MIMICK THIS SOLUTION STRATEGY ON:

$$\dot{x} + \boxed{2t}\, x = t$$

RELIC OF A DIFFERENTIATION

WHICH?

$$\left[e^{t^2}\right]'_t = \underline{2t}\, e^{t^2}$$



$$e^{t^2}\dot{x} + 2t\, e^{t^2} x = t\, e^{t^2}$$

$$\int \left(e^{t^2}\cdot x\right)'_t \, dt = \int t\, e^{t^2}\, dt \qquad \boxed{u = t^2}$$

$$e^{t^2}\cdot x = \frac{1}{2} e^{t^2} + C$$

$$x(t) = \frac{1}{2} + C e^{-t^2}$$

WOW, IT WORKED: e^{t^2} IS

AN INTEGRATING FACTOR!

2. HOUR

$$\dot{x} + a(t)x = b(t)$$

CAN WE ALWAYS FIND AN INTEGRATING FACTOR?

YES! THE FACTOR IS

$$e^{\int a(t)\,dt}$$

THIS LOOKS SCARY, BUT IN PRACTICE NO BIG DEAL.

EXAMPLE: SOLVE THE FOLLOWING INTIAL VALUE PROBLEM

$$\dot{x} + \underline{3t^2}x = e^{-t^3} \qquad x(0)=2$$

SOLUTION:

$$e^{\int a(t)\,dt} = e^{\int \underline{3t^2}\,dt} = e^{t^3}$$

HENCE, WE SHALL MULTIPLY WITH e^{t^3}:

$$e^{t^3}\dot{x} + 3t^2e^{t^3}x = e^{-t^3}\cdot e^{t^3}$$

YES!

$$\int (e^{t^3}x)'\,dt = \int 1\,dt$$

$$e^{t^3}x = t + C$$

$$x = te^{-t^3} + Ce^{-t^3}$$

WE WANTED THE PARTICULAR SOLUTION

$x(0)=2 \Rightarrow x(0) = 0\cdot e^{1} + C\cdot 1 = 2$

$C = 2.$

WE GET:

$$\underline{x_p(t)} = t e^{-t^3} + 2 e^{-t^3}$$

$$= \underline{(t+2) e^{-t^3}}$$

EXACT EQUATIONS

FLASHBACK:

$$\begin{tikzcd}
 & f(x,y) \arrow[dl, "\frac{\partial}{\partial x}"'] \arrow[dr, "\frac{\partial}{\partial y}"] & \\
\frac{\partial f}{\partial x} \arrow[dr, "\frac{\partial}{\partial y}"'] & & \frac{\partial f}{\partial y} \arrow[dl, "\frac{\partial}{\partial x}"] \\
 & \frac{\partial^2 f}{\partial y \partial x} &
\end{tikzcd}$$

THAT $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ IS

CALLED YOUNGS LEMMA.

CONNECTION TO DIFFERENTIAL EQUATIONS:

$$h(x,t) = C$$

DIFFERENTIATE WITH RESPECT TO t $\quad \frac{\partial h}{\partial t}$

$$h(t,x) = c$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\frac{\partial x}{\partial t} = 0$$

Hm! $\cancel{\partial t}$ $\cancel{f(t)+}$

$$\boxed{\frac{\partial h}{\partial t}} + \boxed{\frac{\partial h}{\partial x}}\,\dot{x} = 0$$

THIS IS A DIFFERENTIAL EQUATION,

OF A VERY SPECIAL TYPE:

$$\begin{tikzcd}[column sep=small]
 & h(t,x) \arrow[dl] \arrow[dr] & \\
\boxed{\frac{\partial h}{\partial t}} \arrow[dr] & & \boxed{\frac{\partial h}{\partial x}} \arrow[dl] \\
 & \frac{\partial^2 h}{\partial x \partial t} = \frac{\partial^2 h}{\partial t \partial x} &
\end{tikzcd}$$

THE POINT IS THAT $\frac{\partial h}{\partial t}$ DIFFERENTIATED WITH RESPECT TO x IS EQUAL TO $\frac{\partial h}{\partial x}$ DIFFERENENTIATED WITH RESPECT TO t.

