LECTURE 9: FIRST ORDER DIFFERENTIAL EQUATIONS

Review: We shall see how to solve the following types of first order differential equations:

(1) ODEs solvable by direct integration:

$$y' = a(t) \implies y = \int a(t)\,dt$$

(2) Separable ODEs:

$$y' = a(y)\cdot b(t), \text{ where } \begin{cases} a(y): \text{ function in } y \\ b(t): \text{ — '' — in } t \end{cases}$$

$$\frac{1}{a(y)}\cdot y' = b(t)$$

$$\Updownarrow$$

$$\frac{1}{a(y)}\,dy = b(t)\,dt \quad \text{(separated form)}$$

$$\Updownarrow$$

$$\int \frac{1}{a(y)}\,dy = \int b(t)\,dt \quad \text{(implicit solution)}$$

$$\Updownarrow$$

$$y = \ldots \quad \text{(explicit solution)}$$

(3) Linear first order ODEs

$$y' + a(t)\cdot y = b(t)$$

(4) Exact ODE's

$$a(y,t) + b(y,t)\cdot y' = 0, \text{ where } \frac{\partial a}{\partial y} = \frac{\partial b}{\partial t}$$

We will solve these equations today.

Example of seperable ODE:

Solve: $x' = x\cdot(1-x) \quad (\longleftrightarrow y' = y\cdot(1-y))$

$$x' = \underbrace{x\cdot(1-x)}_{a(x)}\cdot\underbrace{1}_{b(t)}$$

$$\frac{1}{x\cdot(1-x)}x' = 1$$

$$\frac{1}{x(1-x)}dx = 1\cdot dt$$

$$\int \frac{1}{x(1-x)}dx = \int 1\cdot dt$$

$$\int 1\,dt = t + C$$

$$\int \frac{1}{x(1-x)}dx = \,?$$

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$$

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

$$\begin{cases} 1 = A\cdot(1-x) + Bx \\ x=1: \quad 1 = B \Rightarrow B = 1 \\ x=0: \quad 1 = A \Rightarrow A = 1 \end{cases}$$

$$\int \frac{1}{x(1-x)}\,dx = \int 1\,dt$$

$$\int \frac{1}{x} + \frac{1}{1-x}\,dx = t + C$$

$$\ln|x| + \ln|1-x| \cdot (-1) = t + C$$

$$\boxed{\ln|x| - \ln|1-x| = t + C} \quad \text{(implicit form)}$$

$$\ln\left|\frac{x}{1-x}\right| = t + C$$

$$\left|\frac{x}{1-x}\right| = e^{t+C} = e^t \cdot e^C$$

$$\frac{x}{1-x} = \pm e^C \cdot e^t = K \cdot e^t \quad | (1-x) \cdot$$

$$x = (1-x) \cdot Ke^t = K \cdot e^t - x \cdot Ke^t$$

$$x + x \cdot Ke^t = Ke^t$$

$$x \cdot (1 + Ke^t) = Ke^t \implies \underline{\underline{x = \frac{Ke^t}{1 + Ke^t}}} \quad \text{general solution}$$

Linear first order ODE's

Defn: A first order ODE is <u>linear</u> if it can be written

$$\boxed{y' + a(t) \cdot y = b(t)} \iff y' = -a(t) \cdot y + b(t)$$

Ex:

(i) $x' + 2tx = t^2$ is <u>linear</u>: $x' = -2t \cdot x + t^2$

(ii) $y' = y + e^{2t}$ is <u>linear</u> $y' = 1 \cdot y + e^{2t}$

(iii) $y' + y^2 = 0$ is <u>not</u> linear $y' = -y^2$ quadratic in y

(iv) $y' - e^y = 2t$ is <u>not</u> linear $y' = e^y + 2t$ not linear in y.

Ex: $\boxed{y' + 2y = 7}$

$$\left.\begin{aligned} a(t) &= 2 \\ b(t) &= 7 \end{aligned}\right\} \text{ constant function}$$

Idea: $(u \cdot v)' = u' \cdot v + u \cdot v'$

$(c \cdot y)' = c' \cdot y + c \cdot y'$

$\underline{c = e^{2t}}$: $(y \cdot e^{2t})' = y' \cdot e^{2t} + y \cdot e^{2t} \cdot 2$

$= \underbrace{(y' + 2y) \cdot e^{2t}}$

$\cdot e^{2t} \mid \; y' + 2y = 7$ $\qquad a = 2 \rightarrow e^{2t} \leftarrow$ integrating factor

$(y' + 2y) \cdot e^{2t} = 7e^{2t}$

$(y \cdot e^{2t})' = 7e^{2t}$

$y \cdot e^{2t} = \int 7 e^{2t} \, dt$

$y \cdot e^{2t} = 7\left(\frac{1}{2} \cdot e^{2t}\right) + C$

$y \cdot e^{2t} = \frac{7}{2} e^{2t} + C \quad \mid \cdot e^{-2t}$

$\underline{\underline{y = \frac{7}{2} + C e^{-2t}}}$ general solution

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Constant coefficients; general case

