Extra Lecture 3

GRA 6035

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Mathematics

Plan:

Matrices

(a) Compute determinant and rank
(b) Finding eigenvalues and eigenvectors
(c) Determine if a matrix is diagonalizable

(a)

Compute determinant and rank of A in these examples:

Examples:

i) $A = \begin{pmatrix} 3 & -4 & -8 \\ -2 & 1 & 4 \\ 2 & -2 & -5 \end{pmatrix}$

ii) $A = \begin{pmatrix} 2 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 9 \end{pmatrix}$

iii) $A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix}$

iv) $A = \begin{pmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{pmatrix}$ for $\begin{cases} \text{general } t \\ t=-2 \end{cases}$

v) $A = \begin{pmatrix} 1-a & 1 & -2a \\ 1 & 0 & 1 \\ 1 & a & -a \end{pmatrix}$ for $\begin{cases} \text{general } a \\ a=1 \end{cases}$

vi) $A = \begin{pmatrix} t & t & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix}$ for $\begin{cases} \text{general } t \\ t=1 \end{cases}$

vii) $A = \begin{pmatrix} a-2 & 2(a-2) & -1 \\ 1 & a-1 & 2 \\ 2 & 2 & a+2 \end{pmatrix}$ for $\begin{cases} \text{general } a \\ a=0 \end{cases}$

Solution:

i) $\begin{vmatrix} 3 & -4 & -8 \\ -2 & 1 & 4 \\ 2 & -2 & -5 \end{vmatrix} = 3(-5+8) + 2(20-16) + 2(-16+8)$

$= 9 + 8 - 16 = \underline{1}$

$|A| \neq 0 \Rightarrow rk\ A = \underline{3}$

ii) $\begin{vmatrix} 2 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 9 \end{vmatrix} = 9 \cdot (4-9) = \underline{-45}$

$|A| \neq 0 \Rightarrow rk\ A = \underline{3}$

iii) $\begin{vmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{vmatrix} = 4 \cdot (6-4) + 2(-4) = 8 - 8 = \underline{0}$

$|A| = 0 \Rightarrow rk\ A < 3$

$\begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix} = 12 - 4 = 8 \neq 0 \Rightarrow rk\ A = \underline{2}$

iv) $\begin{vmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{vmatrix} = 1 \cdot (-16-4t) - 2(-4t-8) - t(-t^2+8)$

$= \cancel{-16} - 4t + \cancel{8t} + \cancel{16} + t^3 - \cancel{8t}$

$= \underline{t^3 - 4t = t(t^2-4) = t(t-2)(t+2)}$

$|A| = 0 \iff t = 0, -2, 2$

$\underline{rk A = 3}$ when $t \neq 0, -2, 2$

$\underline{t=0}$: $A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 4 & 0 \\ 0 & -4 & -4 \end{pmatrix}$ $\quad \begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4 \neq 0$

$\Rightarrow \underline{rk A = 2}$ for $t=0$

$\underline{t=-2}$: $A = \begin{pmatrix} 1 & -2 & -2 \\ 2 & 4 & 2 \\ 2 & -4 & -4 \end{pmatrix}$ $\quad \begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix} = 4+4 = 8 \neq 0$

$\Rightarrow \underline{rk A = 2}$ for $t=-2$

$\underline{t=2}$: $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 4 & -2 \\ -2 & -4 & -4 \end{pmatrix}$ $\quad \begin{vmatrix} 2 & -2 \\ 4 & -2 \end{vmatrix} = -4+8 = 4 \neq 0$

$\Rightarrow \underline{rk A = 2}$ for $t=2$

$\underline{\text{Concl}}$: $\quad rk A = \begin{cases} 3, & t \neq 0, -2, 2 \\ 2, & t = 0, -2, 2 \end{cases}$

v) $\begin{vmatrix} 1-a & 1 & -2a \\ 1 & 0 & 1 \\ 1 & a & -a \end{vmatrix} = -1(-a-1) - a((1-a)+2a)$

$= a+1 - a(1+a) = 1-a^2$

$= \underline{1-a^2} = \underline{(1-a)(1+a)}$

$|A| = 0 \iff a = -1, +1$

$rk A = 3$ for $a \neq \pm 1$

$\underline{a=-1}$: $A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ $\quad \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

