LECTURE I

GRA 6035

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Lecture plan:
1. Intro to GRA 6035
2. Linear Systems
3. Gaussian elimination
4. Rank of a matrix

Notes: [LSGE] Ch. 1-3 (+ [FMEA] Ch. 1.3-1.4)

1 Intro to GRA 6035

- Lectures
- Problem Sessions
- Exams
  - Midterm 30/09
  - Final ~~10/12~~ 12/12
  - ~~(temporary date, not finalized)~~
- Reading
- Prerequisites

See syllabus / It's Learning

2 Linear Systems

A linear equation in $x_1, x_2, \ldots, x_n$ (variables) is of the form

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$$

where $a_1, \ldots, a_n$ and b are fixed numbers (parameters).

Ex: $x_1 + x_2 = 7$

$x + 2y - z = 13$

Linear equations have graphs that are straight lines (two vars) / planes (three vars).

A <u>linear system</u> in $x_1, x_2, \dots, x_n$ (variables) is a collection of one or more linear equations in these variables.

<u>In general:</u> $m \times n$ linear system

$$\underbrace{\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}\right\}}_{n \text{ variables}} \; m \text{ equations}$$

<u>Ex:</u>

$$\begin{aligned} x + y &= 4 \\ x - y &= 2 \end{aligned} \qquad (2\times 2 \text{ lin. system})$$

A <u>solution</u> of a linear system in $x_1, x_2, \dots, x_n$ is an n-tuple $(s_1, s_2, \dots, s_n)$ such that

$$x_1 = s_1, \; x_2 = s_2, \; \dots, \; x_n = s_n$$

solve all equations simultaneously.

Ex:

$$x+y=4$$
$$x-y=2$$

Substitution

$x+y=4 \Rightarrow y=4-x$

$x-y=2$

$x-(4-x)=2$

$2x-4=2$

$x=\underline{3}$

$y=4-3=\underline{1}$

$(x,y)=\underline{\underline{(3,1)}}$

Elimination

$$\begin{array}{l} x+y=4 \\ x-y=2 \\ \hline 2x \quad =6 \end{array}$$

$x=\underline{3}$

← { var. y is eliminated

$x+y=4$

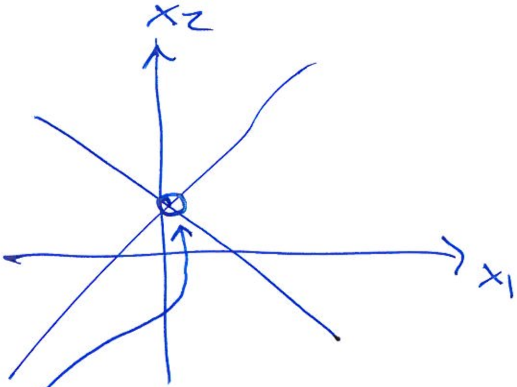
$y=4-x$

$=4-3=\underline{1}$

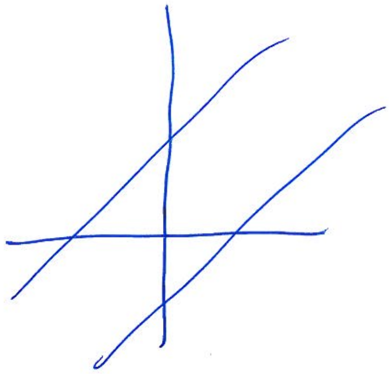
$\begin{array}{l} x+y=4 \\ x-y=2 \end{array} \rightarrow \begin{array}{l} x+y=4 \\ 2x \quad =6 \end{array}$

Solution types:

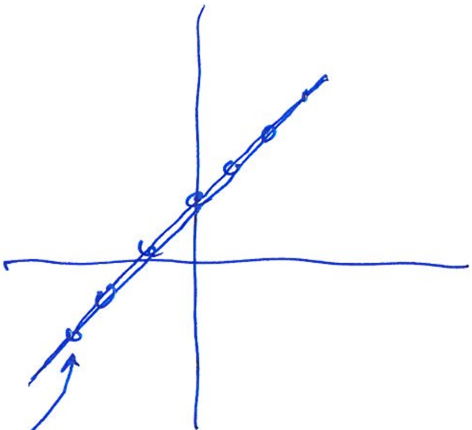
Ex: 2×2 system $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$



non-parallel lines
one solution



parallel lines, but different
no solutions



parallel lines, the same
infinitely many solutions
(all points on the line)

Theorem:

Any linear system ($m \times n$) has either

- one solution
- no solutions
- infinitely many solutions

③ Gaussian elimination

– Operate on <u>systems</u> not equations

$$\begin{aligned} x+y&=4\\ x-y&=2 \end{aligned} \quad \rightarrow \quad \begin{aligned} x+y&=4\\ 2x&=6 \end{aligned}$$

It is useful to have a shorter notation.

