Lecture 3
Eigenvalues and Eigenvectors

Eivind Eriksen

BI Norwegian School of Management
Department of Economics

September 10, 2010

A motivating example: Unemployment

Unemployment rates change over time as individuals gain or lose their employment. We consider a simple model, called a Markov model, that describes the dynamics of unemployment using transitional probabilities. In this model, we assume:

- If an individual is unemployed in a given week, the probability is p for this individual to be employed the following week, and $1 - p$ for him or her to stay unemployed
- If an individual is employed in a given week, the probability is q for this individual to stay employed the following week, and $1 - q$ for him or her to be unemployed

Markov model for unemployment

Let x_t be the ratio of individuals employed in week t, and let y_t be the ratio of individuals unemployed in week t. Then the week-on-week changes are given by these equations:

$$\begin{aligned} x_{t+1} &= qx_t + py_t \\ y_{t+1} &= (1-q)x_t + (1-p)y_t \end{aligned}$$

Note that these equations are linear, and can be written in matrix form as $\mathbf{v}_{t+1} = A\mathbf{v}_t$, where

$$A = \begin{pmatrix} q & p \\ 1-q & 1-p \end{pmatrix}, \quad \mathbf{v}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

We call A the transition matrix and $\mathbf{v}_t$ the state vector of the system. What is the long term state of the system? Are there any equilibrium states? If so, will these equilibrium states be reached?

Long term state of the system

The state of the system after t weeks is given by:

- $\mathbf{v}_1 = A\mathbf{v}_0$
- $\mathbf{v}_2 = A\mathbf{v}_1 = A(A\mathbf{v}_0) = A^2\mathbf{v}_0$
- $\mathbf{v}_3 = A\mathbf{v}_2 = A(A^2\mathbf{v}_0) = A^3\mathbf{v}_0$
- $\Rightarrow \boxed{\mathbf{v}_t = A^t\mathbf{v}_0}$

For white males in the US in 1966, the probabilities where found to be $p = 0.136$ and $q = 0.998$. If the unemployment rate is 5% at $t = 0$, expressed by $x_0 = 0.95$ and $y_0 = 0.05$, the situation after 100 weeks would be

$$\begin{pmatrix} x_{100} \\ y_{100} \end{pmatrix} = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix}^{100} \cdot \begin{pmatrix} 0.95 \\ 0.05 \end{pmatrix} = \quad ?$$

We need eigenvalues and eigenvectors to compute A^{100} efficiently.

Steady states

Definition

A steady state is a state vector $\mathbf{v} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ *with* $x, y \geq 0$ *and* $x + y = 1$ *such that* $A\mathbf{v} = \mathbf{v}$*. The last condition is an equilibrium condition*

Example

Find the steady state when $A = \left(\begin{smallmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{smallmatrix}\right)$.

Solution

The equation $A\mathbf{v} = \mathbf{v}$ *is a linear system, since it can be written as*

$$\begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0.998 - 1 & 0.136 \\ 0.002 & 0.864 - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Steady states

Solution (Continued)

So we see that the system has one degree of freedom, and can be written as

$$-0.002x + 0.136y = 0 \Rightarrow \begin{cases} x = 68y \\ y = \textit{free variable} \end{cases}$$

The only solution that satisfies $x + y = 1$ *is therefore given by*

$$x = \frac{68}{69} \cong 0.986, \quad y = \frac{1}{69} \cong 0.014$$

In other words, there is an equilibrium or steady state of the system in which the unemployment is 1.4%. The question if this steady state will be reached is more difficult, but can be solved using eigenvalues.

Diagonal matrices

An $n \times n$ matrix is diagonal if it has the form

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

It is easy to compute with diagonal matrices.

Example

Let $D = \left(\begin{smallmatrix} 5 & 0 \\ 0 & 3 \end{smallmatrix}\right)$. *Compute* D^2, D^3, D^n *and* D^{-1}.

Computations with diagonal matrices

Solution

$$D^2 = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}^2 = \begin{pmatrix} 5^2 & 0 \\ 0 & 3^2 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}^3 = \begin{pmatrix} 5^3 & 0 \\ 0 & 3^3 \end{pmatrix} = \begin{pmatrix} 125 & 0 \\ 0 & 27 \end{pmatrix}$$

$$D^n = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 5^n & 0 \\ 0 & 3^n \end{pmatrix}$$

$$D^{-1} = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 5^{-1} & 0 \\ 0 & 3^{-1} \end{pmatrix} = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/3 \end{pmatrix}$$

Definitions: Eigenvalues and eigenvectors

Let A be an $n \times n$ matrix.

Definition

If there is a number $\lambda \in \mathbb{R}$ and an n-vector $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \lambda\mathbf{x}$, then we say that λ is an eigenvalue for A, and $\mathbf{x}$ is called an eigenvector for A with eigenvalue λ.

Note that eigenvalues are numbers while eigenvectors are vectors.

Definition

The set of all eigenvectors of A for a given eigenvalue λ is called an eigenspace, and it is written $E_\lambda(A)$.