$y' + a \cdot y = b \qquad (a, b \text{ constants}; \; a \neq 0)$

$y' + ay = b \quad \mid \cdot e^{at}$ $\qquad$ Integrating factor: e^{at}

$(y' + ay) \cdot e^{at} = b \cdot e^{at}$

$(y \cdot e^{at})' = b e^{at}$

$y \cdot e^{at} = \int b e^{at} \, dt$

$y \cdot e^{at} = b\left(\frac{e^{at}}{a}\right) + C$

$y \cdot e^{at} = \frac{b}{a} \cdot e^{at} + C \quad \mid \cdot e^{-at}$

$\underline{\underline{y = \frac{b}{a} + C \cdot e^{-at}}}$ general solution

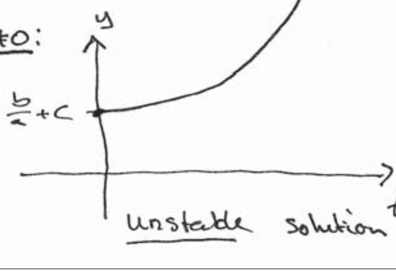
Check: $(y \cdot e^{at})' = y' \cdot e^{at} + y \cdot e^{at} \cdot a = y' \cdot e^{at} + ay \cdot e^{at}$

$= (y' + ay) \, e^{at}$

$a > 0$: $e^{-at} \to 0$ as $t \to \infty$

Stable solution

$a < 0; C \neq 0$: $e^{-at} \to \infty$ as $t \to \infty$



unstable solution

Ex: Model $\begin{cases} D = a - bP \\ S = \alpha + \beta P \\ P' = \lambda(D-S) \end{cases}$ a, b, α, β positive constants, λ constant

$$P' = \lambda \cdot (D-S) = \lambda((a-bP) - (\alpha + \beta P))$$

$$P' = \lambda(a - bP - \alpha - \beta P) = P \cdot (-\lambda b - \lambda\beta) + (\lambda a - \lambda\alpha)$$

$$P' + \lambda(b+\beta) \cdot P = \lambda(a-\alpha)$$

constants

General solution: $y' + ay = b \Rightarrow y = \frac{b}{a} + Ce^{-at}$

$$P = \frac{\cancel{\lambda}(a-\alpha)}{\cancel{\lambda}(b+\beta)} + C \cdot e^{-\lambda(b+\beta)t}$$

$$P = \frac{a-\alpha}{b+\beta} + C \cdot e^{-\lambda(b+\beta)t}$$

stable solution; $P \to \frac{a-\alpha}{b+\beta}$ as $t \to \infty$

$\lambda(b+\beta) > 0$

Numerical values:

$D = 5000 - 4P$ $\quad a = 5000 \quad b = 4$

$S = 1000 + 6P$ $\quad \alpha = 1000 \quad \beta = 6$

$P' = 0.5(D-S)$ $\quad \lambda = 0.5$

$P(0) = 900$

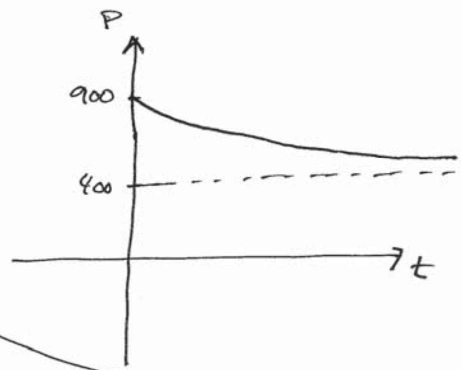
$$P = \frac{5000-1000}{10} + C \cdot e^{-0.5 \cdot 10 t} = 400 + C \cdot e^{-5t}$$

$P(0) = 900$: $900 = 400 + C \cdot 1$

$C = 500$

Particular solution:

$$P = 400 + 500 \cdot e^{-5t}$$



Non-constant coefficients:

Ex: $x' - 2tx = t \quad | \cdot q$ $\qquad a(t) = -2t$, $b(t) = t$

What is the integrating factor?

$(x' - 2tx)q = t \cdot q$

$(x \cdot q)' = tq$

Integrating factor: $q(t) = q$

$(x \cdot q)' = x' \cdot q + x \cdot q'$

$= x' \cdot q + x \cdot (-2tq)$

So $q' = -2t \cdot q$ seperable

$q = e^{\int -2t\,dt} = e^{-t^2}$

$(x \cdot e^{-t^2})' = t \cdot e^{-t^2}$

$x \cdot e^{-t^2} = \int t e^{-t^2} dt$ $\qquad u = -t^2$, $du = -2t\,dt$

$x \cdot e^{-t^2} = \int e^u \cdot (-\frac{1}{2}) du$

$xe^{-t^2} = -\frac{1}{2} \cdot e^u + C = -\frac{1}{2}e^{-t^2} + C$

$x = -\frac{1}{2} + C \cdot e^{t^2}$

In general: $e^{\int a(t)\,dt}$ is the integrating factor

③ General linear first order ODE:

$$y' + a(t)\cdot y = b(t)$$

Integrating factor: $e^{\int a(t)\,dt}$

$$\left(y' + a(t)\cdot y\right) e^{\int a(t)dt} = b(t)\, e^{\int a(t)dt}$$

$$\left(y\cdot e^{\int a(t)dt}\right)' = b(t)\, e^{\int a(t)dt}$$

$$y\cdot e^{\int a(t)dt} = \int b(t)\, e^{\int a(t)dt}\, dt$$

$$y = \frac{\int b(t)\, e^{\int a(t)dt}\, dt}{e^{\int a(t)\,dt}}$$

Ex: $\boxed{y' + 3\cancel{t}^2\cdot y = e^{-t^3}, \quad y(0) = 2}$

$$y' + 3t^2\cdot y = e^{-t^3}$$

I.F.: $e^{\int 3t^2 dt} = e^{t^3}$

$$(y' + 3t^2 y)e^{t^3} = e^{-t^3}\cdot e^{t^3}$$

$$\left(y\cdot e^{t^3}\right)' = 1$$

$$y\cdot e^{t^3} = \int 1\,dt = t + C$$

$$y = te^{-t^3} + C\cdot e^{-t^3}$$ general solution

$y(0)=2$: $\quad 2 = 0\cdot e^{-0} + C\cdot e^{-0}$

$$2 = C \Rightarrow C = 2$$

$$y = te^{-t^3} + 2e^{-t^3}$$

$$= (t+2)e^{-t^3}$$ particular solution

④ Exact ODE's : $\boxed{a(y,t) + y' \cdot b(y,t) = 0}$ (*)

Defn: An equation of the form (*) such that

$$\frac{\partial a}{\partial y} = \frac{\partial b}{\partial t}$$

Example: i) $\underbrace{1 + ty^2}_{\substack{a(y,t) \\ = 1+ty^2}} + \underbrace{t^2 y}_{\substack{b(y,t) \\ = t^2 y}} y' = 0$

$$\underbrace{\frac{\partial a}{\partial y} = t \cdot 2y = 2ty \qquad \frac{\partial b}{\partial t} = 2t \cdot y}_{\text{Equality} \Leftrightarrow \text{exact ODE}}$$

Example: $(1+ty^2) + (t^2 y) y' = 0$ is exact

$$\begin{cases} a(y,t) = 1 + ty^2 \\ b(y,t) = t^2 y \\ \dfrac{\partial a}{\partial y} = \dfrac{\partial b}{\partial t} = 2ty \end{cases}$$

Idea: $a(t,y) + b(t,y) \cdot y' = 0$

can be written as

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = 0$$

for some function $u = u(y,t)$

$$\left.\begin{aligned} a(t,y) + b(t,y) \cdot y' &= 0 \\ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} \cdot y' &= 0 \\ \frac{du}{dt} &= 0 \end{aligned}\right\} \quad u = t + \tfrac{1}{2} t^2 y^2 + C$$

By the chain rule:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} \cdot 1 + \frac{\partial u}{\partial y} \cdot y' = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} \cdot y'$$

Example: $(1+ty^2) + (t^2 y) y' = 0$

Want to find $u = u(y,t)$ such that
$$\begin{cases} \dfrac{\partial u}{\partial t} = 1 + t y^2 & (1) \\[2mm] \dfrac{\partial u}{\partial y} = t^2 y & (2) \end{cases}$$

(1) $u = \int 1 + t y^2 \, dt$

$= t + \frac{1}{2} t^2 y^2 + c(y)$, $c(y)$ any function in y

(2) $\dfrac{\partial u}{\partial y} = 0 + \frac{1}{2} t^2 \cdot 2y + c'(y)$

$= t^2 y + c'(y) = t^2 y \Rightarrow c'(y) = 0 \Rightarrow c(y) = C$ is a constant.

Solution: $u(y,t) = t + \frac{1}{2} t^2 y^2 + C$

$$\frac{du}{dt} = 1 + \left(\tfrac{1}{2}t^2\right)' \cdot y^2 + \left(\tfrac{1}{2}t^2\right) \cdot 2y \cdot y' + 0$$

$$= (1 + ty^2) + t^2 y \cdot y' = \partial u/\partial t + \partial u/\partial y \cdot y'$$

$$\frac{du}{dt} = 0 \Leftrightarrow u = t + \tfrac{1}{2} t^2 y^2 + C = C'$$

$$t + \tfrac{1}{2} t^2 y^2 = K \quad , \quad K = C' - C$$

$$\tfrac{1}{2} t^2 y^2 = K - t$$

$$y^2 = \frac{K-t}{\frac{1}{2}t^2} = \frac{2(K-t)}{t^2}$$

$$y = \pm \frac{\sqrt{2(K-t)}}{t}$$ general solution

In general: $u(y,t) = C$ (solution in implicit form)

$y = \dots$ (— " — explicit form)