HELLO!

DIFFERENTIAL EQUATIONS

LAST TIME:

WE LOOKED AT SOME EASY EXAMPLES OF DIFFERENTIAL EQUATIONS, STATED SOME SOLUTIONS, AND SHOWED THAT THEY WERE SOLUTIONS.

NOW IT IS TIME TO GET SERIOUS, ⇒ HOW DO WE SOLVE DIFFERENTIAL EQUATIONS?

1 LEARN TO SOLVE **SEPARABLE DIFFERENTIAL EQUATIONS**

DEFINITION (FIRST ORDER)
AN FIRST ORDER DIFFERENTIAL EQUATION IS AN EQUATION OF THE FORM

$$\dot{x} = F(t, x)$$

(REMEMBER: $x = x(t)$ x IS A FUNCTION OF t)

DEFINITION **SEPARABLE DIFFERENTIAL EQUATION**

A FIRST ORDER EQUATION IS SAID TO BE SEPARABLE IF WE CAN WRITE ~~A~~ THE EQUATION

$$\dot{x} = \underline{f(t)}\, g(x)$$

(THAT IS SEPARATE THE t's AND THE x's)

EXAMPLE

WHICH OF THESE DIFFERENTIAL EQUATIONS ARE SEPARABLE:

1. $\dot{x} = xt$ SEPARABLE $\underline{f(t)} = t$, $g(x) = x$
2. $\dot{x} = x + t$ NOT SEP.
3. $\dot{x} = xt + t^2$ NOT SEP.
4. $\dot{x} = xt^2 + x^2t^2 = t^2(x + x^2)$ SEP. $f(t) = t^2$
   $g(x) = x + x^2$

HOW DO WE SOLVE A SEPARABLE DIFFERENTIAL EQUATION?

ANSWER: LEIBNITZ - NOTATION VERY HANDY... SO USE THIS.

$$dt \cdot \Big| \quad \frac{dx}{dt} = f(t) \cdot g(x)$$

$$\left[\frac{dx}{dt} = \dot{x} = x'(t) \right]$$

$$dx = f(t)\, g(x)\, dt$$

$$\int \frac{1}{g(x)}\, dx = \int f(t)\, dt$$

AND THIS GIVES US THE SOLUTION.

LET US SEE HOW THIS WORKS OUT IN AN EXAMPLE:

OBSERVATION: THIS SOLUTION STRATEGY BRINGS DIFFERENTIAL EQUATIONS OVER TO THE BALL PARK OF **INTEGRATION!**

WE NEED TO REFRESH OUR MEMORY ON INTEGRATION TECHNIQUES.

INTEGRATION TOOL BOX

1. $$\int x^n dx = \frac{1}{n+1} x^{n+1} + C \qquad n \neq -1$$

(RECALL : $\int x^{\frac{1}{2}} dx = \frac{1}{1+\frac{1}{2}} x^{\frac{1}{2}+1} + C$

$= \frac{2}{3} x^{\frac{3}{2}} + C$)

ABSOLUTE VALUE IMPORTANT

2. $$\int \frac{1}{x} dx = \ln|x| + C$$

3. $$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

4. INTEGRATION BY PARTS:

$$\int \underline{u} \cdot \underline{v'} \, dx = \underline{u} \cdot \underline{v} - \int \underline{u'} \cdot \underline{v} \, dx$$

DIFFERENTIATE

$$\int \underline{x} \, \underline{e^x} \, dx = \underline{x} \cdot \underline{e^x} - \int \underline{1} \cdot \underline{e^x} \, dx$$

INTEGRATE

$$= x \cdot e^x - e^x + C$$

5. PARTIAL FRACTION EXSPANSION

$$\int \frac{1}{x^2+5x+6}\,dx = \int \frac{1}{x+2} - \frac{1}{x+3}\,dx$$

$$= \ln|x+2| - \ln|x+3| + C$$

SET $\dfrac{1}{x^2+5x+6} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$

AND SOLVE FOR A AND B.

6. AN INTEGRAL LIKE

$$\int \frac{x^3+2x+5}{x^2+5x+6}\,dx$$ IS SOLVED

BY POLYNOMIAL DIVISION

SO IT BECOMES:

$$\int \left(\text{POLYNOMIAL} + \boxed{\frac{ax+b}{x^2+5x+6}}\right) dx$$

THIS IS SOLVED BY USING 5.