A matrix is a rectangular array of numbers.

The coefficient matrix of a linear system

$$A=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \leftarrow \quad \begin{aligned} x+y&=4\\ x-y&=2 \end{aligned}$$

The augmented matrix

$$\hat{A}=\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \end{array}\right) \quad \leftarrow \quad \begin{aligned} x+y&=4\\ x-y&=2 \end{aligned}$$

Notation:

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & 0 & 6 \end{array}\right)$$

means

$$\begin{aligned} x+y&=4\\ x-y&=2 \end{aligned} \quad \rightarrow \quad \begin{aligned} x+y&=4\\ 2x&=6 \end{aligned}$$

- Allowed operations = Operations that preserve the solutions of the system

Row operations = Elementary row operation

1. Interchange two rows
2. Multiply a row with a non-zero constant number
3. Change a row by adding to it a multiple of another row

Ex:

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & -1 & 5 \end{array}\right) \longleftarrow \begin{array}{l} x + 2y = 4 \\ 3x - y = 5 \end{array}$$

$$(\ )\updownarrow \qquad (\ ).2 \qquad (\ ) \,\cdot 2$$

$$\left(\begin{array}{cc|c} 3 & -1 & 5 \\ 1 & 2 & 4 \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 6 & -2 & 10 \end{array}\right) \qquad \left(\begin{array}{cc|c} 7 & 0 & 14 \\ 3 & -1 & 5 \end{array}\right)$$

Many other notations are possible, such as:

$$\begin{cases} \text{Row } 1 := \text{Row}1 + 2\,\text{Row}2 \\ \text{Row } 2 = \text{unchanged} \end{cases}$$

Fact:

- Row operations are allowed (preserve solutions)
- All linear systems can be solved using elementary row operations.

Target: to eliminate as many variables as possible using elementary row op.

Ex. I:

$$\begin{aligned} x + y &= 4 \\ x - y &= 2 \end{aligned} \qquad \begin{aligned} x + y &= 4 \\ -2y &= -2 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\,-1\,} \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -2 & -2 \end{array}\right)$$

want to get 0 in this position

Row operation:

$R_1 \leftarrow R_1$

$R_2 \leftarrow R_2 + (-1) \cdot R_1$

Back substitution:

$$\begin{aligned} -2y &= -2 \\ y &= \underline{1} \end{aligned} \qquad \begin{aligned} x + 1 &= 4 \\ x &= \underline{3} \end{aligned}$$

Ex. 2:

$$\boxed{\begin{aligned} S &+ 0.05\,C = 5{,}000 \\ 0.4S + F &+ 0.4\,C = 40{,}000 \\ 0.1S + 0.1F &+ C = 10{,}000 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5{,}000 \\ 0.4 & 1 & 0.4 & 40{,}000 \\ 0.1 & 0.1 & 1 & 10{,}000 \end{array}\right) \begin{array}{c} -0.4 \\ -0.1 \end{array} \rightarrow \begin{array}{c} \\ \\ -0.1 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5{,}000 \\ 0 & 1 & 0.38 & 38{,}000 \\ 0 & 0.1 & 0.995 & 9{,}500 \end{array}\right)$$

eliminate S from eq. (2) and (3)

eliminate F from eq. (3)

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5,000 \\ 0 & 1 & 0.38 & 38,000 \\ 0 & 0 & 0.957 & 5,700 \end{array}\right) \quad \leftarrow \quad \left\{ \text{we cannot eliminate any more vars} \right.$$

This means:

$$\begin{aligned} S \qquad + 0.05\,C &= 5,000 \\ F + 0.38\,C &= 38,000 \\ 0.957\,C &= 5,700 \quad \Rightarrow \quad \underline{C = 5,956} \end{aligned}$$

$$F = 38,000 - 0.38 \cdot (5,956) = \underline{35,737}$$

$$S = 5,000 - 0.05 \cdot (5,956) = \underline{4,702}$$

Solution:

~~S = 4,782~~
~~F = 35,737~~
~~C = 4,~~

$$\begin{aligned} S &= 4,702 \\ F &= 35,737 \\ C &= 5,956 \end{aligned}$$

(rounded to the nearest dollar)

Echelon forms:

Pivot = the first non-zero number in a row in the matrix.