$\Rightarrow rk A = 2$ for $a=-1$

$\underline{a=1}$: $A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ $\quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \Rightarrow rk\, \underline{A=2}$ for $a=1$

Concl: $\quad \text{rk } A = \begin{cases} 3 & , a \neq -1, +1 \\ 2 & , a = -1, +1 \end{cases}$

vi) $|A| = \begin{vmatrix} t & t & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{vmatrix} = t \cdot (t^2 - 1) = \underline{t(t+1)(t-1)}$

$|A| = 0 \iff t = 0, t = 1, t = -1$

$\text{rk } A = 3$ if $t \neq 0, 1, -1$

$\underline{t=0}$: $\quad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \Rightarrow \text{rk } A = \underline{2}$ for $t = 0$

$\underline{t=1}$: $\quad A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow \text{rk } A = 2$ for $t = 1$

$\underline{t=-1}$: $\quad A = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix} = 1 \neq 0 \Rightarrow \text{rk } A = 2$ for $t = -1$

Concl: $\quad \text{rk } A = \begin{cases} 3 & , t \neq 0, 1, -1 \\ 2 & , t = 0, 1, -1 \end{cases}$

vii) $\begin{vmatrix} a-2 & 2(a-2) & -1 \\ 1 & a-1 & 2 \\ 2 & 2 & a+2 \end{vmatrix} = (a-2)\big((a-1)(a+2) - 4\big)$

$-1\big(2(a-2)(a+2) + 2\big) + 2\big(4(a-2) + (a-1)\big)$

$= \underline{(a-2)}(a^2 + a - 6) - \underline{(a-2)}(2a+4) - 2 + \underline{(a-2)} \cdot 8 + 2a - 2$

$= (a-2)(a^2 + a - 6 - 2a - 4 + 8) + 2a - 4$

$= (a-2)(a^2 + a - 6 - 2a - 4 + 8 + 2)$

$= (a-2)(a^2-a) = \underline{(a-2)a(a-1)}$

$|A| = 0 \Leftrightarrow a = 2, 0, 1$

$rk\, A = 3$ for $a \neq 2, 0, 1$

$\underline{a=2}$: $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}$ $\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = +1 \neq 0 \Rightarrow \underline{rk\, A = 2}$ for $a=2$

$\underline{a=0}$: $A = \begin{pmatrix} -2 & -4 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ $\begin{vmatrix} -2 & -4 \\ 1 & -1 \end{vmatrix} = 6 \neq 0 \Rightarrow rk\, A = \underline{2}$ for $a=0$

$\underline{a=1}$: $A = \begin{pmatrix} -1 & -2 & -1 \\ 1 & 0 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ $\begin{vmatrix} -1 & -2 \\ 1 & 0 \end{vmatrix} = 2 \neq 0 \Rightarrow rk\, A = 2$ for $a=1$

$\underline{\text{Concl}}$: $rk\, A = \begin{cases} 3, & a \neq 2, 1, 0 \\ 2, & a = 2, 1, 0 \end{cases}$

6 Eigenvalues and eigenvectors

Defn. equation: $A \cdot \underline{x} = \lambda \cdot \underline{x}$

($\underline{x}$ is eigenvector
λ is eigenvalue)

Std. method: (when only the matrix A is known)

i) Eigenvalues: Solve

$$|A - \lambda I| = 0$$

ii) For each λ, you solve

$$(A - \lambda I)\underline{x} = \underline{0} \iff A\underline{x} = \lambda \underline{x}$$

At least one free variable!

If a vector $\underline{v}$ is given: Check if $\boxed{A\underline{v} = \lambda \underline{v}}$ for some λ. If so, $\underline{v}$ is an eigenvector and λ is the corr. eigenvalue.

Ex. Find all eigenvalues and eigenvectors in i) - vii).

