LECTURE 11: Difference Equations

Review: Differential equations

First order:

- solvable by direct integration $(y' = t)$
- separable $(y' = y \cdot t)$
- first order linear $(y' + 2ty = t^3)$
- exact $(2ty + t^2 y' = 0)$

Second order:

- solvable by direct integration (twice) $(y'' = 6t)$ or by reduction to known first order differential equations $(y'' + y' = 2t)$
- linear second order with constant coefficients
  - homogeneous $y'' - 7y' + 12y = 0$
  - inhomogeneous $y'' - 7y' + 12y = 4$

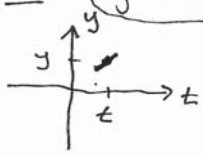
Key technical tool:

- integration!
  - substitution
  - integration by parts

Solutions of differential equations are functions $y = f(t)$

A differential equation gives predictions of the future.

Ex: $y' + 2y = 7 \rightarrow y' = 7 - 2y$

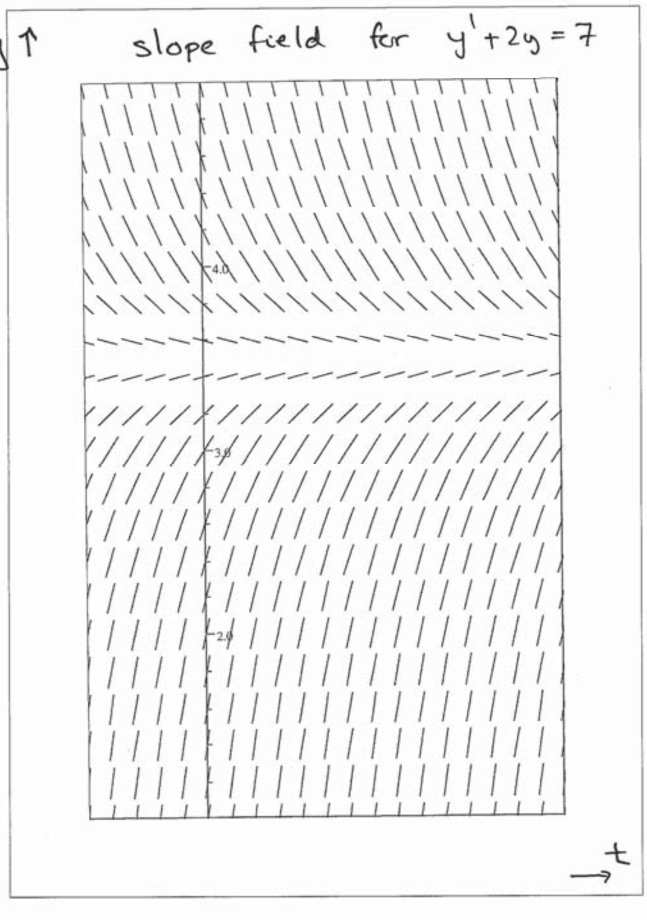


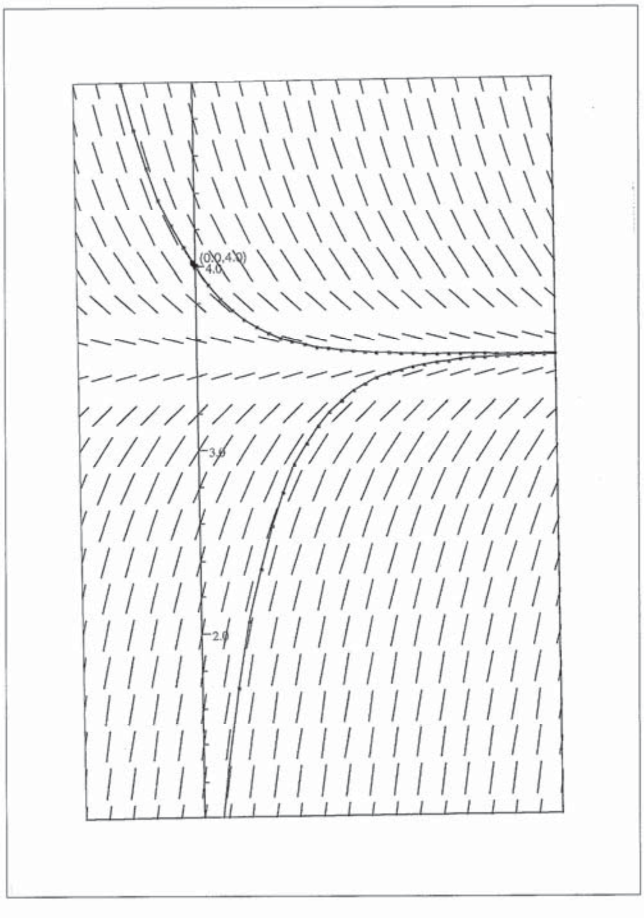
For each point (t, y) in the plane, we can compute $y' = 7 - 2y$ and use this to draw the tangent direction in (t, y).

slope field

$t = 1,\ y = 2$

$y' = 7 - 2 \cdot 2 = 3$





Difference equations

Example: $y_{t+1} = 2 \cdot y_t \,, \quad y_0 = 1$

$$y_0 = \underline{1}$$
$$y_1 = 2 \cdot y_0 = 2 \cdot 1 = \underline{2}$$
$$y_2 = 2 \cdot y_1 = 2 \cdot 2 = \underline{4}$$
$$y_3 = 2 \cdot y_2 = 2 \cdot 4 = \underline{8}$$
$$\vdots$$
$y_t = 2^t$ closed form