Echelon form = all entries under a pivot are zero

$$\begin{pmatrix} 7 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ echelon form

Augmented matrix in echelon form → Stop the row operations.

Gaussian elimination:

$$\text{lin. sys.} \longrightarrow \text{augmented matrix} \xrightarrow{\text{row operations}} \text{echelon form} \longrightarrow \text{lin. sys. w/ variables eliminated}$$

lin. sys. w/ variables eliminated

↓

back substitution

↓

solution

Variatión: Gauss-Jordan elimination

A reduced echelon form is a matrix such that

* all entries under a pivot are zero
* — " — over — " —
* all pivots are 1

Ex:

$$\left(\begin{array}{cc|c} 3 & 7 & 4 \\ 0 & 2 & 1 \end{array}\right)$$ echelon form but not reduced echelon form

$$\downarrow$$

$$\left(\begin{array}{cc|c} 1 & 0 & 0.17 \\ 0 & 1 & 0.5 \end{array}\right)$$ reduced echelon form

When we continue with row operations until we get a reduced echelon form, the method is called Gauss-Jordan elimination.

Facts:

- When we use row operations on a given matrix, an echelon form is not unique, but the reduced echelon form is unique

- Pivot positions = positions where there are pivots in an echelon form (after row operations)

The pivot positions are unique; that is, all echelon forms we get to from a given matrix has the same pivot positions

Systems with no solutions:

Ex:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

echelon form

$$\begin{aligned} x + 2y + 3z &= 5 \\ 7z &= 4 \\ 0 &= 3 \end{aligned}$$

$\uparrow$

no solutions

A linear system has no solutions (inconsistent)

$\Updownarrow$

An echelon form has a pivot in the last column. = pivot position in last column.

A pivot position is a position in the matrix where there is a pivot when the matrix is reduced to an echelon form.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 2 & 2 & 10 & 12 \end{array}\right) \longrightarrow \cdots \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

Systems with many solutions

Ex:

$$\begin{array}{ccc|c} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & -2 \end{array}\right) \\ x \quad y \quad z \end{array}$$

echelon form

no pivots position in last col. $\Rightarrow$ there are solutions (consistent)

Look at pivot positions:

Each column on the left side of the line = one variable

Basic variable = columns with pivot positions

Free variable = " without —"—

Ex: x, y basic, z free

We can solve each basic variable interms of the free ones:

$$x + 2y + 3z = 1$$

$$-4y - 8z = -2 \Rightarrow -4y = 8z - 2$$

$$y = -2z + \frac{1}{2}$$

Back substitution: We work backwards

$$x + 2\left(-2z + \frac{1}{2}\right) + 3z = 1$$

$$x = z$$

$$\boxed{\begin{array}{l} x = z \\ y = -2z + 1/2 \\ z = \text{free} \end{array}}$$

Summary:

(a)

System is consistent $\longleftrightarrow$ no pivot position in the last col.
(at least one sol'n.)

system is inconsistent $\longleftrightarrow$ pivot position in the last column.
(no sol'n.)

b) Assume that the system is consistent

all variables are basic $\longleftrightarrow$ pivot positions in all coloumns to the left of the line
(one solution)

Some variables are free $\longleftrightarrow$ at least one column to the left of the line without a pivot position.
(infinitely many soln.)

degrees of freedom
= # free variables

④ Rank

The rank of a matrix is the number od pivot positions

$$\boxed{\operatorname{rk} A = \#\text{pivot positions in } A}$$

A $m \times n$-matrix: $\operatorname{rk} A \le m$, $\operatorname{rk} A \le n$

$\operatorname{rk} A = 0 \iff$ all entries in A are zero

Ex:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 2 & 4 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

row operations

three pivot positions

$$\operatorname{rk} \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 2 & 4 & 7 \end{pmatrix} = \underline{\underline{3}}$$

Next Lecture: Thursday 01/09 at 17.00 in C1-010

- More details for this lecture: See "Linear Systems and Gaussian Elimination" (notes in It's Learning)
- Work through exercises from Problem Sheet 1.