LECTURE 12: Difference equations + Revision

First order difference equations:

i) $x_{t+1} = ax_t + b \Leftrightarrow x_{t+1} - ax_t = b$:

Solution: $x_t = \begin{cases} a^t \cdot \left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a} &, a \neq 1 \\ x_0 + bt &, a = 1 \end{cases}$

ii) $x_{t+1} = ax_t + b_t \Leftrightarrow x_{t+1} - ax_t = b_t$:

Solution: $x_t = x_t^h + x_t^p$
$= a^t \cdot C + x_t^p$
$= a^t (x_0 - x_0^p) + x_t^p$

Second order difference equations:

i) Homogeneous: $x_{t+2} + ax_{t+1} + bx_t = 0$

Char. eqn: $r^2 + ar + b = 0 \Rightarrow r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

$a^2 - 4b > 0$: $r_1 \neq r_2 \Rightarrow x_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t$

$a^2 - 4b = 0$: $r \Rightarrow x_t = C_1 \cdot r^t + C_2 \cdot t r^t = (C_1 + C_2 t) r^t$

$a^2 - 4b < 0$: $x_t = (\sqrt{b})^t \cdot (C_1 \cdot \cos \theta t + C_2 \sin \theta t)$

with $\theta = \arccos(-a/2b)$

ii) Inhomogeneous: $x_{t+2} + ax_{t+1} + bx_t = c_t$, $c_t \neq 0$

Solution: $x_t = \underbrace{x_t^h}_{\substack{\text{Solution of} \\ x_{t+2} + ax_{t+1} + bx_t = 0 \\ = \text{homogeneous part}}} + x_t^p$ particular solution of inhomogeneous diff. eqn.

Example I: First order

$x_{t+1} = \frac{1}{2} x_t + t \Leftrightarrow \boxed{x_{t+1} - \frac{1}{2} x_t = t}$

Solution: $x_t = x_t^h + x_t^p = C \cdot \left(\frac{1}{2}\right)^t + 2t - 4$

x_t^h: $x_{t+1} - \frac{1}{2} x_t = 0$ — $r - \frac{1}{2} = 0 \Rightarrow r = \frac{1}{2}$; $x_{t+1} = \frac{1}{2} x_t$

$x_t = C \cdot \left(\frac{1}{2}\right)^t$

$x_t^h = C \cdot \left(\frac{1}{2}\right)^t$

x_t^p: $x_{t+1} - \frac{1}{2} x_t = t \rightarrow \boxed{\begin{cases} b_t = t \\ b_{t+1} = t+1 \end{cases}}$

Guess: $\begin{cases} x_t = At + B \\ x_{t+1} = A(t+1) + B = At + A + B \end{cases}$

$(At + A + B) - \frac{1}{2}(At + B) = t$

$(At - \frac{1}{2}At) + (A + B - \frac{1}{2}B) = t$

$(A - \frac{1}{2}A)t + (A + B - \frac{1}{2}B) = 1 \cdot t + 0$

$\frac{1}{2}A = 1$, $A + \frac{1}{2}B = 0 \Rightarrow A = 2, B = -4 \Rightarrow x_t^p = 2t - 4$

Example 2: Second order

$x_{t+2} - 4x_{t+1} + 4x_t = 0$, $x_0 = 2$, $x_1 = 3$

Solution:

Char. eqn: $r^2 - 4r + 4 = 0$

$$r = \frac{4 \pm \sqrt{4^2 - 4\cdot 1\cdot 4}}{2} = \underline{2}$$

(double root)

$$X_t = \underline{C_1 \cdot 2^t + C_2 \cdot t \cdot 2^t}$$

$X_0 = 2$: $C_1 \cdot 2^0 + C_2 \cdot 0 \cdot 2^0 = 2 \Rightarrow C_1 = 2$

$X_1 = 3$: $C_1 \cdot 2^1 + C_2 \cdot 1 \cdot 2^1 = 3$ $\quad 2C_1 + 2C_2 = 3$

$\underline{C_1 = 2}$, $C_2 = \underline{-1/2}$

$$X_t = 2 \cdot 2^t + (-1/2)\, t \cdot 2^t$$
$$= \underline{\underline{2^{t+1} - t \cdot 2^{t-1}}}$$