Eigenvalues: An example

Example

Let

$$A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Are $\mathbf{u}, \mathbf{v}$ *eigenvectors for* A*? If so, what are the eigenvalues?*

Solution

We compute

$$A\mathbf{u} = \begin{pmatrix} -24 \\ 20 \end{pmatrix}, \quad A\mathbf{v} = \begin{pmatrix} -9 \\ 11 \end{pmatrix}$$

We see that $A\mathbf{u} = -4\mathbf{u}$*, so* $\mathbf{u}$ *is an eigenvector with eigenvalue* $\lambda = -4$*. But* $A\mathbf{v} \neq \lambda\mathbf{v}$*, so* $\mathbf{v}$ *is not an eigenvector for* A*.*

Computation of eigenvalues

It is possible to write the vector equation $A\mathbf{x} = \lambda\mathbf{x}$ as a linear system. Since $\lambda\mathbf{x} = \lambda I\mathbf{x}$ (where $I = I_n$ is the identity matrix), we have that

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Leftrightarrow \quad A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad \boxed{(A - \lambda I)\mathbf{x} = \mathbf{0}}$$

This linear system has a non-trivial solution $\mathbf{x} \neq \mathbf{0}$ if and only if $\det(A - \lambda I) = 0$.

Definition

The characteristic equation of A is the equation

$$\det(A - \lambda I) = 0$$

It is a polynomial equation of degree n in one variable λ.

Computation of eigenvalues

Proposition

The eigenvalues of A are the solutions of the characteristic equation $\det(A - \lambda I) = 0$.

Idea of proof: The eigenvalues are the numbers λ for which the equation $A\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

Example

Find all the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$$

Example: Computation of eigenvalues

Solution

To find the eigenvalues, we must write down and solve the characteristic equation. We first find $A - \lambda I$*:*

$$A - \lambda I = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix}$$

Then the characteristic equation becomes

$$\begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = (2-\lambda)(-6-\lambda) - 3 \cdot 3 = \boxed{\lambda^2 + 4\lambda - 21 = 0}$$

The solutions are $\lambda = -7$ *and* $\lambda = 3$*, and these are the eigenvalues of* A*.*

Computation of eigenvectors

Prodedure

- Find the eigenvalues of A, if this is not already known.
- For each eigenvalue λ, solve the linear system $(A - \lambda I)\mathbf{x} = \mathbf{0}$. The set of all solutions of this linear system is the eigenspace $E_\lambda(A)$ of all eigenvectors of A with eigenvalue λ.

The solutions of the linear system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ can be found using Gaussian elimination, for instance.

Example

Find all eigenvectors for the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$$

Example: Computation of eigenvectors

Solution

We know that the eigenvalues are $\lambda = -7$ and $\lambda = 3$, so there are two eigenspaces E_{-7} and E_3 of eigenvectors. Let us compute E_{-7} first. We compute the coefficient matrix $A - \lambda I$ and reduce it to echelon form:

$$A - (-7)I = \begin{pmatrix} 2-(-7) & 3 \\ 3 & -6-(-7) \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 1/3 \\ 0 & 0 \end{pmatrix}$$

Hence $x_2 = s$ is a free variable, and $x_1 = -\frac{1}{3}x_2 = -\frac{1}{3}s$. We may therefore write all eigenvectors for $\lambda = -7$ in parametric vector form as:

$$E_{-7}(A): \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}s \\ s \end{pmatrix} = s \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} \text{ for all } s \in \mathbb{R}$$

Example: Computation of eigenvectors

Solution

Let us compute the other eigenspace E_3 of eigenvector with eigenvalue $\lambda = 3$. We compute the coefficient matrix $A - \lambda I$ and reduce it to echelon form:

$$A - 3I = \begin{pmatrix} 2-3 & 3 \\ 3 & -6-3 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

Hence $x_2 = s$ is a free variable, and $x_1 = 3x_2 = 3s$. We may therefore write all eigenvectors for $\lambda = 3$ in parametric vector form as:

$$E_3(A): \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3s \\ s \end{pmatrix} = s \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ for all } s \in \mathbb{R}$$

Eigenspaces

When λ is en eigenvalue for A, the linear system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ should have non-trivial solutions, and therefore at least one degree of freedom.

How to write eigenspaces

It is convenient to describe an eigenspace E_λ, i.e. the set of solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$, as the set of vectors on a given parametric vector form.

- This parametric vector form is obtained by solving for the basic variables and expressing each of them in terms of the free variables, for instance using a reduced echelon form.
- If the linear system has m degrees of freedom, then the eigenspace is the set of all linear combinations of m eigenvectors.
- These eigenvectors are linearly independent.