Example: $y_{t+2} = y_{t+1} + y_t$, $y_0 = 0, y_1 = 1$

$$y_0 = \underline{0}$$
$$y_1 = \underline{1}$$
$$y_2 = 0 + 1 = \underline{1}$$
$$y_3 = 1 + 1 = \underline{2}$$
$$y_4 = 1 + 2 = \underline{3}$$
$$y_5 = 2 + 3 = \underline{5}$$
$$\vdots$$
$y_t = ?$ closed form?

Defn: A <u>difference equation</u> is a <u>recurrence relation</u>, i.e. an equation relating terms in a sequence with one or more terms preceeding it.

A <u>solution</u> of a difference equation is a sequence that satisfies the recurrence relation.

Ex: $y_{t+1} = 2 y_t \,, \quad y_0 = 1$

$$\left.\begin{array}{l} y_0 = 1 \\ y_1 = 2 \\ y_2 = 4 \\ y_3 = 8 \\ \vdots \\ y_t = 2^t, \end{array}\right\}$$

$t = 0, 1, 2, 3, \ldots$

The solution is the sequence $1, 2, 4, 8, \ldots, 2^t, \ldots$

<u>Closed form solution:</u>

$$y_t = 2^t \,, \quad t = 0, 1, \ldots$$

Example: You borrow an amount K. The interest per period is r. The repayments are of equal amounts s. What is the balance b_t after t periods?

$$\underbrace{b_{t+1}}_{\substack{\text{balance after}\\ t+1 \text{ periods}}} = (1+r)\cdot \underbrace{b_t}_{\substack{\text{balance after}\\ t \text{ periods}}} - s \quad , \qquad b_0 = K$$

$$\begin{aligned}
b_0 &= K \\
b_1 &= (1+r)\cdot K - s \\
b_2 &= (1+r)\left[(1+r)K - s\right] - s \\
&= (1+r)^2 K - (1+r)s - s \\
&\ \vdots \\
b_t &= \ ?
\end{aligned}$$

Similarity with differential equations:

Balance with interest r

$$b_{t+1} = b_t \cdot (1+r)$$

$$b_{t+1} = b_t + b_t \cdot r$$

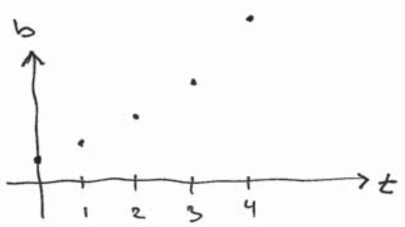
$$b_{t+1} - b_t = b_t \cdot r \quad \xrightarrow{\text{differential equation}} \quad b' = b \cdot r$$

change in balance — change in balance

$\downarrow$ — $\downarrow$

Solution:

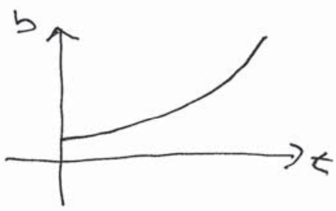
$$b_t = b_0 \cdot (1+r)^t$$



discrete time

Solution:

$$b(t) = b_0 \cdot e^{rt}$$



continuous time

First order difference equations:

$$y_{t+1} = F(y_t, t)$$

Examples:

1) $y_{t+1} = 2y_t$

2) $y_{t+1} = t^2 \cdot y_t - 2t$

3) $b_{t+1} = (1+r)\, b_t - s$ for some numbers r, s

A first order difference equation is linear with constant coefficients if

$$y_{t+1} - a y_t = b \quad (= b_t), \quad \text{where} \begin{cases} a \text{ is a constant} \\ b \text{ is a constant} \\ (\text{or a function of } t) \end{cases}$$

Homogeneous case: $y_{t+1} - a y_t = 0$

$y_{t+1} = a \cdot y_t$

$$\left.\begin{aligned} y_1 &= a \cdot y_0 \\ y_2 &= a \cdot y_1 = a^2 \cdot y_0 \\ y_3 &= a \cdot y_2 = a^3 \cdot y_0 \\ &\vdots \\ &\boxed{y_t = a^t \cdot y_0} \end{aligned}\right\}$$

Solution: $y_t = a^t \cdot y_0 \quad , t = 0, 1, 2, \ldots$

Solution: $y_t = a^t \cdot y_0 + b(a^{t-1} + a^{t-2} + \ldots + a + 1)$

Inhomogeneous case with b constant: $y_{t+1} - a y_t = b$

$y_{t+1} = a y_t + b$

$y_1 = a y_0 + b$

$y_2 = a \cdot y_1 + b = a(a y_0 + b) + b = a^2 y_0 + ab + b = a^2 y_0 + b(a+1)$

$y_3 = a y_2 + b = a(a^2 y_0 + ab + b) + b = a^3 y_0 + a^2 b + ab + b$

$= a^3 y_0 + b(a^2 + a + 1)$

$y_4 = a \cdot (a^3 y_0 + a^2 b + ab + b) + b$

$= a^4 y_0 + a^3 b + a^2 b + ab + b = a^4 y_0 + b(a^3 + a^2 + a + 1)$

Inhomogeneous case:

$y_{t+1} = a \cdot y_t + b$ has solution $y_t = a^t y_0 + b \cdot (a^{t-1} + a^{t-2} + \dots + a + 1)$

$$= a^t y_0 + b \cdot \frac{(a^{t-1} + a^{t-2} + \dots + a + 1)(a-1)}{a-1}$$

$$= a^t y_0 + b \cdot \frac{a^t - 1}{a-1}$$

$$\cancel{y_t = a^t y_0 + b \cdot (a^{t-1})}$$

$$y_t = a^t y_0 + b \cdot \frac{a^t - 1}{a-1}$$

$$\boxed{y_t = a^t y_0 + b \cdot \frac{1-a^t}{1-a}}$$

Examples:

$b_{t+1} = (1+r) b_t - s$

$$b_t = (1+r)^t b_0 + (-s) \cdot \frac{1-(1+r)^t}{1-(1+r)}$$

$$= (1+r)^t K + \frac{s}{r}(1-(1+r)^t)$$

$$\boxed{b_t = (1+r)^t K - \frac{s}{r}((1+r)^t - 1)}$$

Second order linear difference equations with constant coefficients:

Homogeneous case: $y_{t+2} + a y_{t+1} + b y_t = 0$

Inhomogeneous case: $y_{t+2} + a y_{t+1} + b y_t = c_t$

Example: $y_{t+2} = y_{t+1} + y_t$, $y_0 = 0$, $y_1 = 1$

(Fibonacci sequence)

$y_{t+2} - y_{t+1} - y_t = 0$ homogeneous

$y_0 = 0$, $y_1 = 1$, $y_2 = 1$, $y_3 = 2$, $y_4 = 3$, $y_5 = 5$, $y_6 = 8$, $y_7 = 13$, ...

Find closed form of y_t.

General homogeneous case:

$$y_{t+2} + a y_{t+1} + b y_t = 0$$

$$y_{t+1} + a y_t = 0$$

$$\underline{y_t = (-a)^t}$$

Is r^t a solution for some r?

$$y_t = r^t \quad y_{t+1} = r^{t+1} \quad y_{t+2} = r^{t+2}$$

$$r^{t+2} + a \cdot r^{t+1} + b r^t = 0$$

$$r^t \cdot (r^2 + ar + b) = 0$$

Characteristic equation: $r^2 + ar + b = 0$

Characteristic roots: $r = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$

Two roots $r_1 \neq r_2$: $y_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t$

One (double) root r: $y_t = C_1 \cdot r^t + C_2 \cdot t r^t$

No real solutions: $y_t = (\sqrt{b})^t \left(C_1 \cdot \cos \theta t + C_2 \sin \theta t \right)$

$\left(\cos \theta = -\frac{a}{2\sqrt{b}}, \ 0 \le \theta \le \pi \right)$

Example: Fibonacci sequence

$$y_{t+2} - y_{t+1} - y_t = 0 \quad , y_0 = 0, \quad y_1 = 1$$

Char. equation: $r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{1^2 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$y_t = C_1 \cdot \left(\frac{1+\sqrt{5}}{2} \right)^t + C_2 \cdot \left(\frac{1-\sqrt{5}}{2} \right)^t = \underline{\underline{\frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2} \right)^t - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^t}}$$

$y_0 = 0$: $\quad C_1 \cdot 1 + C_2 \cdot 1 = 0 \longrightarrow C_1 + C_2 = 0 \Leftrightarrow C_2 = -C_1$

$y_1 = 1$: $\quad C_1 \cdot \left(\frac{1+\sqrt{5}}{2} \right) + C_2 \cdot \left(\frac{1-\sqrt{5}}{2} \right) = 1$

$$C_1 \left(\frac{1+\sqrt{5}}{2} \right) - C_1 \left(\frac{1-\sqrt{5}}{2} \right) = 1$$

$$C_1 \cdot \left(\cancel{\tfrac{1}{2}} + \frac{\sqrt{5}}{2} - \cancel{\tfrac{1}{2}} + \frac{\sqrt{5}}{2} \right) = 1$$

$$\sqrt{5} \cdot C_1 = 1 \Rightarrow \underline{C_1 = \frac{1}{\sqrt{5}}} \Rightarrow \underline{C_2 = -\frac{1}{\sqrt{5}}}$$

Inhomogeneous case: $y_{t+2} + a y_{t+1} + b y_t = c_t$

Example: $y_{t+2} - 4y_{t+1} + 3y_t = 12$

Solution: $y_t = Y_t^h + Y_t^p = \underbrace{C_1 \cdot 3^t + C_2}_{Y_t^h} + \underbrace{(-6t)}_{Y_t^p} = C_1 \cdot 3^t + C_2 - 6t$

Y_t^h: general solution of the homogeneous equation $Y_{t+2} - 4Y_{t+1} + 3Y_t = 0$