BUT THEN $c(t) = t + k$

k CONSTANT

BUT NOW WE ~~KNO~~ KNOW h:

$$h(t,x) = x^3 + c(t)$$
$$= x^3 + t + k$$

AND THE MOTHER OF THIS EQUATION IS

$$h(t, x) = C$$
$$x^3 + t + k = C \qquad C-k$$
$$x^3 + t = C \qquad (C \text{ A CONSTANT})$$

THIS GIVES THE FOLLOWING SOLUTION:

$$\underline{x(t) = \sqrt[3]{C - t}}$$

DEFINITION

$$f(t,x) + g(t,x)\dot{x} = 0$$

IS SAID TO BE EXACT IF

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial t}$$

EXAMPLE: $1 + 3x^2\dot{x} = 0$

a. VERIFY IT IS EXACT!

b. SOLVE IT!

SOLUTION

a. $f(t,x) = 1 \Rightarrow \frac{\partial f}{\partial x} = 0$

$g(t,x) = 3x^2 \Rightarrow \frac{\partial g}{\partial t} = 0$

$\| \Rightarrow$ EXACT

b. WE WANT TO FIND $h(t,x)$ SUCH THAT $\boxed{\text{I. } \frac{\partial h}{\partial t} = 1}$

$\text{II } \frac{\partial h}{\partial x} = 3x^2 \Rightarrow x^3 + C(t)$

THIS IS THE KEY FACT. THE CONSTANT IS A FUNCTION OF t.

THIS IS A SMALL COMPLICATION!

$h(t,x) = x^3 + C(t)$

EASY, WE KNOW $\boxed{\frac{\partial h}{\partial t} = 1}$,

$$\Rightarrow [(x^3 + C(t)]'_t = C'(t) = 1$$

EXACT DIFFERENTIAL

SOLUTION RECIPY:

$$1 + 3x^2 \dot{x} = 0$$

STEP 1: VERIFY IT IS EXACT

$$f(t,x) = 1 \qquad \frac{\partial f}{\partial x} = 0$$

$$g(t,x) = 3x^2 \qquad \frac{\partial g}{\partial t} = 0$$

STEP 2: FIND h SUCH THAT

I. $\frac{\partial h}{\partial t} = f(t,x)$

II. $\frac{\partial h}{\partial x} = g(t,x)$

BY INTEGRATING ONE OF THEM

$$\text{II} \quad \frac{\partial h}{\partial x} = 3x^2 \Rightarrow h(x,t) = x^3 + \underline{C(t)}$$

STEP 3 FIND $C(t)$ BY USING THAT $\frac{\partial h(x,t)}{\partial t} = f(t,x)$

$$h(x,t) = x^3 + C(t)$$

$$\frac{\partial h}{\partial t} = C'(t) = f(t,x) = 1$$

THIS DETERMINES $C(t)$ BY INTEGRATION.

$$C(t) = t + K$$

STEP 4. WE NOW HAVE

$h(t, x)$

$$h(t, x) = x^3 + t + K$$

WHICH GIVES US

$h(t, x) = C$

$$x^3 + t = C$$

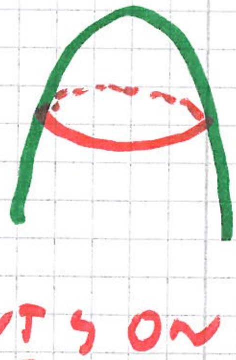
STEP 5

SOLVE FOR x !

$$x = \sqrt[3]{C - t}$$

COMMENT : THE MYSTERIOUS C :

$$h(t, x) = C$$



FLASHBACK:

IMPLISITT DIFF

TO FIND TANGENTS ON HEIGHT CURVES (NIVÅ KURVER)

WE DID: $x^2 + y^2 = C$

$$2x + 2y \cdot y'_x = 0$$

THIS C DOES NOT MATTER

CONNECTION TO WHAT WE DO HERE:

$$h(t, x) = C$$

AGAIN THIS C DOES NOT MATTER, WE COULD CHOOSE 0, AS LONG AS WE KEEP THE INGRATION CONSTANT IN THE COMPUTATION OF $h(x,t)$:

$$x^3 + C(t)$$

$$\frac{\partial h}{\partial t} = C'(t) = 1$$

$$C(t) = t + \underline{K}$$

KEEP THIS

AND WE GET

$$\cancel{h(t)} \; h(t, x) = x^3 + t + k = \underline{0}$$

$$x^3 + t = -k$$

WHICH IS THE SAME
SOLUTION AS

$$x^3 + t = C \quad (C = -K)$$

SECOND ORDER DIFFERENTIAL EQUATIONS.