Example: How to write eigenspaces

Example

We want to write down the eigenspace of a matrix A with eigenvalue λ. We first find the reduced echelon form of $A - \lambda I$. Let's say we find this matrix:

$$\begin{pmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we see that $x_2 = s$ and $x_4 = t$ are free variables and that the general solution can be found when we solve for x_1 and x_3 and express each of them in terms of $x_2 = s$ and $x_4 = t$:

Linearly independent eigenvectors

Example (Continued)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 + 4x_4 \\ \textit{free} \\ -3x_4 \\ \textit{free} \end{pmatrix} = \begin{pmatrix} -2s + 4t \\ s \\ -3t \\ t \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

Hence the correpsponding eigenspace is all linear combinations of the two linearly independent vectors

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

Diagonalization

We have seen that it is easier to compute with diagonal matrices. Most matrices are not diagonal, but sometimes a non-diagonal matrix can be diagonalized:

Definition

An $n \times n$ matrix A is diagonalizable if there exists a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Note that the equation $A = PDP^{-1}$ can also be re-written as

$$A = PDP^{-1} \quad \Leftrightarrow \quad D = P^{-1}AP \quad \Leftrightarrow \quad AP = PD$$

The last equation means that D consists of eigenvalues for A (on the diagonal) and that P consists of eigenvectors for A (as columns).

Criterion for diagonalization

Let A be an $n \times n$ matrix, let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the k distinct eigenvalues of A, and let $m_i \geq 1$ be the degrees of freedom of the linear system $(A - \lambda_i I)\mathbf{x} = \mathbf{0}$ for $i = 1, 2, \dots, k$.

Proposition

The $n \times n$ matrix A is diagonalizable if and only if $m_1 + m_2 + \cdots + m_k = n$. In this case, a diagonalization of A can be chosen in the following way:

1. *D is a diagonal matrix with the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ on the diagonal (with λ_i repeated m_i times)*
2. *P is a matrix consisting of eigenvectors as columns (with m_i linearly independent eigenvector for each eigenvalue λ_i)*

Idea of proof: When we form D and P from eigenvalues and eigenvectors, we know that $AP = PD$, so the question is whether we have enough eigenvectors; P is invertible if and only if it consists of n linearly independent eigenvectors.

Which matrices are diagonalizable?

Remarks

- An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.
- If A has n distinct eigenvalues, then it is diagonalizable
- If A is symmetric, then it is diagonalizable

Example

Diagonalize the following matrix, if possible:

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$$

Example: Diagonalization

Solution

We have computed the eigenvalues and eigenvector of the matrix A earlier. Since $\lambda_1 = -7$ *and* $\lambda_2 = 3$ *are the eigenvalues, we choose*

$$D = \begin{pmatrix} -7 & 0 \\ 0 & 3 \end{pmatrix}$$

Since there was one degree of freedom for each of the eigenvalues, we have $m_1 + m_2 = 1 + 1 = 2$, *and A is diagonalizable. To find P, we use the eigenspaces we found earlier:*

$$E_{-7} : \mathbf{x} = s \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}, \quad E_3 : \mathbf{x} = s \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \Rightarrow \quad P = \begin{pmatrix} -\frac{1}{3} & 3 \\ 1 & 1 \end{pmatrix}$$

Application: Computation of powers

Example

Compute A^{1000} when

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$$

Solution

We have found a diagonalization of A earlier, with

$$D = \begin{pmatrix} -7 & 0 \\ 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} -\frac{1}{3} & 3 \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{10} \begin{pmatrix} -3 & 9 \\ 3 & 1 \end{pmatrix}$$

where P^{-1} has been computed from P. We use this to find a formula for the power A^{1000}:

$$A^{1000} = (PDP^{-1})^{1000} = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) = PD^{1000}P^{-1}$$

Application: Computation of powers

Solution (Continued)

From this formula we compute that

$$A^{1000} = \begin{pmatrix} -\frac{1}{3} & 3 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} (-7)^{1000} & 0 \\ 0 & 3^{1000} \end{pmatrix} \cdot \frac{1}{10} \begin{pmatrix} -3 & 9 \\ 3 & 1 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 7^{1000} + 9 \cdot 3^{1000} & -3 \cdot 7^{1000} + 3 \cdot 3^{1000} \\ -3 \cdot 7^{1000} + 3 \cdot 3^{1000} & 9 \cdot 7^{1000} + 3^{1000} \end{pmatrix}$$

Problem

Compute the unemployment rate after 100 weeks in the example from slide 4.

Example: Diagonalization

Example

Diagonalize the following matrix, if possible:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution

Since the matrix is upper triangular, the eigenvalues are the elements on the diagonal; $\lambda_1 = 1$ *(double root) and* $\lambda_2 = 3$*. For* $\lambda = 1$*, and get*

$$A - 1I = \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & -6 \\ 0 & 0 & 2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and hence $m_1 = 1$ *degrees of freedom.*

Example: Diagonalization

Solution (Continued)

For $\lambda = 3$*, we get*

$$A - 3I = \begin{pmatrix} -2 & 2 & 4 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

and hence $m_2 = 1$ *degrees of freedom. Since* $m_1 + m_2 = 2 < n = 3$*,* A *is not diagonalizable.*

We see that it is the eigenvalue $\lambda = 1$ that is the problem in this example. Even though $\lambda = 1$ appears twice as an eigenvalue (double root), there is only one degree of freedom and therefore not enough linearly independent eigenvectors.