Example 3: Inhomogeneous second order

$x_{t+2} + 2x_{t+1} + x_t = 5$

Solution: $X_t = x_t^h + x_t^p = \underline{\underline{c_1 \cdot (-1)^t + c_2 t (-1)^t + 5/4}}$

x_t^h: $x_{t+2} + 2x_{t+1} + x_t = 0$

$r^2 + 2r + 1 = 0$

$\underline{\underline{r = -1}}$ (double root)

$x_t^h = \underline{C_1 \cdot (-1)^t + C_2 \cdot t \cdot (-1)^t}$

x_t^p: $x_{t+2} + 2x_{t+1} + x_t = 5$

Guess: $x_t = A$

$A + 2A + A = 5 \Rightarrow A = \underline{5/4}$

$x_t^p = \underline{5/4}$

Example: System of first order eqn's

$$\left.\begin{aligned} x_{t+1} &= x_t + 2y_t \\ y_{t+1} &= 3x_t \end{aligned}\right\} \quad \text{with} \quad \begin{aligned} x_0 &= 1 \\ y_0 &= 0 \end{aligned}$$

Solution:

$$x_{t+1} = x_t + 2y_t = x_t + 2\cdot(3x_{t-1}) = x_t + 6x_{t-1}$$

or

$$x_{t+2} = x_{t+1} + 2y_{t+1} = x_{t+1} + 2\cdot(3x_t) = x_{t+1} + 6x_t$$

$$x_{t+2} = x_{t+1} + 6x_t$$

$$x_{t+2} - x_{t+1} - 6x_t = 0$$

$$r^2 - r - 6 = 0 \Rightarrow r = \frac{1 \pm \sqrt{1-4\cdot(-6)}}{2} = \frac{1\pm 5}{2}$$

$$= \underline{3, -2}$$

$$\underline{x_t = c_1 \cdot 3^t + c_2 \cdot (-2)^t}$$

$$y_t = 3\cdot x_{t-1} = 3\cdot\left(c_1\cdot 3^{t-1} + c_2\cdot(-2)^{t-1}\right) = \underline{c_1\cdot 3^t + 3c_2(-2)^{t-1}}$$

Exam Dec. '09:

1. a) $\left\{ \begin{pmatrix} 1 \\ t \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -t \\ -1 \\ -1 \end{pmatrix} \right\}$

$\det(A) \neq 0$
$\Updownarrow$
lin. independent

$$\begin{vmatrix} 1 & 2 & -t \\ t & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix} = 1\cdot(-1+3) - t(-2+3t) + 1(-2+t)$$

$$= 2 + 2t - 3t^2 - 2 + t = -3t^2 + 3t$$

$$\det(A) = 0 \Leftrightarrow -3t^2 + 3t = 0 \Leftrightarrow -3t(t-1) = 0$$

$$\underline{t = 0,\ t = 1}$$

$t = 0, 1$: linearly dependent

$t \neq 0, 1$: — " — independent

b) $c_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$

$$c_1 + 2c_2 = 0 \Rightarrow c_1 = 2$$

$$0\cdot c_1 + c_2 = -1 \Rightarrow c_2 = -1$$

$$\underline{\underline{c_1 = 2,\ c_2 = -1}}$$

c) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ diagonalizable ?

Eigenvalues: $\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)\cdot(1-\lambda) = 0$

$\lambda = 1$ (doubleroot)

Do we have two lin. independent eigenvectors ?

$\lambda = 1$: $(A - \lambda I) = \begin{pmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{pmatrix}_{\lambda=1} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \leftarrow rk = 1$

free var $= 2-1 = 1$

Eigenvectors: $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \cdot \underline{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

1 linearly ind. eigenvector $\Rightarrow$ too few $\Rightarrow$ not diag.

2. $f(x_1, x_2, x_3) = x_1 + x_2^2 + x_3^3 - x_1 x_2 - 3x_3$

a) Stationary points: $(2,1,1), (2,1,-1)$

$f'_{x_1} = f'_1 = 1 - x_2 = 0 \qquad x_2 = 1$

$f'_2 = 2x_2 - x_1 = 0 \qquad x_1 = 2$

$f'_3 = 3x_3^2 - 3 = 0 \qquad x_3^2 = 1 \rightarrow x_3 = \pm 1$

b) Classification: local max / local min / ~~stationary~~ saddle point.