Char. eqn: $r^2 - 4r + 3 = 0$

$r = 3, \; r = 1$

$Y_t^h = C_1 \cdot 3^t + C_2 \cdot 1^t$

$= C_1 \cdot 3^t + C_2$

Y_t^p: particular solution to $Y_{t+2} - 4Y_{t+1} + 3Y_t = 12$

Try $y_t = A$ (a constant):

$$\left.\begin{aligned} Y_{t+2} &= A \\ Y_{t+1} &= A \\ Y_t &= A \end{aligned}\right\} \quad \begin{aligned} A - 4A + 3A &= 12 \\ 0A &= 12 \end{aligned}$$

(no solution)

Try: $y_t^p = A \cdot t = -6t$

$$\left.\begin{aligned} Y_{t+2} &= A \cdot (t+2) \\ Y_{t+1} &= A(t+1) \\ Y_t &= A \cdot t \end{aligned}\right\} \quad \begin{aligned} &(At + 2A) - 4(At + A) + 3At = 12 \\ &(At - 4At + 3At) + (2A - 4A) = 12 \\ &0 \cdot t + (-2A) = 12 \\ &A = -6 \end{aligned}$$

Example: $x_{t+2} - 7x_{t+1} + 12x_t = t^2$

Homogeneous:

$$\left.\begin{aligned} x_{t+2} - 7x_{t+1} + 12x_t &= 0 \\ r^2 - 7r + 12 &= 0 \\ r = 3, \; r &= 4 \end{aligned}\right\} \Rightarrow Y_t^h = C_1 \cdot 3^t + C_2 \cdot 4^t$$

Particular: $x_{t+2} - 7x_{t+1} + 12x_t = t^2$

$$\left.\begin{aligned} &t^2 \\ &(t+1)^2 = t^2 + 2t + 1 \end{aligned}\right\} \quad Y_t = At^2 + Bt + C$$

$Y_t^p = \frac{1}{6}t^2 + \frac{5}{18}t + \frac{17}{54}$

$Y_t = C_1 \cdot 3^t + C_2 \cdot 4^t + \frac{1}{6}t^2 + \frac{5}{18}t + \frac{17}{54}$

$$\left(A(t+2)^2 + B(t+2) + C\right) - 7\left(A(t+1)^2 + B(t+1) + C\right) + 12(At^2 + Bt + C) = t^2$$

~~Try B=C=0~~:

$$\left.\begin{aligned} &A(t^2 + 4t + 4) - 7A(t^2 + 2t + 1) + 12At^2 \\ &+ B(t+2) - 7B(t+1) + 12Bt \end{aligned}\right\} = t^2$$

$C - 7C + 12C$

$$\underset{\substack{\shortparallel \\ 1}}{(6A)} \cdot t^2 + \underset{\substack{\shortparallel \\ 0}}{(-10A + 6B)}\, t + \underset{\substack{\shortparallel \\ 0}}{(-3A - 5B + 6C)} = t^2$$

$A = \frac{1}{6}$

$B = \frac{10}{36} = \frac{5}{18}$

$C = \frac{3A + 5B}{6} = \cancel{\frac{\;}{36}}$

$= \frac{3}{36} + \frac{25}{6 \cdot 18} = \frac{34}{108}$

Example: $$Y_{t+1} = aY_t + b$$
$$Y_{t+1} - aY_t = b$$

$$Y_t = Y_t^h + Y_t^p = C \cdot a^t + \frac{b}{1-a} \quad (*)$$

$$Y_t = a^t \cdot y_0 + b \cdot \frac{1-a^t}{1-a}$$
$$= a^t\left(y_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$$

Look at: Exam December 2009.

Hom: $Y_{t+1} - aY_t = 0$

Char. eqn: $r - a = 0$
$$r = a$$
$$Y_t^h = C \cdot a^t$$

Part: $Y_{t+1} - aY_t = b$

Try: $Y_t = A$: $$A - aA = b$$
$$A \cdot (1-a) = b$$
$$A = \frac{b}{1-a}$$