EXAMPLE

$$\dot{x} = \frac{2t}{3x^2}$$

SEPARABLE? YES! $\underbrace{2t}_{f(t)} \cdot \underbrace{\frac{1}{3x^2}}_{g(x)}$

SOLUTION:

$$3x^2 dt \cdot \Big| \quad \frac{dx}{dt} = \frac{2t}{3x^2}$$

$$\int 3x^2\, dx = \int 2t\, dt$$

$$x^3 = t^2 + C$$

$$\underline{x(t) = \sqrt[3]{t^2 + C}}$$

RECALL
$(x^3)'_x = 3x^2$
$(t^2)'_t = 2t$

(IF WE DO NOT BELIEVE IN THE SOLUTION STRATEGY; WE COULD TEST THE SOLUTION)

EXAMPLE (A LITTLE MORE CHALLENGING)

SOLVE $\frac{dx}{dt} = x(1-x)$

SEPARABEE!

$$\int \frac{1}{x(1-x)}\,dx = \int 1 \cdot dt$$

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x} = \frac{A(1-x)+Bx}{x(1-x)}$$

SO:

$$1 = A\cancel{(1-x)}\cancel{+Bx}$$

RECALL = FOR ALL x!

x = 0 GIVES A = 1

x = 1 GIVES B = 1

$$= \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = t + C$$

$$\ln|x| - \ln|1-x| = t + C$$

$$e^{\ln\left|\frac{x}{1-x}\right|} = e^{t+C}$$

$$\left|\frac{x}{1-x}\right| = e^{t+C}$$

$$\frac{x}{1-x} = \boxed{\pm e^{C}} \cdot e^{t}$$

WE CAN DEFINE A NEW KONSTANT K

$$\frac{x}{1-x} = K \cdot e^{t}$$

(THIS K NOW BOTH POSITIVE AND NEGATIVE)

EXAMPLE CONTINUED

$$\frac{x}{1-x} = ke^t$$

(WE WANT TO FIND x!)

$$x = ke^t(1-x)$$

$$x = ke^t - ke^t x$$

$$x + ke^t x = ke^t$$

$$(1+ke^t)x = ke^t$$

$$x = \frac{ke^t}{1+ke^t}$$

DIVIDE BY ke^t

$$\cancel{x = \frac{1}{ke^t + 1}}$$

$$x = \frac{1}{\frac{1}{ke^t} + 1}$$

$$x = \frac{1}{\frac{1}{k} \cdot e^{-t} + 1}$$

SMART COSMETICS

$$x = \frac{1}{\boldsymbol{k} e^{-t} + 1}$$

$$\boldsymbol{k} = \frac{1}{k}$$

THAT EXAMPLE HAD SOME TECHNICAL DETAIL, LET US DO AN EASY ONE THAT FACILITATE SOME DEEPER INSIGHTS:

EXAMPLE

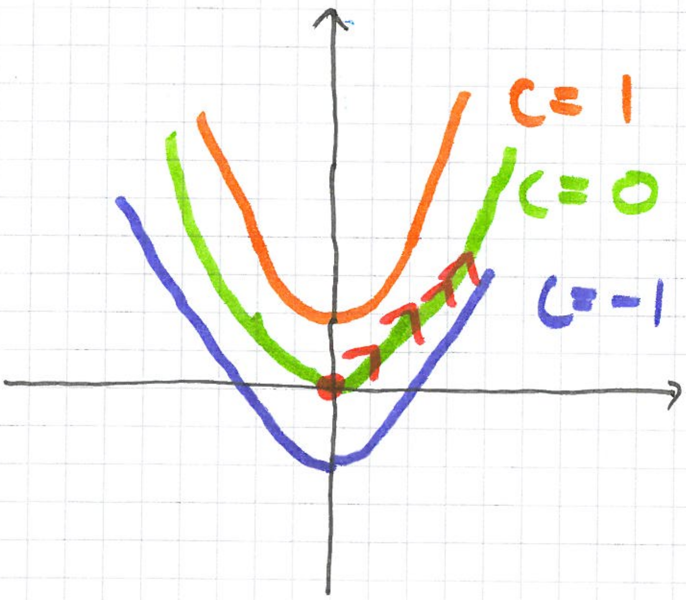
$$\dot{x} = 2t \qquad \text{(SEPARABLE)}$$

$$\frac{dx}{dt} = 2t$$

$$\int dx = \int 2t \, dt$$

$$x = t^2 + C$$

THIS IS THE **GENERAL** SOLUTION OF $\dot{x} = 2t$.