Hint: In the cases with parameters, use the numerical value of the parameter

Solution:

i)
$$\begin{vmatrix} 3-\lambda & -4 & -8 \\ -2 & 1-\lambda & 4 \\ 2 & -2 & -5-\lambda \end{vmatrix} = (3-\lambda)\big((1-\lambda)(-5-\lambda)+8\big) + 2\big(-4(-5-\lambda)-16\big)$$

$$+ 2\big(-16 + 8(1-\lambda)\big) = (3-\lambda)(\lambda^2 - 4\lambda + 3) + 2(4\lambda + 4) + 2(-8\lambda - 8)$$

$$= (3-\lambda)(\lambda^2 - 4\lambda + 3) + (-8\lambda - 8) = (3-\lambda)(\lambda^2 - 4\lambda + 3) - 8(\lambda + 1)$$

Solución:

i) $\begin{vmatrix} 3-\lambda & -4 & -8 \\ -2 & 1-\lambda & 4 \\ 2 & -2 & -5-\lambda \end{vmatrix} = 0$

$(3-\lambda)\cdot\left((1-\lambda)(-5-\lambda) + 8 \right) + 2\left(-4(-5-\lambda) - 16\right) + 2\left(-16 + 8 - 8\lambda\right)$

$= (3-\lambda)(1-\lambda)(-5-\lambda) + \left(24 - \cancel{8\lambda} + 8 + \cancel{8\lambda} - 16 - 16\lambda\right)$

$= (3-\lambda)(1-\lambda)(-5-\lambda) + (16 - 16\lambda)$

$= (3-\lambda)(1-\lambda)(-5-\lambda) + 16(1-\lambda)$

$= (1-\lambda)\cdot\left((3-\lambda)(-5-\lambda) + 16\right) = (1-\lambda)\cdot(\lambda^2 + 2\lambda + 1)$

$= (1-\lambda)\cdot(\lambda+1)^2 = 0$

Eigenvalues: $\lambda = 1$, $\lambda = -1$

Eigenvectors:

$\lambda = 1$: $\begin{pmatrix} 2 & -4 & -8 \\ -2 & 0 & 4 \\ 2 & -2 & -6 \end{pmatrix} \to \begin{pmatrix} 2 & -4 & -8 \\ 0 & -4 & -4 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\left.\begin{array}{lll} 2x - 4z = 0 & & x = 2z \\ y + z = 0 & & y = -z \\ z \text{ free} & & z = z \end{array}\right\} \quad \underline{x} = z \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\underline{\lambda=-1}$: $\begin{pmatrix} 4 & -4 & -8 \\ -2 & 2 & 4 \\ 2 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} 4 & -4 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\left.\begin{array}{r} x-y-2z=0 \\ y \text{ free} \\ z \text{ free} \end{array}\right\} \quad \begin{array}{l} x=y+2z \\ y=y \\ z=z \end{array} \qquad \underline{x} = \begin{pmatrix} y+2z \\ y \\ z \end{pmatrix}$$

$$= y\cdot\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\cdot\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

ii)

$$\begin{vmatrix} 2-\lambda & -3 & 0 \\ -3 & 2-\lambda & 0 \\ 0 & 0 & 9-\lambda \end{vmatrix} = (9-\lambda)\cdot\left((2-\lambda)(2-\lambda)-9\right)$$

$$= (9-\lambda)(\lambda^2-4\lambda-5) = (9-\lambda)(\lambda-5)(\lambda+1) = 0$$

Eigenvalues: $\underline{\lambda=9,\ \lambda=5,\ \lambda=-1}$

<u>Eigenvectors:</u>

$\underline{\lambda=9}$: $\begin{pmatrix} -7 & -3 & 0 \\ -3 & -7 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -7 & -3 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad *\neq 0$

$x=y=0$
$z=$ free

$$\underline{x} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{\lambda=5}$: $\begin{pmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{array}{l} x+y=0 \\ z=0 \end{array} \quad \begin{array}{l} x=-y \\ y=y \text{ (free} \\ z=0 \end{array}$

$$\underline{x} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$\underline{\lambda = -1}$:

$$\begin{pmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} x - y = 0 \\ z = 0 \\ (y \text{ free}) \end{array}$$

$$\underline{x} = \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} = y \cdot \underline{\underline{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}}$$

iii)

$$\begin{vmatrix} 4-\lambda & -2 & 0 \\ -2 & +3-\lambda & -2 \\ 0 & -2 & 2-\lambda \end{vmatrix} = (4-\lambda)\Big((3-\lambda)(2-\lambda) - 4 \Big) + 2\Big(-2(2-\lambda) \Big)$$

