Lecture 2
The rank of a matrix

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Linear dependence

To decide if a set of m-vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ are linearly independent, we have to solve the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

We know that $\mathbf{x} = \mathbf{0}$ is one solution, the *trivial solution*. Are there other (non-trivial) solutions?

- If yes, then the vectors are *linearly dependent*. We can use a non-trivial solution to express one vector as a linear combination of the others.
- If no, then the vectors are *linearly independent*

Linear systems and vector equations

A linear system of m equations is the same as a single vector equation of m-vectors. We may therefore re-write a vector equation as a linear system, and also re-write a linear system as a vector equation.

Example

Write the following linear system as a vector equation:

$$\begin{array}{ccccccc} 2x_1 & + & 2x_2 & - & x_3 & = & 0 \\ 4x_1 & & & + & 2x_3 & = & 0 \\ & & 6x_2 & - & 3x_3 & = & 0 \end{array}$$

Linear systems and vector equations

Solution

We re-write the three equations as one equation of 3*-vectors:*

$$x_1 \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We may write this as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}$$

Note that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are the columns of the coefficient matrix of the linear system, and $\mathbf{0}$ is the last (augmented) column of the augmented matrix.

Criterion for linear independence

Theorem

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be n-vectors, and let A be the $n \times n$ matrix with these vectors as columns. Then $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ are linearly independent if and only if

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \neq 0$$

Idea for proof: The linear system $A\mathbf{x} = \mathbf{0}$ has a unique solution (that is, only the trivial solution) if and only if $\det(A) \neq 0$.

An example

Example

Show that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly independent when

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

Solution

Since we have

$$\begin{vmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -3 \end{vmatrix} = (-4) \cdot 0 + (-2) \cdot 12 = -24 \neq 0$$

it follows that the vectors are linearly independent.

Rank of a matrix

Let A be any $m \times n$ matrix. Then A consists of n column vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$, which are m-vectors.

Definition

The rank of A is the maximal number of linearly independent column vectors in A, i.e. the maximal number of linearly independent vectors among $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$. If $A = 0$, then the rank of A is 0.

We write $\text{rk}(A)$ for the rank of A. Note that we may compute the rank of any matrix — square or not.

Rank of 2×2 matrices

Let us first see how to compute the rank of a 2×2 matrix:

Example

The rank of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by:

- $\text{rk}(A) = 2$ *if* $\det(A) = ad - bc \neq 0$, *since both column vectors are independent in this case*
- $\text{rk}(A) = 1$ *if* $\det(A) = 0$ *but* $A \neq 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, *since both column vectors are not linearly independent, but there is a single column vector that is linearly independent (i.e. non-zero)*
- $\text{rk}(A) = 0$ *if* $A = 0$

How do we compute $\text{rk}(A)$ for an $m \times n$ matrix A?

Computing rank using Gauss elimination

Gauss elimination

Use elementary row operations to reduce A to echelon form. The rank of A is the number of pivots or leading coefficients in the echelon form. In fact, the pivot columns (i.e. the columns with pivots in them) are linearly independent.

Note that it is not necessary to find the reduced echelon form — any echelon form will do since only the pivots matter.

Possible ranks

Counting possible number of pivots, we see that

- $\text{rk}(A) \leq m$ and $\text{rk}(A) \leq n$

for any $m \times n$ matrix A.

Rank: Example using Gauss elimination

Example

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Solution

We use elementary row operations:

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix}$$

Since the echelon form has pivots in the first three columns, A has rank $\text{rk}(A) = 3$. *The first three columns of A are linearly independent.*

Computing rank using determinants

Definition

Let A be an $m \times n$ matrix. A minor of A of order k is a determinant of a $k \times k$ sub-matrix of A.

We obtain the minors of order k from A by first deleting $m - k$ rows and $n - k$ columns, and then computing the determinant. There are usually many minors of A of a given order.

Example

Find the minors of order 3 of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Computing minors

Solution

We obtain the determinants of order 3 by keeping all the rows and deleting one column from A. So there are four different minors of order 3. We compute one of them to illustrate:

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1 \cdot (-4) + 2 \cdot 0 = -4$$

The minors of order 3 are called the maximal minors of A, since there are no 4×4 sub-matrices of A. There are $3 \cdot 6 = 18$ minors of order 2 and $3 \cdot 4 = 12$ minors of order 1

Computing rank using minors

Proposition

Let A be an $m \times n$ matrix. The rank of A is the maximal order of a non-zero minor of A.

Idea of proof: If a minor of order k is non-zero, then the corresponding columns of A are linearly independent.

Computing the rank

Start with the minors of maximal order k. If there is one that is non-zero, then $\text{rk}(A) = k$. If all maximal minors are zero, then $\text{rk}(A) < k$, and we continue with the minors of order $k-1$ and so on, until we find a minor that is non-zero. If all minors of order 1 (i.e. all entries in A) are zero, then $\text{rk}(A) = 0$.