Compute the Hessian:

$$f'' = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6x_3 \end{pmatrix}$$

Leading principal minors

$D_1 = 0$

$D_2 = -1 < 0$

$\Downarrow$

not local max/min

Local max	Local min.
$\Updownarrow$	$\Updownarrow$
Negative semidef.	Pos. semidef.
$\Updownarrow$	$\Updownarrow$
$\Delta_1 \le 0,\ \Delta_2 \ge 0,\ \Delta_3 \le 0$	$\Delta_1, \Delta_2, \Delta_3 \ge 0$

Conclusion: $\left.\begin{matrix}(2,1,1)\\(2,1,-1)\end{matrix}\right\}$ are saddle points

$$f(x,y;p) = -px + e^{x+y^2}$$

c) Is f convex/concave as a function of (x,y).

f'' pos semidef. $\Leftrightarrow$ f convex $\cup$

f'' neg. semidef. $\Leftrightarrow$ f concave $\cap$

$$\left.\begin{aligned} f'_x &= -p + e^{x+y^2} \\ f'_y &= e^{x+y^2}\cdot 2y \end{aligned}\right\} \quad f'' = \begin{pmatrix} e^{x+y^2} & e^{x+y^2}\cdot 2y \\ e^{x+y^2}\cdot 2y & e^{x+y^2}\cdot 2y\cdot 2y + e^{x+y^2}\cdot 2 \end{pmatrix}$$

$$= e^{x+y^2}\cdot \begin{pmatrix} \boxed{1} & 2y \\ 2y & 4y^2+2 \end{pmatrix}$$

Leading Principal Min:

$$D_1 = 1\cdot e^{x+y^2} > 0$$

$$D_2 = \left(1\cdot(4y^2+2) - (2y)^2\right)\left(e^{x+y^2}\right)^2$$

$$= \left(\cancel{4y^2} + 2 - \cancel{4y^2}\right) e^{2(x+y^2)}$$

$$= 2e^{2(x+y^2)} > 0 \quad \Rightarrow \text{f is convex in } (x,y)$$

d) $f(x,y;p) = -px + e^{x+y^2}$

Stationary pts:

$$f'_x = -p + e^{x+y^2} = 0 \Rightarrow e^x = p \Rightarrow \begin{cases} x = \ln p, & p > 0 \\ \text{no sol'n}, & p \le 0 \end{cases}$$

$$f'_y = e^{x+y^2}\cdot 2y = 0 \Rightarrow y = 0$$

Assume $p > 0$:

$x^*(p) = \ln p$, $y^*(p) = 0$ global min. (since f convex)

$$f^*(p) = f(\ln p, 0; p) = -p\ln p + e^{\ln p + 0} = -p\ln p + p$$

e) $$\frac{d}{dp}\left(f^*(p)\right) = -1\cdot\ln p - p\cdot\frac{1}{p} + 1 = -\ln p$$

|| envelope thm.

$$\left.\frac{\partial f}{\partial p}\right|_{(x,y)=(x^*(p),y^*(p))} = -x^*(p) = -\ln p$$

3. a) $x' + at \cdot x = 2t \quad , \; a \neq 0$

$(x' + atx) e^{\frac{1}{2}at^2} = 2t e^{\frac{1}{2}at^2}$ $\quad \boxed{IF = e^{\int at\,dt} = e^{\frac{1}{2}at^2}}$

$\left(x \cdot e^{\frac{1}{2}at^2}\right)' = 2t \cdot e^{\frac{1}{2}at^2}$

$x \cdot e^{\frac{1}{2}at^2} = \int 2t\, e^{\frac{1}{2}at^2}\, dt = \int 2\cancel{t} \cdot e^u \cdot \frac{du}{a\cancel{t}}$