$$= (4-\lambda) \cdot (3-\lambda)(2-\lambda) + (4-\lambda)(-4) - 4(2-\lambda)$$

$$= (4-\lambda)(3-\lambda)(2-\lambda) + 4\lambda - 16 - 8 + 4\lambda$$

$$= (4-\lambda)(3-\lambda)(2-\lambda) + 8\lambda - 24 = (4-\lambda)(\underline{3-\lambda})(2-\lambda) + 8(\underline{\lambda-3})$$

$$= (\lambda-3) \cdot \Big(-(4-\lambda)(2-\lambda) + 8 \Big) = (\lambda-3) \cdot (-\lambda^2 + 6\lambda) = \underline{(\lambda-3)(-\lambda)(\lambda-6) = 0}$$

Eigenvalues: $\underline{\lambda_1 = 3}$ $\quad \underline{\lambda_2 = 0}$ $\quad \underline{\lambda_3 = 6}$

Eigenvectors:

$\underline{\lambda = 3}$:

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 \\ 0 & -4 & -2 \\ 0 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} x - 2y = 0 \\ -2y - z = 0 \\ (z \text{ free}) \end{array}$$

$$\left.\begin{array}{l} x = +2y = +2(-\frac{1}{2}z) = -z \\ y = -\frac{1}{2}z \\ z \text{ free} \end{array}\right\} \quad \underline{x} = \begin{pmatrix} -z \\ -1/2\, z \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix}}}$$

$\underline{\lambda=0}$: $\begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix} \to \begin{pmatrix} -2 & 3 & -2 \\ 0 & 4 & -4 \\ 0 & -2 & 2 \end{pmatrix} \to \begin{pmatrix} -2 & 3 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ $-2x+3y-2z=0$, $-2y+2z=0$ (z free)

$$\left.\begin{aligned} x &= \frac{3y-2z}{2} = \frac{1}{2}z \\ y &= z \\ z &= z \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} \frac{1}{2}z \\ z \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}}}$$

$\underline{\lambda=6}$: $\begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} -2 & -2 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} -2 & -2 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ $-2x-2y=0$, $-y-2z=0$ (z free)

$$\left.\begin{aligned} x &= -y = 2z \\ y &= -2z \\ z &= z \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} 2z \\ -2z \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}}$$

iv) With $t=-2$:

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & t & -2 \\ 2 & 4-\lambda & -t \\ -t & -4 & -4-\lambda \end{vmatrix} \overset{t=-2}{=} \begin{vmatrix} 1-\lambda & -2 & -2 \\ 2 & 4-\lambda & 2 \\ 2 & -4 & -4-\lambda \end{vmatrix}$$

$$= (1-\lambda)\cdot\big((4-\lambda)(-4-\lambda)+8\big) - 2\big(-2(-4-\lambda)-8\big) + 2\big(-4+2(4-\lambda)\big)$$

$$= (1-\lambda)(4-\lambda)(-4-\lambda) + 8(1-\lambda) - 2(\cancel{8}+2\lambda-\cancel{8}) + 2(-4+8-2\lambda)$$

$$= (1-\lambda)(4-\lambda)(-4-\lambda) + (8+8-8\lambda-4\lambda-4\lambda) = (1-\lambda)(4-\lambda)(-4-\lambda)+16-16\lambda$$

$$= (1-\lambda)(4-\lambda)(-4-\lambda)+16(1-\lambda) = (1-\lambda)\big((4-\lambda)(-4-\lambda)+16\big) = (1-\lambda)(-16+\lambda^2+16)$$

$$= \lambda^2(1-\lambda) = 0 \quad \Rightarrow$$

Eigenvalues: $\underline{\lambda=0}$, $\underline{\lambda=1}$

(mult. 2)

Eigenvectors:

$\lambda=0$: $\begin{pmatrix} 1 & -2 & -2 \\ 2 & 4 & 2 \\ 2 & -4 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 8 & 6 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

$x = \frac{1}{2}z$

(z free)