Rank: Examples using minors

Example

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Solution

The maximal minors have order 3, and we found that the one obtained by deleting the last column is $-4 \neq 0$. Hence $\text{rk}(A) = 3$.

Rank: Examples using minors

Example

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}$$

Rank: Examples using minors

Solution

The maximal minors have order 3*, so we compute the* 4 *minors of order* 3*. The first one is*

$$\begin{vmatrix} 1 & 2 & 1 \\ 9 & 5 & 2 \\ 7 & 1 & 0 \end{vmatrix} = 7 \cdot (-1) + (-1) \cdot (-7) = 0$$

The other three are also zero. Since all minors of order 3 *are zero, the rank must be* $\text{rk}(A) < 3$*. We continue to look at the minors of order two. The first one is*

$$\begin{vmatrix} 1 & 2 \\ 9 & 5 \end{vmatrix} = 5 - 18 = -13 \neq 0$$

It is not necessary to compute any more minors, and we conclude that $\text{rk}(A) = 2$*. In fact, the first two columns of* A *are linearly independent.*

Application: Linear independence

Example

Show that the vectors are linearly independent:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

Solution

The vectors are linearly independent if and only if $\text{rk}(A) = 2$*, where* A *is the matrix with* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *as columns. Since we have*

$$\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0$$

it follows that $\text{rk}(A) = 2$.

Rank and linear systems

Theorem

Let $A_{\mathbf{b}} = (A|\mathbf{b})$ *be the augmented matrix of a linear system* $A\mathbf{x} = \mathbf{b}$ *in* n *unknowns. Then we have:*

1. *The linear system is consistent if and only if* $\text{rk}\, A_{\mathbf{b}} = \text{rk}\, A$.
2. *If the linear system is consistent, then it has a unique solution if and only if* $\text{rk}(A) = n$*. Moreover, if* $\text{rk}(A) < n$*, then the system has* $n - \text{rk}(A)$ *free variables.*

Idea of proof: Think of the pivots in the reduced echelon form of the system.

Linear system: Example using rank

Example

Is the following linear system consistent? Does it have a unique solution?

$$\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & - & x_3 & = & 1 \\ 4x_1 & & & + & 2x_3 & = & 2 \\ & & 6x_2 & - & 3x_3 & = & 4 \end{array}$$

Linear system: Example using rank

Solution

We form the matrices

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -1 \end{pmatrix}, \quad A_{\mathbf{b}} = \begin{pmatrix} 2 & 2 & -1 & 1 \\ 4 & 0 & 2 & 2 \\ 0 & 6 & -1 & 4 \end{pmatrix}$$

We compute that $\det(A) = -24 \neq 0$, *so* $\operatorname{rk}(A) = 3$ *and* $\operatorname{rk} A_{\mathbf{b}} = 3$ *(since the determinant is a maximal minor of the augmented matrix). Hence the system is consistent. In fact,* $n - \operatorname{rk} A = 3 - 3 = 0$, *so the system has a unique solution.*

Linear system: Explicit solutions using minors

Interpretation of minors

We consider a consistent linear system $A\mathbf{x} = \mathbf{b}$ and let $k = \text{rk}(A)$. Then there is a non-zero minor of order k. We can interpret this minor in the following way:

- The deleted rows are not essential, and we may disregard them. Hence we only regard the rows (equations) that are *in* the minor.
- The variables corresponding to deleted columns represent free (independent) variables. The variables corresponding to columns *in* the minor are basic (dependent) variable.
- We may write down the solution of the system by solving the equations *in* the minor for the basic (dependent) variables.

Linear system: Example solved using minors

Example

Solve the following (consistent) linear system using minors:

$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_4 &= 0 \\ x_2 + x_3 &= 0 \\ x_1 + 3x_3 + 3x_4 &= 0 \end{aligned}$$

We remark that this system is consistent, since it has the trivial solution $\mathbf{x} = \mathbf{0}$.

Linear system: Example solved using minors

Solution

We compute the rank of the coefficient matrix

$$A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 3 & 3 \end{pmatrix}$$

After some computations, we see that all maximal (order 3) minors are zero. However, the minor of order 2 obtained by deleting the last row and the last two columns from A is

$$\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

This means that $\operatorname{rk} A = 2$, *and that the linear system has* $4 - 2 = 2$ *free variables.*

Linear system: Example solved using minors

Solution

The free variables are x_3, x_4, *and we may express* x_1, x_2 *in terms of the free variables using the first two equations:*

$$\begin{aligned} x_1 - x_2 &= -2x_3 - 3x_4 \\ x_2 &= -x_3 \end{aligned}$$

This gives

$$\begin{aligned} x_1 &= -3x_3 - 3x_4 \\ x_2 &= -x_3 \\ x_3 &= \textit{ free variable} \\ x_4 &= \textit{ free variable} \end{aligned}$$