SECOND ORDER MEANS THAT WE HAVE $\ddot{x}$ $\left(\ddot{x}(t) = x''(t)\right)$

EXAMPLE (EASY)

$\ddot{x} = 5$, HOW DO WE SOLVE THIS?

$$\int \ddot{x}\,dt = \int 5\,dt$$

$$\dot{x} = 5t + C_1$$

WE WILL GET TWO CONSTANTS

$$\int \dot{x}\,dt = \int (5t + C_1)\,dt$$

$$x(t) = \frac{5}{2}t^2 + C_1 t + C_2$$

HERE COMES THE SECOND

WE WANT TO LEARN HOW TO SOLVE

LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

THEY LOOK LIKE THIS:

$$\ddot{x} + a\dot{x} + bx = f(t)$$

IF $f(t) \equiv 0$ THEN THE EQUATION IS CALLED **HOMOGENEOUS**.

IF $f(t) \not\equiv 0$ THEN IT IS **<u>IN</u>HOMOGE....** .

WE WILL FIRST SOLVE EQUATIONS WHERE $f(t) \equiv 0$.

EXAMPLE:

$$\ddot{x} - 5\dot{x} + 6x = 0$$

A SHOT IN THE DARK:

WHAT ABOUT SOMETHING LIKE e^{rt} WHERE r IS A NUMBER?

LET US TRY:

$$x = \underline{e^{rt}}$$

$$\dot{x} = \underline{re^{rt}}$$

$$\ddot{x} = \underline{r^2e^{rt}}$$

INSERT INTO: $\ddot{x} - 5\dot{x} + 6x = 0$

$$\underline{r^2e^{rt}} - 5\underline{re^{rt}} + 6\underline{e^{rt}} = 0$$

$$e^{rt}(r^2 - 5r + 6) = 0$$

:-) WOW! IF $r^2 - 5r + 6 = 0$ THEN WE HAVE A SOLUTION

$$r^2 - 5r + 6 = 0$$

$$\Downarrow$$

$$(r-3)(r-2) \qquad r = 3, \quad r = 2$$

$$\ddot{x} - 5\dot{x} + 6x = 0$$

WE BELIEVE THAT $r = 3$

$r = 2$

GIVES SOLUTIONS

$x = e^{3t}$ IS A SOLUTION ?

CHECK : $\cancel{\ddot{x} = 3}$

$$x = e^{3t}$$

$$\dot{x} = 3e^{3t} \Rightarrow 9e^{3t} - 5 \cdot 3e^{3t}$$

$$\ddot{x} = 9e^{3t} \qquad + 6e^{3t}$$

$$= (9 - 15 + 6)e^{3t}$$

$$= 0 \quad \text{YES!}$$

LIKEWISE FOR $r = 2$. $x = e^{2t}$

THIS GIVES US A GENERAL SOLUTION

$$x(t) = C_1 e^{3t} + C_2 e^{2t}$$

(WHY? $C_1 e^{3t}$ AND $C_2 e^{2t}$ BOTH SATISFY $\ddot{x} - 5x + 6x = 0$,)

$$\boxed{\begin{array}{l}\text{SOLUTION OF}\\ \ddot{x} + a\dot{x} + b\,x = 0\end{array}}$$

DEFINITION

THE ~~CHA~~ CHARACTERISTIC EQUATION (POLYNOMIAL)

OF

$$\ddot{x} + a\dot{x} + bx = 0$$

IS

$$r^2 + ar + b = 0$$

[WHERE DOES THIS POLYNOMIAL COME FROM?

$$\left.\begin{aligned} x &= e^{rt} \\ \dot{x} &= re^{rt} \\ \ddot{x} &= r^2 e^{rt} \end{aligned}\right\} \quad \begin{array}{l} \ddot{x} + a\dot{x} + b = 0 \\ \text{BECOMES} \\ (r^2 + ar + b)e^{rt} = 0 \end{array}$$

]

THE SOLUTION OF $\ddot{x} + a\dot{x} + bx = 0$

IS INTIMATELY LINKED TO

THE CHARACTERISTIC POLYNOMIAL.

THEOREM

SOLUTION OF $\ddot{x} + a\dot{x} + bx = 0$

1) $r^2 + ar + b = 0$ HAS TWO DISTINCT ROOTS, r_1, r_2. $(r_1 \neq r_2)$

THEN

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

(A, B CONSTANTS)