$4y+3z=0$

$\Downarrow$

$y = -\frac{3}{4}z$

$\left.\begin{array}{l} x = \frac{1}{2}z \\ y = -\frac{3}{4}z \\ z = z \end{array}\right\} \quad \underline{x} = \begin{pmatrix} \frac{1}{2}z \\ -\frac{3}{4}z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1/2 \\ -3/4 \\ 1 \end{pmatrix}$

$\lambda=1$: $\begin{pmatrix} 0 & -2 & -2 \\ 2 & 3 & 2 \\ 2 & -4 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & -7 & -7 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$2x - z = 0$

$y + z = 0$

(z free)

$\left.\begin{array}{l} x = \frac{1}{2}z \\ y = -z \\ z = z \end{array}\right\} \quad \underline{x} = \begin{pmatrix} \frac{1}{2}z \\ -z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1/2 \\ -1 \\ 1 \end{pmatrix}$

V) With a=1:

$|A-\lambda I| = \begin{vmatrix} 1-a-\lambda & 1 & -2a \\ 1 & -\lambda & 1 \\ 1 & a & -a-\lambda \end{vmatrix} \overset{a=1}{=} \begin{vmatrix} -\lambda & 1 & -2 \\ 1 & -\lambda & 1 \\ 1 & 1 & -1-\lambda \end{vmatrix}$

$= -\lambda(-\lambda(-1-\lambda)-1) - 1(-1-\lambda+2) + 1(1-2\lambda)$

$= \lambda^2(-1-\lambda) + \cancel{\lambda} \cancel{-1} + \cancel{\lambda} \cancel{+1} \cancel{-2\lambda} = \lambda^2(-1-\lambda) = 0$

Eigenvalue: $\underline{\lambda=0}$ $\quad \underline{\lambda=-1}$

(mult. 2)

Eigenvectors:

$\lambda=0$: $\begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x+z=0$, $y-2z=0$ (z free)

$$\left.\begin{aligned} x &= -z \\ y &= 2z \\ z &= z \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} -z \\ 2z \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}}}$$

$\lambda=-1$: $\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x+y=0$, $z=0$ (y free)

$$\left.\begin{aligned} x &= -y \\ y &= y \\ z &= 0 \end{aligned}\right\} \quad \underline{x} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \cdot \underline{\underline{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}}$$

vi) With $t=1$:

$$|A-\lambda I| = \begin{vmatrix} t-\lambda & t & 1 \\ 0 & t-\lambda & 0 \\ 1 & 0 & t-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\cdot\left((1-\lambda)^2-1\right)$$

$$= (1-\lambda)(\lambda^2-2\lambda) = (1-\lambda)\lambda(\lambda-2) = 0$$

Eigenvalues: $\underline{\lambda=1}$, $\underline{\lambda=0}$, $\underline{\lambda=2}$

Eigenvectors:

$\lambda=1$: $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x=0$, $y=-z$ (z free) $\quad \underline{x} = \begin{pmatrix} 0 \\ -z \\ z \end{pmatrix} = z\underline{\underline{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}}$

$\lambda = 0$: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x = -z$, $y = 0$, z free $\quad \underline{x} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 2$: $\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x = z$, $y = 0$, z free $\quad \underline{x} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(Ex. vi) is also possible to solve for general t)

vii) With $a = 0$:

$$|A - \lambda I| = \begin{vmatrix} a-2-\lambda & 2(a-2) & -1 \\ 1 & a-1-\lambda & 2 \\ 2 & 2 & a+2-\lambda \end{vmatrix} \overset{a=0}{=} \begin{vmatrix} -2-\lambda & -4 & -1 \\ 1 & -1-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\left((-1-\lambda)(2-\lambda) - 4\right) - 1\left(-4(2-\lambda) + 2\right) + 2\left(-8 + (-1-\lambda)\right)$$

$$= (-2-\lambda)(-1-\lambda)(2-\lambda) + (-4)(-2-\lambda) - 1(-6 + 4\lambda) + 2(-9-\lambda)$$

$$= (-2-\lambda)(-1-\lambda)(2-\lambda) + (-2\lambda - 4) = (-2-\lambda)(-1-\lambda)(2-\lambda) + 2 \cdot (-2-\lambda)$$

$$= (-2-\lambda) \cdot \left((-1-\lambda)(2-\lambda) + 2\right) = -(\lambda+2)(\lambda^2 - \lambda) = -(\lambda+2)\lambda(\lambda-1) = 0$$