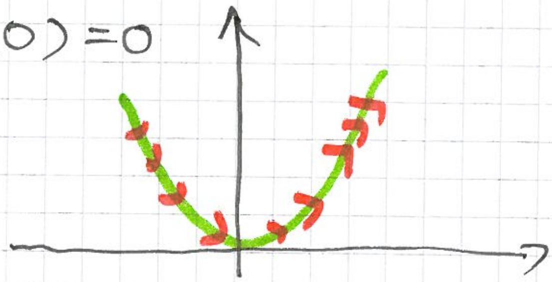


ASSUME THAT $\dot{x} = 2t$ IS THE EQUATION ~~OF A BUMBLE BEES~~ IS THE FLIGHT OF A BUMBLE BEE

1. EVERY C GIVES A UNIQUE CURVE.

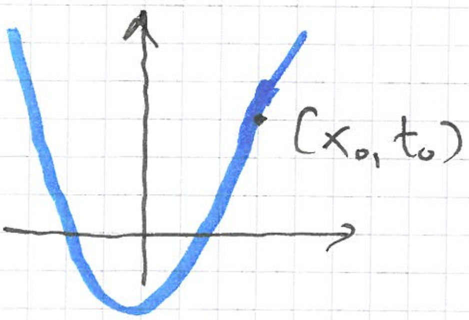
2. IF WE KNOW THAT THE BUMBLE BEE WAS AT $X(0)$, THAT IS AT $\dot{X}(0)$ AT TIME $t=0$, WE KNOW EVERYTHING ABOUT THE BUMBLE BEE.

(ASSUME $X(0)=0$



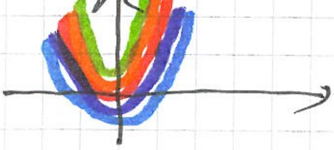
THIS IS THE FLIGHT OF THE BUMBLE BEE.

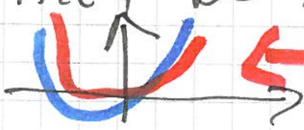
3. ~~EVE~~ GIVEN A POINT (x_0, t_0),



THERE EXISTS $\boxed{\text{JUST ONE}}$ C SUCH THAT (x_0, t_0) LIES ON $x(t) = t^2 + C$.

(REMARK: THE CURVES, ~~THE~~ THE SOLUTION CURVES, FILL THE ENTIRE (x,t)-SPACE AND THEY DO NOT INTERSECT.





JUSTIN B IS WRONG

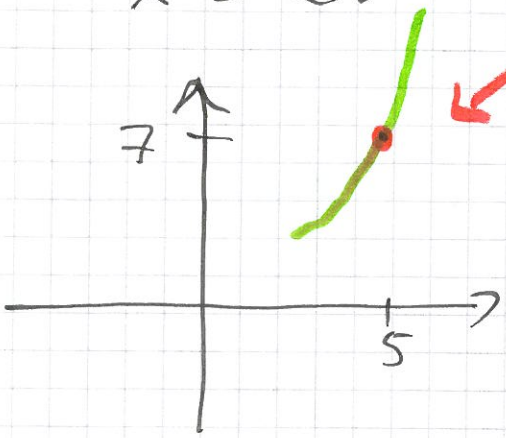
OK!

THESE INSIGHTS ALLOW US

TO SOLVE PROBLEMS LIKE THIS:

$\dot{x} = 2t$ $\qquad x(5) = 7$

↳ BEE IS AT $x=7$ WHEN $t=5$



JUST ONE CURVE, AND WE FIND IT THE FOLLOWING WAY:

(1) $$\frac{dx}{dt} = 2t$$

$$\int dx = \int 2t\,dt$$

$$x = t^2 + C$$ [GENERAL SOLUTION]

(2) DETERMINE C: USING $x(5) = 7$

$$x(5) = \underbrace{5^2 + C = 7}_{\text{THIS IS THE EQUATION WE NEED TO SOLVE!}}$$

$$25 + C = 7$$

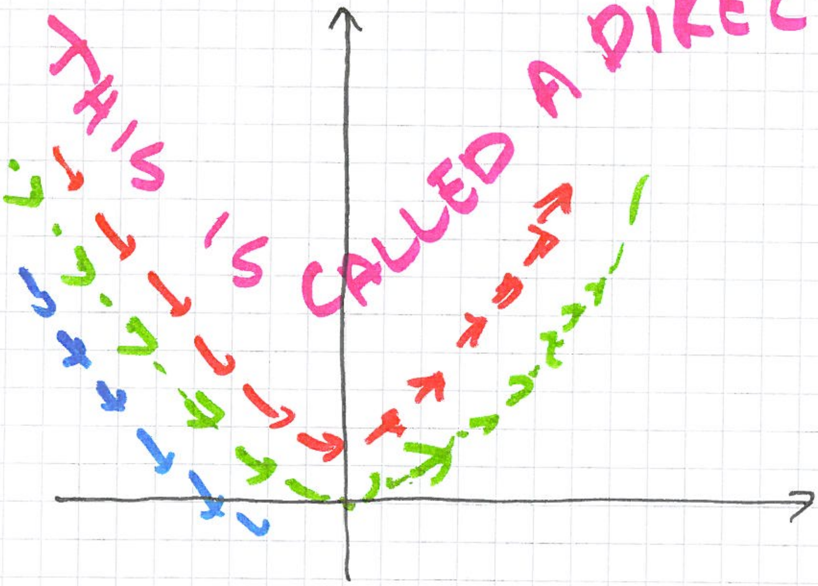
$$C = 7 - 25 = -18$$

$$x(t) = t^2 - 18$$ [SPECIAL SOLUTION! WITH $x(5) = 7$]

JUST ONE MORE THING:

WE CAN THINK ABOUT THESE SOLUTIONS AS CURRENTS...