$u = \frac{1}{2}at^2$

$du = \frac{1}{2} \cdot a \cdot 2t \cdot dt = at \cdot dt$

$x \cdot e^{\frac{1}{2}at^2} = \int \frac{2}{a} \cdot e^u \, du$

$x \cdot e^{\frac{1}{2}at^2} = \frac{2}{a} \cdot e^u + C \qquad dt = \frac{du}{at}$

$x \cdot e^{\frac{1}{2}at^2} = \frac{2}{a} \cdot e^{\frac{1}{2}at^2} + C$

$\underline{x = \frac{2}{a} + C \cdot e^{-\frac{1}{2}at^2}}$

If $a = 0$: $\quad x' = 2t \Rightarrow x = \int 2t\, dt = \underline{t^2 + C}$

b) $x'' + 2x' + x = 4e^t \quad , \; x(0) = 1, \; x'(0) = 2$

$\underline{x = x_h + x_p} = \underline{\underline{C_1 e^{-t} + C_2 t e^{-t} + e^t}} = \underline{\underline{t e^{-t} + e^t}}$

x_h: $\quad r^2 + 2r + 1 = 0 \rightarrow r = \underline{-1}$

$\quad x_h = \underline{C_1 e^{-t} + C_2 t e^{-t}}$

x_p: $\quad \left.\begin{aligned} x &= Ae^t \\ x' &= Ae^t \\ x'' &= Ae^t \end{aligned}\right\} \quad \begin{aligned} Ae^t + 2Ae^t + Ae^t &= 4e^t \\ 4Ae^t &= 4e^t \\ \underline{A} &\underline{= 1} \end{aligned}$

$x_p = \underline{e^t}$

$\underline{x(0) = 1}$: $\quad C_1 \cdot 1 + C_2 \cdot 0 \cdot e^0 + e^0 = 1$

$C_1 + 1 = 1 \Rightarrow \underline{C_1 = 0} \rightarrow \cancel{x = te}$

$x = C_2 t e^{-t} + e^t$

$x' = C_2 e^{-t} + C_2 t e^{-t} \cdot (-1) + e^t$

$\underline{x'(0) = 2}$: $\quad C_2 \cdot e^0 + C_2 \cdot 0 \cdot e^0 \cdot (-1) + e^0 = 2$

$C_2 + 1 = 2 \Rightarrow \underline{C_2 = 1}$

c) $x_{t+1} - x_t = r x_t + s \quad , \quad x_0 = \underline{100s}$

$x_{t+1} - (1+r)x_t = s$

$x_t = C \cdot (1+r)^t + \frac{s}{1-(1+r)} = C \cdot (1+r)^t - \frac{s}{r}$

$x_0 = C \cdot 1 - \frac{s}{r} = 100s \Rightarrow C = \frac{s}{r} + 100s = s\left(100 + \frac{1}{r}\right)$

$$\underline{\underline{x_t = s \cdot \left(100 + \frac{1}{r}\right)(1+r)^t - \frac{s}{r}}}$$

d) $x_{t+2} + 2x_{t+1} + x_t = 4t+4 \quad , \quad x_0 = 1 \, , \, x_1 = 1$

$x_t = x_t^h + x_t^p = \underline{\underline{C_1 \cdot (-1)^t + C_2 t (-1)^t + t}}$

$C_1 \cdot 1 + 0 + 0 = 1$

$C_1 \cdot (-1) + C_2 (-1) + 1 = 1$

$\cancel{-1} - C_2 \cancel{+1} = 1$

$\underline{C_1 = 1} \quad C_2 = -1$

$= \underline{\underline{(-1)^t - t(-1)^t + t}}$

$\underline{x_t^h}$: $r^2 + 2r + 1 = 0$

$r = -1 \Rightarrow x_t^h = \underline{C_1 \cdot (-1)^t + C_2 \cdot t \cdot (-1)^t}$

$\underline{x_t^p}$: $x_t = At + B$

$x_{t+1} = A(t+1) + B = At + A + B$

$x_{t+2} = A(t+2) + B = At + 2A + B$

$(At + 2A + B) + 2(At + A + B) + (At + B) = 4t + 4$

$4A \cdot t + (4A + 4B) = 4t + 4$

$A = 1 \quad B = 0 \Rightarrow x_t^p = \underline{t}$