Eigenvalues: $\lambda = -2$, $\lambda = 0$, $\lambda = 1$

Eigenvectors:

$\lambda = -2$: $\begin{pmatrix} 0 & -4 & -1 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x = -y - 2z = +\frac{1}{4}z - 2z = -\frac{7}{4}z$, $y = -\frac{1}{4}z$, (z free)

$$\underline{x} = \begin{pmatrix} -7/4\, z \\ -1/4\, z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -7/4 \\ -1/4 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 0}$: $\begin{pmatrix} -2 & -4 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 \\ 0 & -6 & 3 \\ 0 & 4 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$x - y + 2z = 0$ $\qquad x = y - 2z = \frac{1}{2}z - 2z = -\frac{3}{2}z$

$2y - z = 0$ $\qquad y = \frac{1}{2}z$

$(z \text{ free})$ $\qquad z = z$

$$\underline{x} = \begin{pmatrix} -3/2\, z \\ 1/2\, z \\ z \end{pmatrix} = z \underline{\underline{\begin{pmatrix} -3/2 \\ 1/2 \\ 1 \end{pmatrix}}}$$

$\underline{\lambda = 1}$: $\begin{pmatrix} -3 & -4 & -1 \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 \\ 0 & -10 & 5 \\ 0 & 6 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$x = 2y - 2z = z - 2z = -z$

$y = \frac{1}{2}z$

$(z \text{ free})$

$$\underline{x} = \begin{pmatrix} -z \\ \frac{1}{2}z \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} -1 \\ 1/2 \\ 1 \end{pmatrix}}}$$

<u>Hints:</u> Problems with eigenvalues / eigenvectors

a) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det(A)$ – use this to check your calculations

b) If $\det(A) = 0$, then $\lambda = 0$ is eigenvalue since

$|A| = 0 \iff \lambda_1 \lambda_2 \lambda_3 = 0 \iff \lambda_1 = 0$ or $\lambda_2 = 0$ or $\lambda_3 = 0$

c) If λ is eigenvalue, then the corresponding system $(A - \lambda I) \cdot \underline{x} = \underline{0}$ has <u>at least</u> one degree of freedom.

(c) Diagonalizable matrices

A diagonalizable $\Longleftrightarrow$ there are $\underline{n}$ lin. independent eigenvectors for A
(A $n \times n$ matrix)

Check: Find all eigenvalues and all eigenvectors

$$\text{A diagonalizable} \Longleftrightarrow \begin{array}{l} n_1 = \#\text{ degrees of freedom for } \lambda_1 \\ + n_2 = \text{— '' —} \quad \lambda_2 \\ + n_3 = \text{— '' —} \quad \lambda_3 \\ \hline n = 3 \end{array}$$

If A is symmetric, it is always diagonalizable
If A has n distinct eigenvectors, it is always diagonalizable.

Exc: Determine if A is diagonalizable in i) – vii).
(Use specific numerical value for parameters)

i) $\underline{\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -1}$ ($\lambda = -1$ mult. 2)
Count degrees of freedom:

$$\underset{\substack{\uparrow \\ \text{for } \lambda = 1}}{1} + \underset{\substack{\uparrow \\ \lambda = -1}}{2} = 3 \Rightarrow \text{A } \underline{\text{diagonalizable}}$$

ii) A symmetric $\Rightarrow$ A $\underline{\text{diagonalizable}}$ (Also $1+2=3$)

iii) A has three distinct eigenvalues $\Rightarrow$ A $\underline{\text{diagonalizable}}$ $\begin{cases} \text{(Also, A symmetric)} \\ \text{(Also, } 1+1+1=3) \end{cases}$

iv) $\underline{\text{with } t=-2}$: A $\underline{\text{not}}$ diagonalizable $(1+1=2 \neq 3)$

v) <u>With $a=1$</u>: A <u>not</u> <u>diagonalizable</u> $(1+1=2\neq 3)$

vi) <u>With $t=1$</u>: A <u>diagonalizable</u>
since three distinct eigenvalues (Also, $1+1+1=3$)

vii) <u>With $a=0$</u>: A <u>diagonalizable</u>
since three distinct eigenvalues (Also $1+1+1=3$)