THIS IS CALLED A DIRECTION DIAGRAM



THIS REMINDS US OF OC~~AN~~EAN CURRENTS... LIKE WATER FLOWING.

THE POINT IS:

A DIRECTION DIAGRAM TELLS US WHERE "THE SYSTEM / THE SOLUTIONS" ARE FLOWING, AND AT TIMES WE CAN CONSTRUCT THIS DIAGRAM WITHOUT SOLVING THE DIFFERENTIAL EQUATION.

WHY ARE WE INTERESTED IN DIFF. EQUATIONS IN ECONOMICS?

ANSWER:

ECONOMICS IS OBSESSED WITH THE NOTION OF EQUILIBRIUM/ ~~STEA~~ STEADY STATE, SO WHAT WE ARE USUALLY INTERESTED IN IS (NOT THE SOLUTION) BUT

$$\lim_{t \to \infty} x(t)$$

THAT IS THE ASYMPTOTIC BEHAVIOR OF THE SOLUTION.

(BACK TO THIS IN A JIFFY...)

LINEAR FIRST ORDER DIFFERENTIAL EQUATIONS

DEFINITION

A **LINEAR FIRST ORDER DIFFERENTIAL EQUATION** IS AN DIFFERENTIAL EQUATION THAT CAN BE WRITTEN IN THE FOLLOWING WAY:

$$\dot{x} + a(t)x = b(t)$$

DEFINITION EXPLORATION:

1. $\dot{x} + 2tx = 4t$ **YES** $a(t) = 2t$, $b(t) = 4t$
2. $\dot{x} - x = e^{2t}$ **YES** $a(t) = -1$, $b(t) = e^{2t}$
3. $(t^2+1)\dot{x} + e^t x = t\ln t$ **YES*** $a(t) = \frac{e^t}{t^2+1}$, $b(t) = \frac{t\ln t}{t^2+1}$
4. $\dot{x} - x^2 = 0$ **NO** (NOT LINEAR IN X)
5. $x - e^x = 2t$ **NO** (NOT LINEAR IN X)

WANT TO GET RID OF THIS TERM...

*** OBSERVATION:** $\underline{(t^2+1)}\dot{x} + e^t x = t\ln t$

GIVES $\dot{x} + \frac{e^t}{t^2+1}x = \frac{t\ln t}{t^2+1}$

HOW DO WE SOLVE THESE DIFFERENTIAL EQUATIONS?

SOLUTION IDEA ....

LET US LOOK AT AN EASY EQUATION OF THIS TYPE:

$$\underline{\dot{x} + 2x} = 3 \qquad \left[\begin{array}{l} a(t) = 2 \\ b(t) = 3 \end{array}\right]$$

IN THE DEFINITION

WINNIE THE POOH:

THIS REMINDS ME OF SOMETING...

LOOK AT: xe^{2t}

WHAT IF WE DIFFERENTIATE THIS?

$$\left[xe^{2t}\right]'_t = \dot{x}e^{2t} + x \cdot 2e^{2t}$$

$$= \underline{(\dot{x} + 2x)}e^{2t}$$

WOW! THE LEFT HAND SIDE RESEMBLES THE DERIVATIVE OF A PRODUCT!

WE CAN USE THIS TO SOLVE THE DIFF. EQUATION

REALLY?

YES, WE DO IT THIS WAY

$e^{2t} \cdot \,\Big|\; \dot{x} + 2x = 3$

$$\dot{x}e^{2t} + x\,2e^{2t} = 3e^{2t}$$

$$\int \underbrace{(x \cdot e^{2t})'_t}_{} dt \overset{\shortparallel}{=} \int 3e^{2t}\, dt$$

CHECK:
$(x \cdot e^{2t})'_t$
$= \dot{x} \cdot e^{2t}$
$+ x \cdot 2e^{2t}$
YES!

$$x \cdot e^{2t} = 3 \cdot \frac{1}{2} e^{2t} + C$$

VOILA!

X IS FREED!

$$x = \frac{3}{2} + Ce^{-2t}$$

(TO CONTINUED IN THE NEXT EPISODE!)